MARGULIS NUMBERS FOR HAKEN MANIFOLDS

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ABSTRACT. For every closed orientable hyperbolic Haken 3-manifold and, more generally, for any orientable hyperbolic 3-manifold M which is homeomorphic to the interior of a Haken manifold, the number 0.286 is a Margulis number. If $H_1(M;\mathbb{Q}) \neq 0$, or if M is closed and contains a semi-fiber, then 0.292 is a Margulis number for M.

1. INTRODUCTION

If M is an orientable hyperbolic n-manifold, we may write $M = \mathbb{H}^n/\Gamma$ where $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^n)$ is discrete and torsion-free. The group Γ is determined, up to conjugacy in $\mathrm{Isom}_+(\mathbb{H}^n)$, by the hyperbolic structure on M.

For any $\gamma \in \Gamma$ and any $P \in \mathbb{H}^3$ we shall write $d_P(\gamma) = \mathrm{dist}(P, \gamma \cdot P)$. (Here and throughout the paper, dist denotes the hyperbolic distance on $\mathbb{H}^n$.) We will define a *Margulis number* for M, or for Γ, to be a positive real number μ with the following property:

1.0.1. *For any $P \in \mathbb{H}^n$, the subgroup of Γ generated by $\{x \in \Gamma \mid d_P(x) < \mu\}$ has an abelian subgroup of finite index.*

If $n \leq 3$ then any subgroup of Γ which contains an abelian subgroup of finite index is itself abelian. Hence in this case 1.0.1 may be replaced by the following simpler condition:

1.0.2. *For any $P \in \mathbb{H}^n$, if x and y are elements of Γ such that $\max(d_P(x), d_P(y)) < \mu$, then x and y commute.*

The Margulis Lemma [5, Chapter D] implies that for every $n \geq 2$ there is a positive constant which is a Margulis number for every hyperbolic n-manifold. The largest such number, $\mu(n)$, is called the *Margulis constant* for hyperbolic n-manifolds.

Margulis numbers play a central role in the geometry of hyperbolic manifolds. If M is a hyperbolic n-manifold, which for simplicity we take to be closed, and μ is a Margulis number for M, then the points of M where the injectivity radius is less than $\mu/2$ form a disjoint union of "tubes" about closed geodesics whose geometric structure can be precisely described. Topologically they are open $(n-1)$-ball bundles over S^1. This observation and the Margulis Lemma can be used to show, for example, that for every $V > 0$ there is a finite collection of compact orientable 3-manifolds $M_1, \ldots, M_N$, whose boundary components are tori, such that every closed, orientable hyperbolic 3-manifold of volume at most V can be obtained by a Dehn filling of one of the M_i.

Both authors are partially supported by NSF grant DMS-0906155.

The value of $\mu(3)$ is not known; the best known lower bound is $0.104\ldots$, due to Meyerhoff [16]. In this paper we will derive a larger lower bound for the Margulis numbers of closed orientable hyperbolic Haken 3-manifolds. In fact, our results apply to any orientable 3-manifold which is homeomorphic to the interior of a Haken 3-manifold.

Recall that a compact orientable irreducible 3-manifold N is *Haken* if it contains an incompressible surface or if it is homeomorphic to B^3. We will say that N is *strictly Haken* if it contains an incompressible surface which is not a fiber or a semi-fiber. (We will review the definitions of incompressible surface, fiber and semi-fiber in Section 4.)

Our main result is the following.

Theorem 1.1. *Let M be an orientable hyperbolic 3-manifold which is homeomorphic to the interior of a Haken manifold. (In particular M may be a closed Haken manifold.) Then 0.286 is a Margulis number for M. If $H_1(M;\mathbb{Q}) \neq 0$, or if M is closed and contains a semi-fiber, then 0.292 is a Margulis number for M.*

Remark 1.2. The condition $H_1(M;\mathbb{Q}) \neq 0$ always holds unless M is closed or $M = \mathbb{H}^3$. This is because B^3 is the only non-closed Haken manifold with first Betti number 0.

To prove Theorem 1.1 for a hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ which is homeomorphic to the interior of a given Haken manifold (or strict Haken manifold) N, we must obtain a lower bound for $\max(d_P(x), d_P(y))$ whenever P is a point of $\mathbb{H}^3$ and x and y are non-commuting elements of Γ.

If we assume for simplicity that N contains a separating incompressible surface, then the choice of such a surface allows us to identify $\Gamma \cong \pi_1(M)$ with an amalgamated free product $A \star_C B$. The proof of the lower bound for $\max(d_P(x), d_P(y))$ breaks up into various cases depending on the normal forms for x and y in $A \star_C B$. In some cases one can show that suitable short words in x and y generate a free group of rank 2, and one can then obtain the desired bound by combining the "$\log 3$ Theorem" and its refinements ([8], [4], [1], [6], [2]) with some delicate hyperbolic trigonometry; in other cases one can show that certain short words in x and y generate a free *semigroup* of rank 2, and one can then obtain a bound using packing arguments of the type used in [18, Section 5].

In cases where the $\log 3$ Theorem and packing arguments do not give a suitable lower bound, a third method is required. We shall illustrate this method by describing it in a simple case. Suppose that we have $x \in A$, $y \in B$, and that neither x^2 nor y^2 lies in C. Let X (respectively Y) denote the set of all elements of $\Gamma = A \star_C B$ whose normal form begins with an element of A (respectively B). Then Γ is the disjoint union of X, Y and C, and we have (i) $x^{\pm 1}Y \subset X$ and $y^{\pm 1}X \subset Y$, (ii) $xY \cap x^{-1}Y = \emptyset$, and (iii) $yX \cap y^{-1}X = \emptyset$. The argument now proceeds in a way that is partially analogous to the proof of the $\log 3$ theorem given in [8] for the special case of a rank-2 free group for which the normalized area measure is a Patterson-Sullivan measure on the sphere at infinity S_∞. The decomposition $\Gamma = X \cup Y \cup C$ gives rise to a decomposition of the area measure on S_∞, and the set-theoretic conditions (i)—(iii) give information about how the terms in the decomposition transform under x and y. This information is used to obtain a lower bound for $\max(d_P(x), d_P(y))$; here the hyperbolic trigonometry is combined with an argument analogous to the one used in [8].

The transition from information about measures on S_∞ to information about the quantity $\max(d_P(x), d_P(y))$ uses Lemma 5.5 of [8], a result that was also applied in [4]. The statement of the lemma given in [8] contains some minor errors and an irrelevant hypothesis. We provide a corrected and improved statement, with a complete proof, in Section 2 of the present paper, along with an account of the small additional arguments needed for the applications of the lemma given in [8] and [4].

In the sketch given above of a special case of the argument involving decompositions of measures, the group-theoretical argument was stated in terms of free products with amalgamation. In the general case, it is best to use the language of group actions on trees. The needed background concerning incompressible surfaces and actions on trees is given in Sections 3 and 4. The key argument involving decompositions of measures is given in section 6. The packing arguments that apply in the case where short words in x and y generate a free semigroup are given in Section 5. The hyperbolic trigonometry needed for the arguments described above appears in Section 7. In Section 8, these ingredients are assembled to prove the main theorem.

We are grateful to David Futer for pointing out that our methods apply to the case of a non-compact hyperbolic manifold, and to Mark Kidwell and David Krebes for pointing out some errors in the original manuscript. We thank the anonymous referee for several helpful suggestions which improved the exposition.

2. Measures and displacements

The purpose of this section is to prove the following result, which is a corrected and improved version of Lemma 5.5 of [8] and will be needed in Section 6 of the present paper.

