A note on the nonzero spectra of irreducible matrices

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Abstract

In this note we extend the necessary and sufficient conditions of Boyle-Handleman 1991 and Kim-Ormes-Roush 2000 for a nonzero eigenvalue multiset of primitive matrices over $\mathbb{R}_+$ and $\mathbb{Z}_+$, respectively, to irreducible matrices.

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1 Introduction

Denote by $\mathbb{R}^{n \times n} \supset \mathbb{R}_+^{n \times n}$ the algebra of real valued $n \times n$ matrices and the cone of $n \times n$ nonnegative matrices, respectively. For $A \in \mathbb{R}^{n \times n}$ denote by $\Lambda(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$ the eigenvalue multiset of $A$, i.e. $\det(zI-A) = \prod_{i=1}^{n} (z-\lambda_i(A))$. An outstanding problem in matrix theory, called NIEP, is to characterize a multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ which is an eigenvalue multiset of some $A \in \mathbb{R}^{n \times n}$. Denote by $\rho(\Lambda) := \max\{|\lambda|, \lambda \in \Lambda\}$, and by $\Lambda(r)$ all elements in $\Lambda$ satisfying $|\lambda| = r \geq 0$. For $\lambda \in \Lambda$ denote by $m(\lambda) \in \mathbb{N}$ the multiplicity of $\lambda$ in $\Lambda$. The obvious necessary conditions for $\Lambda = \Lambda(A)$ for some $A \in \mathbb{R}_+^{n \times n}$ are the trace conditions:

$$s_k(\Lambda) := \sum_{i=1}^{n} \lambda_i^k \geq 0 \text{ for } k = 1, \ldots,$$

since $s_k(\Lambda(A)) = \text{tr} A^k$. The following theorem is deduced straightforward from [2, Thm 2]. (See §2.)

Theorem 1.1 Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a multiset of complex numbers. Assume that the inequalities in (1.1) hold except for a finite number values of $k$. Then

1. $\bar{\Lambda} = \Lambda$.

2. $\rho(\Lambda) \in \Lambda$.

3. $m(\rho(\Lambda)) \geq m(\lambda)$ for all $\lambda \in \Lambda(\rho(\Lambda))$.

4. Assume that $\rho(\Lambda) > 0$ and let $\{\lambda_1, \ldots, \lambda_p\}$ be all distinct elements of $\Lambda$ such that $|\lambda_i| = \rho(\Lambda)$ and $m(\lambda_i) = m(\rho(\Lambda))$ for $i = 1, \ldots, p$. Then $\zeta \Lambda(\rho(\Lambda)) = \Lambda(\rho(\Lambda))$ for $\zeta = e^{2\pi\sqrt{-1}/p}$. 

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By considering the diagonal elements of $A^k$ and comparing them with the diagonal elements of $A^{km}$ Loewy and London added the following additional necessary conditions [7].

**Theorem 1.2** Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be an eigenvalue multiset of some $A \in \mathbb{R}^{n \times n}_+$. Then in addition to the inequalities (1.1) the following inequalities hold.

$$s_{km}(\Lambda) \geq \frac{1}{n^{k-1}}(s_m(\Lambda))^k \text{ for } m, k - 1 = 1, \ldots.$$ (1.2)

In particular,

$$\text{if } s_m(\Lambda) > 0 \text{ then } s_{km}(\Lambda) > 0 \text{ for } k = 2, \ldots.$$ (1.3)

The inequalities (1.1) and (1.2) imply that $\Lambda$ is an eigenvalue multiset of some $A \in \mathbb{R}^{n \times n}_+$ in the following cases: $n = 3$; $n = 4$ and $\Lambda$ is a multiset of real numbers. For $n = 4$ and nonreal $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ the conditions (1.1) and (1.2) are not sufficient [7]. The necessary and sufficient conditions are given in [9]. The inequality $ns_4(\Lambda) \geq (s_2(\Lambda))^2$ in (1.2) can be improved to $(n-1)s_4(\Lambda) \geq (s_2(\Lambda))^2$ if $s_1(\Lambda) = 0$ and $n$ is odd [8].

**Definition 1.3** A multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ is called a Frobenius multiset if the following conditions hold.

1. $\bar{\Lambda} = \Lambda$.
2. $\rho(\Lambda) \in \Lambda$.
3. $m(\lambda) = 1$ for each $\lambda \in \Lambda(\rho(\Lambda))$.
4. Assume that $\#\Lambda(\rho(\Lambda)) = p$. Then $\zeta\Lambda = \Lambda$ for $\zeta = e^{2\pi i/p}$.