Lemma 2.1. *Let a and b be numbers in $[0,1]$ which are not both equal to 0 and are not both equal to 1. Let γ be a loxodromic isometry of $\mathbb{H}^3$ and let z be a point in $\mathbb{H}^3$. Suppose that ν is a measure on S_∞ such that*

(1) $\nu \le A_z$,
(2) $\nu(S_\infty) \le a$, and
(3) $\int_{S_\infty} \lambda_{\gamma,z}^2 \, d\nu \ge b$.

Then we have $a > 0$, $b < 1$, and

$$\mathrm{dist}(z, \gamma \cdot z) \ge \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}.$$

In [8, Lemma 5.5], the numerator and denominator were interchanged in the conclusion. The issue of guaranteeing that the denominator in the corrected expression is non-zero was not addressed in the statement or proof, and the hypothesis which assures that this denominator is non-zero was omitted. Furthermore, the statement of [8, Lemma 5.5] contained the hypothesis $a \le 1/2$, which is not needed for the proof.

For the applications of [8, Lemma 5.5] given in [8] and [4], we will explain after the proof of Lemma 2.1 how the latter result can be applied to replace [8, Lemma 5.5].

Proof of Lemma 2.1. Throughout this proof we use the conventions of [8]. We let h denote the constant $\operatorname{dist}(z, \gamma \cdot z)$, and set $c = \cosh h$ and $s = \sinh h$. Since γ is loxodromic we have $h > 0$, so that $c > s$. We let λ denote the function $\lambda_{\gamma,z}$. We identify $\overline{\mathbb{H}}^3$ conformally with the unit ball in $\mathbb{R}^3$ in such a way that z is the origin (so that S_∞ has the round metric centered at z) and $\gamma^{-1} \cdot z$ is on the positive vertical axis.

According to [8, 2.4], we have $\lambda(\zeta) = \mathcal{P}(z, \gamma^{-1} \cdot z, \zeta)$ for all $\zeta \in S_\infty$. Hence by [8, 2.1.1], λ is given by the formula

$$\lambda(\zeta) = (c - s\cos\phi)^{-1}$$

where $\phi = \phi(\zeta)$ is the angle between the positive vertical axis and the ray from the origin through ζ; thus ϕ is the polar angle of ζ in spherical coordinates.

Set $A = A_z$. Since S_∞ has the round metric centered at z, the measure A is obtained by dividing the area measure on the unit sphere by the area 4π of the sphere. In spherical coordinates θ and ϕ on the unit sphere we have $dA = (1/4\pi)\sin\phi\, d\phi d\theta$.

Set $\phi_0 = \arccos(1 - 2a) \in [0, \pi]$, and let $C \subset S_\infty$ denote the spherical cap defined by the inequality $\phi \ge \phi_0$. Then we have

$$A(C) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\phi_0} \sin\phi\, d\phi d\theta = \frac{1}{2}(1 - \cos\phi_0) = a.$$

Thus by hypothesis (ii) we have $A(C) \ge \nu(S_\infty)$. Observe also that since λ is given by the function $(c - s\cos\phi)^{-1}$, which is positive and monotone decreasing for $0 \le \phi \le \pi$, we have $\inf \lambda(C) \ge \sup \lambda(S_\infty - C)$. Since we also have $\nu \le A$ by hypothesis (i), we may apply [8, Lemma 5.4] with $f = \lambda^2$ to obtain

$$\begin{aligned}\int_{S_\infty} \lambda^2 d\nu \le \int_C \lambda^2 dA &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\phi_0} \frac{\sin\phi}{(c - s\cos\phi)^2} d\phi d\theta \\ &= \frac{1}{2}\int_0^{\phi_0} \frac{\sin\phi}{(c - s\cos\phi)^2} d\phi = \frac{1}{2s}\left(\frac{1}{c-s} - \frac{1}{c - s\cos\phi_0}\right),\end{aligned}$$

where the last step follows from the substitution $u = c - s\cos\phi$. Recalling that $\cos\phi_0 = 1 - 2a$ and using hypothesis (iii), we find that

$$b \le \int_{S_\infty} \lambda^2 d\nu \le \frac{a}{(c-s)(c-s+2as)},$$

which by the definitions of c and s gives

$$be^{-2h} + ab - abe^{-2h} \le a. \tag{2.1.1}$$

It follows from (2.1.1) that if $a = 0$ then $b \leq 0$, and that if $b = 1$ then $a \geq 1$; in view of the hypotheses it follows that $a > 0$ and that $b < 1$. It now follows from (2.1.1) that

$$e^{-2h} \leq \frac{a(1-b)}{b(1-a)},$$

which implies the conclusion of the lemma. □

In the applications of [8, Lemma 5.5] given in [8], Lemma 2.1 can be applied without change. In particular, these applications involved only specific values of a and b which are not equal to 1 or 0.

For the application of [8, Lemma 5.5] given in [4], citing Lemma 2.1 requires a bit more care. The application appears in the proof of Theorem 6.1 of [4], and in the notation of that proof we have $a = \alpha_i$ and $b = 1 - \beta_i$ for some given $i \in \{1, \dots, k\}$. In order to apply Lemma 2.1 we must check that we cannot have $a = b = 1$ or $a = b = 0$. If $a = b = 1$ then $\alpha_i = 1$ and $\beta_i = 0$, which is impossible because $\alpha_i \leq \beta_i$ in the context of the proof. Now suppose that $a = b = 0$, so that $\alpha_i = 0$ and $\beta_i = 1$. Since $\sum_{i=1}^{k}(\alpha_i + \beta_i) = 1$ and $k \geq 2$, we may fix an index $j \neq i$ in $\{1, \dots, k\}$, and we have $\alpha_j = \beta_j = 0$. Hence the measures ν_j and ν_j' are identically zero. But we have

$$\int (\lambda_{\xi_j^{-1}, z_0})^2 d\nu_j' = 1 - \int d\nu_j,$$

which is impossible if $\nu_j = \nu_j' = 0$.

3. Group actions on trees

We begin by reviewing some elementary notions related to group actions on trees. Our point of view here is similar to that taken in [9].

3.1. By a *tree* we will mean a 1-connected 1-dimensional simplicial complex. We may regard the set of vertices of a tree as an integer metric space by defining the distance between two vertices to be the number of edges in the arc joining them.

A *line* in a tree T is a subcomplex isomorphic to the real line with the standard triangulation, in which the vertices are the integers.

If Γ is a group, we will define a Γ-*tree* to be a tree equipped with a simplicial action of Γ which has no inversions in the sense of [17]. We will be using basic facts about Γ-trees proved in [17]. In particular, if T is a Γ-tree and γ is a non-trivial element of Γ then either γ has a non-empty fixed tree, in which case it is said to be T-*elliptic*, or γ has a unique invariant line, called its *axis*; in this case γ is said to be T-*hyperbolic*. The stabilizer of an edge e or a vertex s will be denoted Γ_e or Γ_s respectively.

If T is a Γ-tree and γ is a T-elliptic element of Γ, we will denote its fixed tree by $\mathrm{Fix}(\gamma)$. We also set $\mathrm{Per}(\gamma) = \bigcup_{n=1}^{\infty} \mathrm{Fix}(\gamma^n)$. Rewriting $\mathrm{Per}(\gamma)$ as $\bigcup_{n=1}^{\infty} \mathrm{Fix}(\gamma^{n!})$ shows that it is a monotone union of subtrees. Hence $\mathrm{Per}(\gamma)$ is a subtree of T, which we refer to as the *periodic subtree* of γ.

Definition 3.2. Let Γ be a group and let T be a Γ-tree. We shall say that the action of Γ on T is *linewise faithful* if for every line L in T, the subgroup of Γ that fixes L pointwise is trivial. We shall say that the action is *trivial* if some vertex of T is fixed by the entire group Γ.

Proposition 3.3. *Let Γ be a group and let T be a Γ-tree. Suppose that the action of Γ on T is non-trivial. Then for each vertex s of T, the stabilizer Γ_s has infinite index in Γ.*

Proof. If Γ_s has finite index in Γ then the orbit $\Gamma \cdot s$ is finite. Let T_0 denote the smallest subtree containing $\Gamma \cdot s$. Then T_0 is a finite subtree and is Γ invariant. It is clear that the action of Γ on T_0 factors through a finite quotient G of Γ. But it follows from [17, No. I.6.3.1] that if G is any finite group, then for any G-tree T_0 the action of G on T_0 is trivial. This contradicts the non-triviality of the action of Γ on T. $\square$

Definition 3.4. We will say that elements $x_1, \dots, x_n$ of a group Γ are *independent* if they generate a free group of rank n, and *semi-independent* if they generate a free semigroup of rank n. (In other words, $x_1, \dots, x_n$ are semi-independent if distinct positive words in these elements represent distinct elements of Γ.)

Lemma 3.5. *Let Γ be a group and let T be a Γ-tree. Suppose that x and y are T-hyperbolic elements of Γ whose axes are distinct. Then x and y are semi-independent.*

Proof. This follows from [11, p. 687, Proof of Lemma]. $\square$

Proposition 3.6. *Let Γ be a group, and let T be a Γ-tree such that the action of Γ on T is non-trivial and linewise faithful. Then any two non-commuting T-hyperbolic elements of Γ are semi-independent.*

Proof. Suppose that $x_1, x_2 \in \Gamma$ are non-commuting T-hyperbolic elements. Let A_i denote the axis of x_i in T. Then x_i acts by a translation on A_i. If $A_1 = A_2$, it follows that $x_1 x_2 x_1^{-1} x_2^{-1}$ fixes the line $A_1 = A_2$ pointwise. Since the action of Γ on T is linewise faithful, it follows that $x_1 x_2 x_1^{-1} x_2^{-1} = 1$, a contradiction. Hence $A_1 \neq A_2$. It now follows from Lemma 3.5 that x and y are semi-independent. $\square$

4. Incompressible surfaces and actions on trees

Here, and in the sequel, we will often suppress base points when denoting fundamental groups of connected spaces. (We note that, while inclusion homomorphisms are only defined up to conjugacy if we have not chosen a base point, the injectivity of an inclusion homomorphism is independent of such a choice.) In addition, we will often assume that an identification of the fundamental group of a connected 3-manifold M with the group of deck transformations of its universal cover $\widetilde{M}$ has been fixed.

Definition 4.1. Let M be a connected orientable 3-manifold and F an orientable 2-manifold embedded in M. We will say that F is *bicollared* if there exists an embedding $c : F \times [-1, 1] \to M$, proper in the sense of general topology, such that $c(x, 0) = x$ for all $x \in F$, and $c((\operatorname{int} F) \times [-1, 1]) \subset \operatorname{int} M$. The map c will be called a *bicollaring* of F.

4.2. Let F be a bicollared surface in a connected, orientable 3-manifold M. Let $c : F \times [-1,1] \to M$ be a bicollaring of F. If $p : \widetilde{M} \to M$ denotes the universal covering of M, then $\widetilde{F} = p^{-1}(F)$ inherits a bicollaring $\widetilde{c}$. We may partition $\widetilde{M}$ into the components of $\widetilde{M} - \widetilde{c}(\widetilde{F} \times (-1,1))$ and the sets of the form $\widetilde{c}(\widetilde{\Phi} \times \{t\})$, where $\widetilde{\Phi}$ ranges over the components of $\widetilde{F}$ and t ranges over the interval $(-1,1)$. Let T denote the decomposition space defined by this partition, and let $q : \widetilde{M} \to T$ denote the decomposition map. Then T may be regarded as a tree in such a way that the vertices of T are images under q of the components of $\widetilde{M} - \widetilde{c}(\widetilde{F} \times (-1,1))$, and the edges of T are images under q of the components of $\widetilde{F}$. If Γ denotes the group of deck transformations of $\widetilde{M}$, the action of Γ on $\widetilde{M}$ induces a simplicial action, without inversions, on T. Thus T has the structure of a Γ-tree in a natural way; we call it the *dual tree* of the bicollared surface F.

Definition 4.3. Let M be a compact orientable 3-manifold, possibly with non-empty boundary. By an *incompressible surface* in M we mean a connected bicollared surface $F \subset M$ such that

- the inclusion homomorphism $\pi_1(F) \to \pi_1(M)$ is injective;
- F is not the boundary of a 3-ball in M; and
- F is not parallel to a subsurface of ∂M.

Proposition 4.4. *Suppose that F is an incompressible surface in a compact orientable 3-manifold M. Let Γ denote the group of deck transformations of $\widetilde{M}$, and T the dual Γ-tree of F. Then T is a non-trivial Γ-tree.*

Proof. We identify Γ with $\pi_1(M)$. The stabilizer of a vertex of T is then a conjugate of the image under inclusion of $\pi_1(V)$ for some component V of $M - F$.

If the surface F is nonseparating then there is a surjective homomorphism $\pi_1(M) \to \mathbb{Z}$ whose kernel contains the image under inclusion of $\pi_1(M - F)$. This shows that the vertex stabilizers are proper subgroups of Γ in this case.

Otherwise, we may write $M = A \cup B$ where A and B are compact connected submanifolds of M with $A \cap B = F$. Since F is incompressible, the inclusion homomorphisms $\pi_1(F) \to \pi_1(A)$ and $\pi_1(F) \to \pi_1(B)$ are injective. By Van Kampen's theorem, $\pi_1(M)$ may be identified with the free product of $\pi_1(A)$ and $\pi_1(B)$ amalgamated along $\pi_1(F)$. In particular $\pi_1(A)$ and $\pi_1(B)$ are then identified with subgroups of $\pi_1(M)$. We will show that both of these factors are proper subgroups.

Assume for a contradiction that $\pi_1(A) = \pi_1(M)$. The normal form theorem for an amalgamated free product then implies that $\pi_1(F) = \pi_1(B)$, i.e. that the inclusion $F \hookrightarrow B$ induces an isomorphism of fundamental groups. It follows from [12, Theorem 10.5] that there is a homeomorphism from B to $F \times [0,1]$ sending F to $F \times \{0\}$. But this implies that F is parallel to a subsurface of ∂M, a contradiction to the incompressibility of F. Thus $\pi_1(A)$ is a proper subgroup of $\pi_1(M)$. A symmetrical argument shows that $\pi_1(B)$ is a proper subgroup of $\pi_1(M)$.

Since any vertex stabilizer is conjugate to one of the two proper subgroups $\pi_1(A)$ and $\pi_1(B)$, it follows that T is a non-trivial Γ-tree in this case. $\square$

Definition 4.5. An incompressible surface F in a closed orientable 3-manifold M is called a *fiber* if there is a fibration $M \to S^1$ having F as one of the fibers. We will say that F is a *semi-fiber* if F separates M into two components, each of which is the interior of a twisted I-bundle over a non-orientable surface.