The Frobenius theorem for irreducible $A \in \mathbb{R}^{n \times n}_+$, i.e. $(I + A)^{n-1}$ is a positive matrix, claims that $\rho(\Lambda(A)) > 0$ and $\Lambda(A)$ is a Frobenius set. In particular, an irreducible $A \in \mathbb{R}^{n \times n}_+$ is primitive, i.e. $A^{(n-1)^2+1}$ is a positive matrix, if and only if $\Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\}$, see [3, XII.5] and [4, §8.5.9].

We say that a multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_i \neq 0$ for $i = 1, \ldots, n$, is a nonzero eigenvalue multiset of a nonnegative matrix if there exists an integer $N \geq n$ and $A \in \mathbb{R}^{N \times N}_+$, such that $\Lambda$ is obtained from $\Lambda(A)$ by removing all zero eigenvalues. The following remarkable theorem was proved by Boyle and Handelman [1]. Namely, a multiset $\Lambda \subset \mathbb{C}\{0\}$ is a nonzero spectrum of a nonnegative primitive matrix if and only if $\Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\}$, and the inequalities (1.1) and (1.3) hold. See the recent proof of Thomas Laffey [6] of a simplified version of this result. The aim of this note is to extend the theorem of Boyle-Handelman to a nonzero eigenvalue multiset of nonnegative irreducible matrices.

**Theorem 1.4** Let $\Lambda$ be a multiset of nonzero complex numbers. Then $\Lambda$ is a nonzero eigenvalue multiset of a nonnegative irreducible matrix if and only if $\Lambda$ is a Frobenius set, and (1.1) and (1.3) hold.

Similarly, we extend the results of Kim, Ormes and Roush [5] to a nonzero eigenvalue multiset of nonnegative irreducible matrices with integer entries.
2 Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. For \( \Lambda = \{0, \ldots, 0\} \) the theorem is trivial. Assume that \( \rho(\Lambda) > 0 \). Consider the function

\[
f_{\Lambda}(z) = \sum_{i=1}^{n} \frac{1}{1 - \lambda_i z} = \sum_{k=0}^{\infty} s_k(\Lambda) z^k.
\]

Assume that \( s_k(\Lambda) \geq 0 \) for \( k > N \). Then by subtracting a polynomial \( P(z) \) of degree \( N \) at most, we deduce that \( f_0(z) := f(z) - P(z) \) has real nonnegative MacLaurin coefficients. So \( f_0(z) \) is analytic. Hence \( \Lambda = \Lambda \). The radius of convergence of this series is \( R(f_{\Lambda}) = \frac{1}{\rho(\Lambda)} \). The principal part of \( f \) is \( f_1 := \sum_{i, |\lambda_i| = \rho(\Lambda)} \frac{1}{1 - \lambda_i z} \). So \( \pi(f_{\Lambda}(z)) = \{(\lambda_1, m(\lambda_1), 1), \ldots, (\lambda_q, m(\lambda_q), 1)\} \), where \( \lambda_1, \ldots, \lambda_q \) are all pairwise distinct elements of \( \Lambda(\rho(\Lambda)) \), see [2, Dfn. 1]. Then parts 2–4 follow from [2, Thm. 2].

Proof of Theorem 1.4. Assume first that \( \Lambda \) is a nonzero eigenvalue multiset of a nonnegative irreducible matrix. The Frobenius theorem yields that \( \Lambda \) has to be a Frobenius set, and (1.1) and (1.3) hold. Assume now that \( \Lambda \) is a Frobenius set, and (1.1) and (1.3) hold. In view of the Boyle-Handelman theorem it is enough to consider the case

\[
\Lambda(\rho(\Lambda)) = \{\rho(\Lambda), \zeta \rho(\Lambda), \ldots, \zeta^{p-1} \rho(\Lambda)\}, \quad \text{for } \zeta = e^{\frac{2\pi i}{p}} \text{ and } 1 < p \in \mathbb{N}. \tag{2.1}
\]