Proposition 4.6. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold containing an incompressible surface F which is not a fiber or a semi-fiber. Let g denote the genus of F, and let T denote the dual Γ-tree of F. Then for every non-trivial T-elliptic element $\gamma \in \Gamma$, the diameter (as an integer metric space, cf. 3.1) of the set of fixed vertices γ in T is at most $14g - 12$. In particular, the action of Γ on T is linewise faithful.*

Proof. The first assertion follows from [10, Corollary 1.5]. The second assertion follows from the first, since a line in a tree has infinite diameter. $\square$

Lemma 4.7. *Let N be a connected orientable 3-manifold without boundary, and let A_0 and A_1 be disjoint incompressible open annuli in N. Suppose that the inclusion homomorphism $\pi_1(A_0) \to \pi_1(N)$ is an isomorphism. Then the inclusion homomorphism $\pi_1(A_1) \to \pi_1(N)$ is also an isomorphism.*

Proof. In this proof, unlabeled homomorphisms will be understood to be induced by inclusion. For $i = 0, 1$ it follows from incompressibility that the image of $\pi_1(A_i) \to \pi_1(N)$ has finite index in $\pi_1(N)$. If one of the (bicollared) annuli A_i did not separate N then the image of $H_1(A_i; \mathbb{Z}) \to H_1(N; \mathbb{Z})$ would lie in the kernel of a homomorphism of $H_1(N; \mathbb{Z})$ onto $\mathbb{Z}$, a contradiction. Hence each of the A_i separates N, and there is a connected submanifold N_0 of N whose frontier is $A_0 \cup A_1$. Since the A_i are incompressible in N, $\pi_1(N_0) \to \pi_1(N)$ is injective. Since $\pi_1(A_0) \to \pi_1(N)$ is surjective, $\pi_1(N_0) \to \pi_1(N)$ is also surjective, and is therefore an isomorphism. It follows that $\pi_1(A_0) \to \pi_1(N_0)$ is an isomorphism, and that the image of $\pi_1(A_1) \to \pi_1(N_0)$ has finite index in the infinite cyclic group $\pi_1(N_0)$. Hence there is a map $f : S^1 \times [0, 1] \to N_0$ such that for $i = 0, 1$ we have $f(S^1 \times \{i\}) \subset A_i$, and $f|S^1 \times \{i\}$ is homotopically non-trivial. It now follows from Waldhausen's generalized loop theorem [19] that there is a properly embedded planar surface $P \subset N_0$ having at most one boundary component in each A_i, and such that each boundary component of P is homotopically non-trivial in ∂N_0. Since the A_i are incompressible in N, the surface P has no disk components; hence it has exactly one boundary component in each A_i, and is therefore an annulus. The boundary of P must consist of a core curve in A_0 and a core curve in A_1. Since $\pi_1(A_0) \to \pi_1(N)$ is an isomorphism, it now follows that $\pi_1(A_1) \to \pi_1(N)$ is also an isomorphism. $\square$

Remark 4.8. A proof of Lemma 4.7 could also be based on [15, Theorem 2].

Proposition 4.9. *Let F be an incompressible surface in an orientable 3-manifold M. Let Γ denote the group of deck transformations of the universal cover of M, and let T denote* *the dual Γ-tree of F. Suppose that γ is an infinite-order element of Γ such that $\mathrm{Fix}(\gamma) \subset T$ contains at least one edge of T. Then for every integer $n > 0$ we have $\mathrm{Fix}(\gamma^n) = \mathrm{Fix}(\gamma)$.*

Proof. We choose an open edge e_0 in $\mathrm{Fix}(\gamma)$. If $n > 0$ is given, $\mathrm{Fix}(\gamma^n)$ is a subtree containing e_0, and is therefore a union of closed edges. Hence it suffices to show that if an open edge e_1 is fixed by γ^n then it is fixed by γ. We may assume that $e_1 \neq e_0$.

Let C denote the infinite cyclic group $\langle\gamma\rangle$, and for $i = 0, 1$ let C_i denote the stabilizer of e_i in C; then $C_0 = C$ since γ fixes e_0, and C_1 is also infinite cyclic since γ^n fixes e_1. We need to show that $C_1 = C$. Since C stabilizes e_0 and $e_1 \neq e_0$, the C-orbits of e_0 and e_1 are distinct.

We use the notation of Subsection 4.2. For $i = 0, 1$, we have $e_i = q(\widetilde{\Phi}_i \times (-1, 1))$ for some component $\widetilde{\Phi}_i$ of $\widetilde{F}$. The stabilizer of $\widetilde{\Phi}_i$ is C_i. Hence $A_i := \widetilde{\Phi}_i/C_i$ is identified with a surface in the 3-manifold $N := \widetilde{M}/C$, which is incompressible since F is incompressible in M. Since $C_i \cong \pi_1(A_i)$ is infinite cyclic, each A_i is an annulus. Since $C_0 = C$, the inclusion homomorphism $\pi_1(A_0) \to \pi_1(N)$ is surjective. Since the e_i are in distinct C-orbits we have $A_0 \cap A_1 = \emptyset$. It therefore follows from Lemma 4.7 that the inclusion homomorphism $\pi_1(A_1) \to \pi_1(N)$ is surjective. This implies that $C_1 = C$, as required. $\square$

Proposition 4.10. *Let Γ be a group, let T be a Γ-tree, and let γ_0, γ_1 be T-elliptic elements of Γ. Suppose that $\mathrm{Per}(\gamma_0) \cap \mathrm{Per}(\gamma_1) = \emptyset$. Then γ_0 and γ_1 are independent in Γ.*

Proof. The hypotheses immediately imply that γ_0 and γ_1 have infinite order. We will apply the Klein criterion, (or "ping-pong lemma") as stated in [14, Proposition 12.2], taking the groups G_1 and G_2 of the latter result to be the infinite cyclic groups generated by γ_0 and γ_1 respectively. According to this criterion, we need only construct disjoint subsets Ω_0 and Ω_1 of T such that $\gamma_i^n \cdot \Omega_i \subset \Omega_{1-i}$ for all $0 \neq n \in \mathbb{Z}$. For $i = 0, 1$ set $X_i = \mathrm{Per}(\gamma_i)$. By hypothesis we have $X_0 \cap X_1 = \emptyset$. Let C be an open topological arc in T whose endpoints are vertices, one of which lies in X_0 and one in X_1. Among all such arcs we choose C in such a way as to minimize the number of open edges that it contains. Then for $i = 0, 1$, the set $C \cap X_i$ consists of a single vertex s_i. Let e_i denote the open edge which is contained in C and has s_i as an endpoint. (Note that e_0 and e_1 may or may not coincide.) We define Ω_{1-i} to be the component of $T - e_i$ that contains s_i. Since T is a tree, Ω_i is a component of $T - C$. In particular, the sets Ω_0, Ω_1 and C are pairwise disjoint. Since X_i contains s_i and is disjoint from e_i, we have $X_i \subset \Omega_{i-1}$ for $i = 0, 1$. Note that $\Omega_i \cup C$ is connected for $i = 0, 1$.

Since e_i is not contained in $\mathrm{Per}(\gamma_i)$ it follows that $\gamma_i^n \cdot e_i \neq e_i$ for $0 \neq n \in \mathbb{Z}$; thus we have $\gamma_i^n \cdot \overline{e}_i \subset T - e_i$. On the other hand we have $s_i \in X_i = \mathrm{Per}(\gamma_i)$ and hence $\gamma^n \cdot s_i \in X_i \cap \gamma^n \cdot \overline{e}_i$. In particular, $X_i \cup \gamma^n \cdot \overline{e}_i$ is connected, and it follows that the component of $T - e_i$ that contains $\gamma_i^n \cdot \overline{e}_i$ also contains s_i and is therefore equal to Ω_{i-1}. Thus we have shown that

$$\gamma_i^n \cdot \overline{e}_i \subset \Omega_{1-i} \tag{4.10.1}$$

whenever $0 \neq n \in \mathbb{Z}$ and $i \in \{0, 1\}$.

It follows that e_i is disjoint from $\gamma_i^n \cdot (C \cup \Omega_i)$, and hence from its closure $\gamma_i^n \cdot (\overline{C} \cup \Omega_i)$. Since $\overline{C} \cup \Omega_i$ is connected, $\gamma_i^n \cdot (\overline{C} \cup \Omega_i)$ must be contained in a component of $T - e_i$; and since

$s_i = \gamma_i \cdot s_i \in \gamma_i^n \cdot \overline{C}$, this component must be Ω_{1-i}. Thus we have shown that $\gamma_i^n \cdot (\overline{C} \cup \Omega_{1-i}) \subset \Omega_i$ whenever $0 \neq n \in \mathbb{Z}$ and $i \in \{0, 1\}$. In particular, we have $\gamma_i^n \cdot \Omega_i \subset \Omega_{1-i}$, as required. $\square$

Proposition 4.11. *Let $M = \mathbb{H}^3/\Gamma$ be a closed orientable hyperbolic 3-manifold, and let T be a non-trivial Γ-tree. Suppose that γ_0 and γ_1 are non-commuting T-elliptic elements of Γ such that $\mathrm{Fix}(\gamma_0) \cap \mathrm{Fix}(\gamma_1) \neq \emptyset$. Then γ_0 and γ_1 are independent in Γ.*

Proof. Choose a vertex $s \in \mathrm{Fix}(\gamma_0) \cap \mathrm{Fix}(\gamma_1)$. Then $H := \langle \gamma_1, \gamma_2 \rangle$ is a subgroup of Γ_s. Since T is a non-trivial Γ-tree, it follows from 4.4 that $H < \Gamma_s$ has infinite index in Γ. But since M is a closed hyperbolic 3-manifold, it follows from [13, Theorem VI.4.1] that every two-generator subgroup of infinite index in $\Gamma = \pi_1(M)$ is free of rank at most 2; since γ_0 and γ_1 do not commute, H is a free group of rank 2, i.e. γ_0 and γ_1 are independent. $\square$

Proposition 4.12. *Let F be an incompressible surface in the closed orientable hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$, and let T be the dual Γ-tree of F. Suppose that γ_0 and γ_1 are non-commuting elements of Γ such that $\mathrm{Fix}(\gamma_i) \subset T$ contains at least one edge of T for $i = 0, 1$. Then γ_0 and γ_1 are independent in Γ.*

Proof. Since M is a hyperbolic manifold, Γ is torsion-free. Thus it follows from Proposition 4.9 that $\mathrm{Per}(\gamma_i) = \mathrm{Fix}(\gamma_i)$ for $i = 0, 1$. Hence if $\mathrm{Fix}(\gamma_0) \cap \mathrm{Fix}(\gamma_1) = \emptyset$, it follows from Proposition 4.10 that γ_0 and γ_1 are independent. If $\mathrm{Fix}(\gamma_0) \cap \mathrm{Fix}(\gamma_1) \neq \emptyset$, then γ_0 and γ_1 are independent by Proposition 4.11. $\square$

5. Word growth and displacements

We observe that for any $P \in \mathbb{H}^3$ and any isometries x and y of $\mathbb{H}^3$, we have $d_P(xy) \leq d_P(x) + d_P(y)$, $d_P(x^{-1}) = d_P(x)$, and $\mathrm{dist}(x \cdot P, y \cdot P) = d_P(x^{-1}y)$. These facts will be used frequently in this and the following sections.

Definition 5.1. If S is a finite subset of a group Γ, and m is a positive integer, we will denote by $b_m(S)$ the number of elements of Γ that can be expressed as words of length at most m in elements of S. We will set $\omega(S) = \lim_{m\to\infty} b_m(S)^{1/m}$; it is pointed out in [18] that this limit exists.

Proposition 5.2. *Suppose that Γ is a torsion-free discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^n)$ for some integer $n \geq 2$. Let x and y be elements of Γ. Then for any point $P \in \mathbb{H}^n$ we have*

$$\max(d_P(x), d_P(y)) \geq \frac{\log \omega(\{x, y\})}{n-1}.$$

Proof. (Cf. [18, proof of Proposition 5.2]) Set $\omega = \omega(\{x, y\})$. If the conclusion is false, there exist real numbers $C < \omega$ and $\lambda < (\log C)/(n-1)$ such that $d_P(x) < \lambda$ and $d_P(y) < \lambda$. Since $C < \omega$, the definition of $\omega(\{x, y\})$ implies that $b_m(S) > C^m$ for all sufficiently large m. Choose a bounded neighborhood U of P such that $\gamma \cdot U \cap U = \emptyset$ for every $\gamma \in \Gamma - 1$. Set $v = \mathrm{vol}\, U > 0$. If $\gamma \in \Gamma$ is expressible as a word of length $\leq m$ in x and y then $d_P(\gamma) < m\lambda$. Thus for any m there are $b_m(\{x, y\})$ elements $\gamma \in \Gamma$ such that $\gamma \cdot P$ lies in the ball of radius $m\lambda$ about P. Hence if $\Delta = \mathrm{diam}\, U$, there are $b_m(S)$ disjoint sets of the form $\gamma \cdot U$, $\gamma \in \Gamma$, in a ball of radius $m\lambda+\Delta$ about P. But there is a constant K depending on the dimension n such that the volume of any ball of sufficiently large radius r in hyperbolic n-space is bounded above by $K\exp((n-1)r)$. Hence for large m we have $b_m(S)\cdot v < K\exp((n-1)(m\lambda+\Delta))$, and therefore

$$v \cdot C^m < K \cdot \exp((n-1)(m\lambda+\Delta)). \tag{5.2.1}$$

Now taking logarithms of both sides of (5.2.1), dividing by m and taking limits as $m \to \infty$ we get $(n-1)\lambda \geq \log C$. This contradicts our choices of C and λ. ☐

Corollary 5.3. *Suppose that Γ is a torsion-free discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^n)$ for some integer $n \geq 2$. Let x and y semi-independent elements of Γ. Then for any point $P \in \mathbb{H}^n$ we have*

$$\max(d_P(x), d_P(y)) \geq \frac{\log 2}{n-1}.$$

Proof. For any positive integer m there are 2^m positive words in x and y, and since x and y are semi-independent these represent distinct elements of Γ. In particular we have $b_m(\{x,y\}) \geq 2^m$ for each $m > 0$, and hence $\omega(\{x,y\}) \geq 2$. The assertion now follows from Proposition 5.2. ☐

6. Decompositions

If T_1 and T_2 are disjoint subtrees of a tree T then there is a unique open topological arc A which is disjoint from T_1 and T_2, and whose endpoints are vertices, one of which lies in T_1 and the other in T_2. We say that an open edge contained in A *lies between* T_1 and T_2.

Proposition 6.1. *Let Γ be a group and T a Γ-tree. Suppose that x and y are elliptic elements of Γ such that $\mathrm{Fix}(x) \cap \mathrm{Fix}(y) = \emptyset$. Let e lie between $\mathrm{Fix}(x)$ and $\mathrm{Fix}(y)$. Suppose that n is a positive integer such that e is not fixed by x^k nor y^k for $0 < k \leq 2n$. Then there exist disjoint subsets X and Y of Γ such that*

- *Γ is the disjoint union of X, Y and Γ_e;*
- *$x^{\pm k}Y \subset X$ and $y^{\pm k}X \subset Y$ for $0 < k \leq n$; and*
- *$x^iY \cap x^jY = \emptyset$ and $y^iX \cap y^jX = \emptyset$ for any pair of distinct integers i and j with $-n \leq i \leq n$ and $-n \leq j \leq n$.*

Proof. Let A denote the unique open topological arc which is disjoint from $\mathrm{Fix}(x) \cup \mathrm{Fix}(y)$ and has endpoints $a \in \mathrm{Fix}(x)$ and $b \in \mathrm{Fix}(y)$. Thus $e \subset A$. Let the subtrees T_x and T_y be the components of $T - e$ which contain a and b respectively. Let v_x and v_y be the endpoints of e which are contained in T_x and T_y respectively.