Observe first that \( s_k(\Lambda) = 0 \) if \( p \not| k \). Let \( \phi : \mathbb{C} \to \mathbb{C} \) be the map \( z \mapsto z^p \). Since \( \zeta \Lambda = \Lambda \), it follows that for \( z \in \Lambda \) with multiplicity \( m(z) \) the multiplicity of \( z^p \) in \( \phi(\Lambda) \) is \( pm(z) \). Hence \( \phi(\Lambda) \) is a union of \( p \) copies of a Frobenius set \( \Lambda_1 \), where \( \rho(\Lambda_1) = \rho(\Lambda)^p \) and \( \Lambda_1(\rho(\Lambda_1)) = \{\rho(\Lambda_1)\} \). Moreover \( s_{kp}(\Lambda) = ps_k(\Lambda_1) \). Hence \( \Lambda_1 \) satisfies the assumptions of the Boyle-Handelman theorem. Thus there exists a primitive matrix \( B \in \mathbb{R}_{+}^{M \times M} \) whose nonzero eigenvalue multiset is \( \Lambda_1 \). Let \( A = [A_{ij}]_{i=j=1}^{p} \) be the following nonnegative matrix of order \( pM \).

\[
A = \begin{bmatrix}
0_{n \times n} & I_n & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} \\
0_{n \times n} & 0_{n \times n} & I_n & 0_{n \times n} & \cdots & 0_{n \times n} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0_{n \times n} & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & I_n \\
B & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} 
\end{bmatrix}. \tag{2.2}
\]

Then \( A \) is irreducible and the nonzero part of eigenvalue multiset \( \Lambda(A) \) is \( \Lambda \). \qed

3 An extension of Kim-Ormes-Roush theorem

In this section we give necessary and sufficient conditions on a multiset \( \Lambda \) of nonzero complex number to be a nonzero eigenvalue multiset of a nonnegative irreducible matrix with integer entries. Recall the Möbius function \( \mu : \mathbb{N} \to \{-1,0,1\} \). First \( \mu(1) = 1 \). Assume that \( n > 1 \). If \( n \) is not square free, i.e. \( n \) is divisible by \( l^2 \) for some positive integer \( l > 1 \), then \( \mu(n) = 0 \). If \( n > 1 \) is square free, let \( \omega(n) \) be the number of distinct primes that divide \( n \). Then \( \mu(n) = (-1)^{\omega(n)} \). The following theorem is a generalization of the Kim-Ormes-Roush theorem [5].
Theorem 3.1 Let $\Lambda$ be a multiset of nonzero complex numbers. Then $\Lambda$ is a nonzero eigenvalue multiset of a nonnegative irreducible matrix with integer entries if and only if the following conditions hold.

1. $\Lambda$ is a Frobenius set.

2. The coefficients of the polynomial $\prod_{\lambda \in \Lambda} (z - \lambda)$ are integers.

3. $t_k(\Lambda) := \sum_{d|k} \mu(d) s_d(\Lambda) \geq 0$ for all $k \in \mathbb{N}$.

The case $\Lambda(\rho(\Lambda)) = \{\rho(\Lambda)\}$ is the Kim-Ormes-Roush theorem.

Proof. Assume that $\Lambda$ is a nonzero spectrum of a nonnegative irreducible matrix with integer entries, i.e. $A \in \mathbb{Z}_+^{N \times N}$. Then part 1 follows from the Frobenius theorem. Since $\det(zI - A)$ has integer coefficients we deduce part 2. It is known that $t_k(\Lambda) = t_k(\Lambda(A))$ is the number of minimal loops of length $k$ in the directed multigraph induced by $A$, see [1]. Hence part 3 holds.

Suppose that $\Lambda$ satisfies 1–3. In view of the Kim-Ormes-Roush theorem it is enough to assume the case (2.1). We now use the notations and the arguments of the proof of Theorem 1.4. First $t_k(\Lambda) = 0$ if $p \nmid k$. Second $\prod_{\lambda \in \Lambda} (z - \lambda) = \prod_{\lambda \in \Lambda_1} (z^p - \kappa)$. Hence $\prod_{\lambda \in \Lambda_1} (z - \kappa)$ has integer coefficients. A straightforward calculation shows that $t_{pk}(\Lambda) = pt_k(\Lambda_1)$. Hence $t_k(\Lambda_1) \geq 0$. Kim-Ormes-Roush theorem yields the existence of $B \in \mathbb{Z}_+^{M \times M}$ such that $\Lambda_1$ is the nonzero eigenvalue multiset of $B$. Hence $\Lambda$ is the nonzero eigenvalue set of $A \in \mathbb{Z}_+^{pM \times pM}$ given by (2.2).

References