We define

$$X = \{\gamma \in \Gamma \mid \gamma \cdot e \subset T_x\};$$
$$Y = \{\gamma \in \Gamma \mid \gamma \cdot e \subset T_y\}.$$

It is clear that Γ is the disjoint union of X, Y and Γ_e.

Let k be any integer with $0 < k \leq n$. Since $A \supset e$ has $a \in \mathrm{Fix}(x)$ as an endpoint, and since $x^{\pm k}$ does not fix e, we have $e \not\subset x^{\pm k} \cdot A$. Since $a \in x^{\pm k} \cdot A$, the component of $T - e$ which contains $x^{\pm k} \cdot A$ is the one containing a, namely T_x. This shows that $x^{\pm k} \cdot A \subset T_x$. Similarly we have $y^{\pm k} \cdot A \subset T_y$. In particular this implies $x^{\pm k} \cdot e \subset T_x$ and $y^{\pm k} \cdot e \subset T_y$.

Since T_x and T_y are disjoint, we have that $e \not\subset x^{\pm k} \cdot T_y$ and $e \not\subset y^{\pm k} \cdot T_x$. Since the subtree $x^{\pm k} \cdot T_y$ contains the endpoint $x^{\pm k} \cdot v_k$ of $x^{\pm k} \cdot e \subset T_x$, it follows from the connectedness of T_y that $x^{\pm k} \cdot T_y \subset T_x$. Similarly we have $y^{\pm k} \cdot T_x \subset T_y$.

Now let $\gamma \in Y$ be given. We have $\gamma \cdot e \subset T_y$ and hence $x^{\pm 1}\gamma \cdot e \subset x^{\pm 1} \cdot T_y \subset T_x$. Hence $x^{\pm 1}\gamma \in X$. This shows that $x^{\pm 1}Y \subset X$. The proof that $y^{\pm 1}X \subset Y$ is similar.

For the proof of the last part of the statement, we assume that i and j are distinct integers with $-n \leq i \leq n$ and $-n \leq j \leq n$. By symmetry it is enough to show that x^iY is disjoint from x^jY. Since $x^kY = \{\gamma \in \Gamma \mid \gamma \cdot e \subset x^k \cdot T_y\}$, it suffices to show that $x^i \cdot T_y$ is disjoint from $x^j \cdot T_y$. Our hypothesis implies that $x^i \cdot e \neq x^j \cdot e$, and hence that $x^i \cdot v_y$ is distinct from $x^j \cdot v_y$. For $k = i, j$, the tree $x^k \cdot T_y$ is the component of $T - x^k \cdot e$ which contains the vertex $x^k \cdot v_y$. Since e lies on the geodesic between a and T_y, and since a is fixed by x, it follows that the edge $x^j \cdot e$ cannot be contained in $x^i \cdot T_y$, and the edge $x^i \cdot e$ cannot be contained in $x^j \cdot T_y$. In particular $x^i \cdot T_y$ and $x^j \cdot T_y$ are components of $T - (x^i \cdot e \cup x^j \cdot e)$. The subtrees $x^i \cdot T_y$ and $x^j \cdot T_y$ have distinct closest vertices to a, namely $x^i \cdot v_y$ and $x^j \cdot v_y$, so they are not equal. Therefore they must be disjoint.

□

The results in [8] (with Lemma 2.1 replacing [8, Lemma 5.5] as discussed in Section 2) will allow us to translate the combinatorial information provided by Proposition 6.1 into measure-theoretic information.

In the following discussion we will regard a Kleinian group Γ as acting on the sphere at infinity S_∞ of $\mathbb{H}^3$. We will use the same notation as in [8]. In particular, the conformal expansion factor ([8, 2.4]) of an element $\gamma \in \Gamma$ associated to the point $z \in \mathbb{H}^3$ will be denoted $\lambda_{\gamma,z}$; the pull-back ([8, 3.1]) of a measure μ under an isometry γ will be denoted $\gamma^*\mu$; and $\mathcal{A} = (A_z)$ is the area density on S_∞ (see [8, 3.3]).

Proposition 6.2. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and let P be a point in $\mathbb{H}^3$. Assume that T is a Γ-tree and that there are T-elliptic elements x and y of Γ such that $\mathrm{Fix}(x) \cap \mathrm{Fix}(y) = \emptyset$ and neither x^2 nor y^2 has a fixed edge in T. Then there exist Borel measures ν, ν^+, ν^-, η, η^+, η^- on S_∞ such that*

(1) $\nu + \eta \leq A_P$;
(2) $\nu^+ + \nu^- \leq \nu$ *and* $\eta^+ + \eta^- \leq \eta$
(3) $\int_{S_\infty} \lambda^2_{x^{\pm 1},P} d\nu^\pm = \eta(S_\infty)$ *and* $\int_{S_\infty} \lambda^2_{y^{\pm 1},P} d\eta^\pm = \nu(S_\infty)$

Furthermore, in the case that T is the dual tree of an incompressible surface F in the closed 3-manifold $\mathbb{H}^3/\Gamma$, we have $\nu + \eta = A_P$.

Proof. Note that the hypotheses of Proposition 6.1 hold with $n = 1$. Let X and Y be the subsets of Γ given by Proposition 6.1. Thus Γ is the disjoint union of X, Y and Γ_e for some edge e of T. Furthermore, $x^{-1}Y$ and xY are mutually disjoiut subsets of X, while $y^{-1}X$ and yX are mutually disjoiut subsets of X. Since Γ is discrete, the set $\Gamma \cdot P$ is uniformly discrete, in the sense of [8, 4.1]. Set $X' = X - (xY \cup x^{-1}Y)$ and $Y' = Y - (yX \cup y^{-1}X)$. Define $\mathfrak{V}$ to be the collection of all unions of sets in

$$\{X' \cdot P,\ Y' \cdot P,\ xY \cdot P,\ x^{-1}Y \cdot P,\ yX \cdot P,\ y^{-1}X \cdot P,\ \Gamma_e \cdot P\}.$$

We apply [8, Proposition 4.2] with $W = \Gamma \cdot P$, to construct a family $(\mathcal{M}_V)_{V\in\mathfrak{V}}$, of D-dimensional conformal densities, for some $D \in [0, 2]$, such that conditions (i)-(iv) of [8, Proposition 4.2] hold. Set $\mathcal{M}_{X\cdot P} = (\nu_z)$, $\mathcal{M}_{xY\cdot P} = (\nu_z^+)$, $\mathcal{M}_{x^{-1}Y\cdot P} = (\nu_z^-)$, $\mathcal{M}_{Y\cdot P} = (\eta_z)$, $\mathcal{M}_{yX\cdot P} = (\eta_z^+)$, $\mathcal{M}_{y^{-1}X\cdot P} = (\eta_z^-)$, and $\mathcal{M}_{\Gamma_e\cdot P} = (\epsilon_z)$. It follows from conditions (i) and (ii) of [8, Proposition 4.2] that $\mathcal{M}_\Gamma = \mathcal{M}_{X\cdot P} + \mathcal{M}_{Y\cdot P} + \mathcal{M}_{\Gamma_e\cdot P}$ is a Γ-invariant conformal density. Since $M = \mathbb{H}^3/\Gamma$ is a closed manifold, every Γ-invariant superharmonic function on M is constant. Thus by [8, Proposition 3.9], $D = 2$ and $\mathcal{M}_{X\cdot P} + \mathcal{M}_{Y\cdot P} + \mathcal{M}_{\Gamma_e\cdot P} = k\mathcal{A}$ for some constant k. Condition (i) of [8, Proposition 4.2] guarantees that $k > 0$. Thus by normalizing the family $(\mathcal{M}_V)_{V\in\mathfrak{V}}$ appropriately we may assume that $k = 1$.

We define $\nu = \nu_P$, $\nu^\pm = \nu_P^\pm$, $\eta = \eta_P$, and $\eta^\pm = \eta_P^\pm$. Then Conclusion (1) follows from the equality $\mathcal{M}_{X\cdot P} + \mathcal{M}_{Y\cdot P} + \mathcal{M}_{\Gamma_e\cdot P} = k\mathcal{A}$ by specializing to $z = P$.

According to [8, Proposition 4.2 (ii)], we have

$$\mathcal{M}_{X\cdot P} = \mathcal{M}_{X'\cdot P} + \mathcal{M}_{xY\cdot P} + M_{x^{-1}Y\cdot P}$$

and

$$\mathcal{M}_{Y\cdot P} = \mathcal{M}_{Y'\cdot P} + \mathcal{M}_{yX\cdot P} + \mathcal{M}_{y^{-1}X\cdot P}.$$

Specializing to $z = P$, we obtain Conclusion (2).

Now we turn to the proof of Conclusion (3). We will only give the proof that $\int_{S_\infty} \lambda_{x,P}^2 d\nu^+ = \eta(S_\infty)$; the other three statements included in Conclusion (3) are proved by the same argument. The proof is based on [8, Proposition 4.2 (iii)], which asserts that $x_\infty^*(\mathcal{M}_{xY}) = \mathcal{M}_Y$. After substituting $\mathcal{M}_{xY\cdot P} = (\nu_z^+)$ and $\mathcal{M}_{Y\cdot P} = (\eta_z)$, the definition of the pull-back (see [8, 3.4.1]) states that $d(x_\infty^*\nu_z^+) = \lambda_{x,z}^2 d\nu_z^+$. Taking $z = P$ and integrating over S_∞ we obtain

$$\int_{S_\infty} \lambda_{x,P}^2 d\nu^+ = \int_{S_\infty} d(x_\infty^*\nu^+) = \int_{S_\infty} d\eta = \eta(S^\infty)$$

as required for Conclusion (3).

We now consider the case where T is the dual tree of an incompressible surface F in the 3-manifold $\mathbb{H}^3/\Gamma$. For $z \in \mathbb{H}^3$ the support of the measure ϵ_z is contained in the limit set of the Kleinian group Γ_e. The group Γ_e cannot be conjugate to the fundamental group of a fiber or a semi-fiber in M, since in that case the tree T would be a line and the square of any elliptic element of Γ would fix the entire tree T. It follows from work of Thurston, Bonahon and Canary (see [7, Corollary 8.1]) that the group Γ_e is either geometrically finite or a virtual fiber subgroup. If Γ_e were a virtual fiber subgroup then, since Γ_e is equal to the image of $\pi_1(F)$ up to conjugacy, we would deduce, by applying [12, Theorem 10.5] to the components of the manifold obtained by splitting M along F, that F is a fiber or semi-fiber. Thus Γ_e must be geometrically finite, and by a theorem of Ahlfors's [3] the limit set of Γ_e has area measure 0. Since the measure ϵ_P is supported on the limit set of Γ_e, we see that ϵ_P is singular with respect to A_P and we have $\nu + \eta = \nu_P + \eta_P = A_P$, as required for the last sentence of the statement. □

Proposition 6.3. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and let P be a point of $\mathbb{H}^3$. Suppose that T is the dual Γ-tree of an incompressible surface in M, and that x and y are T-elliptic elements of Γ such that $\mathrm{Fix}(x) \cap \mathrm{Fix}(y) = \emptyset$ and neither x^2 nor y^2 has a fixed edge in T. Set $D_x = \exp(2d_P(x))$ and $D_y = \exp(2d_P(y))$. Suppose that ν, η, ν^+, ν^-, η^+, η^- are measures having the properties stated in (the conclusion of) Proposition 6.2, and let α, β, α^+, α^-, β^+ and β^- denote their respective total masses. Then we have*

(1) $\alpha + \beta = 1$;
(2) $\dfrac{\beta(1-\alpha^+)}{\alpha^+(1-\beta)} \le D_x$; $\quad \dfrac{\beta(1-\alpha^-)}{\alpha^-(1-\beta)} \le D_x$;
(3) $\dfrac{\alpha(1-\beta^+)}{\beta^+(1-\alpha)} \le D_y$; $\quad \dfrac{\alpha(1-\beta^-)}{\beta^-(1-\alpha)} \le D_y$.

Proof. Since T is the dual Γ-tree of an incompressible surface, the last sentence of Proposition 6.2 gives that $\nu + \eta = A_P$, which implies (1). We intend to deduce the inequalities in (2) and (3) from Lemma 2.1. For (2), the element γ in the statement of Lemma 2.1 should be replaced by $x^{\pm 1}$, the measure ν by $\nu^{\pm}$, a by $\alpha^{\pm}$ and b by β. For (3), we should replace γ by $y^{\pm 1}$, ν by η, a by $\beta^{\pm}$ and b by α. The only obstruction to this argument is that to apply Lemma 2.1 we must ensure that we do not have $a = b = 0$ or $a = b = 1$ in any of these four cases.

We claim that $\alpha \ne 0$. Otherwise we would have $\alpha^+ = 0$, so the measure ν^+ would be singular. But by Proposition 6.2 we then would have

$$\beta = \eta(S_\infty) = \int_{S_\infty} \lambda_{x,P}^2 d\nu^+ = 0,$$

contradicting the fact that $\alpha + \beta = 1$. Similarly, $\beta \ne 0$. In addition, since $\alpha + \beta = 1$, we cannot have $\alpha^{\pm} = \beta = 1$ nor $\alpha = \beta^{\pm} = 1$. This shows that the possibilities $a = b = 0$ or $a = b = 1$ do not arise, and the argument given above does, in fact, prove the Proposition. □

Proposition 6.4. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and let P be a point of $\mathbb{H}^3$. Suppose that T is the dual Γ-tree of an incompressible surface in M, and that x and y are T-elliptic elements of Γ such that $\mathrm{Fix}(x) \cap \mathrm{Fix}(y) = \emptyset$ and neither x^2 nor y^2 has a fixed edge in T. Set $D_x = \exp(2d_P(x))$ and $D_y = \exp(2d_P(y))$. Then*

$$\frac{\sqrt{8D_x+1}-3}{D_x-1} + \frac{\sqrt{8D_y+1}-3}{D_y-1} \le 2.$$

Proof. Let α, β, α^+, α^-, β^+, β^- in $[0,1]$ be the numbers defined as in Proposition 6.3. By symmetry may assume that $\alpha^+ \le \alpha/2$ and $\beta^+ \le \beta/2$. Since the function $f(t) = (1-t)/t$ is decreasing on the interval $(0,1)$, it follows from Proposition 6.3 that

$$D_x \geq \left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\alpha/2}{\alpha/2}\right) = \left(\frac{\beta}{1-\beta}\right)\left(\frac{2-\alpha}{\alpha}\right) = \frac{(1-\alpha)(2-\alpha)}{\alpha^2}$$

and

$$D_y \geq \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-\beta/2}{\beta/2}\right) = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{2-\beta}{\beta}\right) = \frac{(1-\beta)(2-\beta)}{\beta^2}.$$

Let us set $\overline{D}_x = (1-\alpha)(2-\alpha)/\alpha^2$ and $\overline{D}_y = (1-\beta)(2-\beta)/\beta^2$. Solving the quadratic equations, we find that

$$2\alpha = \frac{\sqrt{8\overline{D}_x+1}-3}{\overline{D}_x-1}; \text{ and } 2\beta = \frac{\sqrt{8\overline{D}_y+1}-3}{\overline{D}_y-1}.$$

A straightforward computation shows that the function $g(t) = (\sqrt{8t+1}-3)/(t-1)$ is decreasing on the interval $(1,\infty)$. Since $\overline{D}_x \leq D_x$ and $\overline{D}_y \leq D_y$ we thus have

$$\frac{\sqrt{8D_x+1}-3}{D_x-1} + \frac{\sqrt{8D_y+1}-3}{D_y-1} \leq 2(\alpha+\beta) = 2.$$

□

Corollary 6.5. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold and let P be a point of $\mathbb{H}^3$. Suppose that T is the dual Γ-tree of an incompressible surface in M, and that x and y are T-elliptic elements of Γ such that $\mathrm{Fix}(x)\cap\mathrm{Fix}(y)=\emptyset$ and neither x^2 nor y^2 has a fixed edge in T. Then*

$$\max(d_P(x), d_P(y)) \geq \frac{1}{2}\log 3.$$

Proof. According to Proposition 6.4 we have

$$\frac{\sqrt{8D_x+1}-3}{D_x-1} + \frac{\sqrt{8D_y+1}-3}{D_y-1} \leq 2,$$

where $D_x = \exp(2d_P(x))$ and $D_y = \exp(2d_P(y))$. In view of the symmetry of the desired conclusion, it suffices to consider the case in which

$$\frac{\sqrt{8D_x+1}-3}{D_x-1} \leq 1.$$

Solving, we find $D_x \leq 3$, and hence $d_P(x) \geq \frac{1}{2}\log 3$, from which the conclusion follows in this case. □

7. Some hyperbolic trigonometry

We shall denote by d_s the spherical distance on the unit sphere $S^2 \subset \mathbb{E}^3$.

Proposition 7.1. *If $\eta_1,\ldots,\eta_n$ are points on S^2, we have*

$$\sum_{1\leq i<j\leq n} \cos d_s(\eta_i,\eta_j) \geq -n/2.$$

Proof. We regard S^2 as the unit sphere in $\mathbb{E}^3$, and we let $v_i \in \mathbb{R}^3$ denote the position vector of η_i. We have

$$\begin{aligned} 0 &\leq \langle \sum_{i=1}^{n} v_i, \sum_{i=1}^{n} v_i \rangle \\ &= \sum_{i=1}^{n} \|v_i\|^2 + \sum_{i \neq j} \langle v_i, v_j \rangle \\ &= n + 2 \sum_{1 \leq i < j \leq n} \langle v_i, v_j \rangle \\ &= n + 2 \sum_{1 \leq i < j \leq n} \cos d_s(\eta_i, \eta_j), \end{aligned}$$

from which the conclusion follows. □

Corollary 7.2. *Let P be a point in $\mathbb{H}^3$, and let $Q_1, \dots, Q_n \in \mathbb{H}^3$ be points distinct from P. Then we have*

$$\sum_{1 \leq i < j \leq n} \cos \angle(Q_i, P, Q_j) \geq -n/2.$$

Proof. We consider the unit sphere Σ in the tangent space to $\mathbb{H}^3$ at P. For $i = 1, \dots, n$ we let r_i denote the ray from P to Q_i, and let $\eta_i \in \Sigma$ denote the unit tangent vector to the ray r_i. Then for any two distinct indices $i, j \in \{1, \dots, n\}$, we have $d_s(\eta_i, \eta_j) = \angle(Q_i, P, Q_j)$. The conclusion now follows from Proposition 7.1. □

Proposition 7.3. *Let ν be a positive real number, and let Q, R, S be points of $\mathbb{H}^n$, for some $n \geq 2$, such that* $\max(\mathrm{dist}(Q, R), \mathrm{dist}(R, S)) \leq \nu$. *Then*

$$\mathrm{dist}(Q, S) \leq \max(\nu, \operatorname{arccosh}(\cosh^2 \nu - \cos(\angle QRS) \sinh^2 \nu)).$$

Proof. Let q and s denote the rays that originate at R and pass, respectively, through Q and S. Let Q' and S' denote the points that lie on q an r respectively, and have distance ν from R. Then $\mathrm{dist}(Q, S)$ is bounded above by the diameter of the triangle Δ with vertices Q', R and S'. This diameter is in turn equal to the maximum of the side lengths of Δ. Two of these side lengths are equal to ν, and the third is equal to $\operatorname{arccosh}(\cosh^2 \nu - \cos(\angle QRS) \sinh^2 \nu))$ by the hyperbolic law of cosines. □

Lemma 7.4. *Let x and y be isometries of $\mathbb{H}^3$. Let P be a point of $\mathbb{H}^3$, and let ν be a positive real number. Suppose that*

$$d_P(x) \leq \nu < d_P(x^2)$$

and that

$$d_P(y) \leq \nu < d_P(y^2).$$

Set

$$A = \frac{\cosh^2 \nu - \cosh \nu}{\sinh^2 \nu}.$$

Then

$$\sum_{(u,v) \in \{\pm 1\} \times \{\pm 1\}} \cos \angle(x^u \cdot P, P, y^v \cdot P) > -2 - 2A.$$

Proof. Noting that $d_P(x^{-1}) = d_P(x) \leq \nu$, and applying Proposition 7.3 with $Q = x^{-1} \cdot P$, $R = P$ and $S = x \cdot P$, we find that

$$\mathrm{dist}(x^{-1} \cdot P, x \cdot P) \leq \max(\nu, \operatorname{arccosh}(\cosh^2 \nu - \cos(\angle(x^{-1} \cdot P, P, x \cdot P)) \sinh^2 \nu)).$$

On the other hand, using the hypothesis we find that

$$\mathrm{dist}(x^{-1} \cdot P, x \cdot P) = \mathrm{dist}(x^2 \cdot P, P) > \nu.$$

Hence $\nu < \operatorname{arccosh}(\cosh^2 \nu - \cos(\angle(x^{-1} \cdot P, P, x \cdot P)) \sinh^2 \nu)$, i.e.

$$\cosh \nu < \cosh^2 \nu - \cos(\angle(x^{-1} \cdot P, P, x \cdot P)) \sinh^2 \nu.$$

In view of the definition of A, this implies that

$$\cos \angle(x^{-1} \cdot P, P, x \cdot P) < A. \tag{7.4.1}$$

The same argument shows that

$$\cos \angle(y^{-1} \cdot P, P, y \cdot P) < A. \tag{7.4.2}$$

Now set $\xi_1 = x$, $\xi_2 = x^{-1}$, $\xi_3 = y$ and $\xi_4 = y^{-1}$. It follows from Corollary 7.2, with $Q_i = \xi_i \cdot P$, that

$$\sum_{1 \leq i < j \leq n} \cos \angle(\xi_i \cdot P, P, \xi_j \cdot P) \geq -n/2. \tag{7.4.3}$$

Note that the left-hand sides of (7.4.1) and (7.4.2) are among the six terms on the left side of (7.4.3). The remaining terms are the quantities $\cos \angle(x^u \cdot P, P, y^v \cdot P)$, where (u, v) ranges over $\{\pm 1\} \times \{\pm 1\}$. Hence the conclusion of the lemma follows from (7.4.1), (7.4.2) and (7.4.3). $\square$

Notation 7.5. We define a function ϕ on $(0, \infty)$ by

$$\phi(t) = \max(3t, 2 \operatorname{arccosh}\Big(2\cosh^2 t - \frac{1}{2}\cosh t - \frac{1}{2}\Big)).$$

Note that the expression $\operatorname{arccosh}\Big(2\cosh^2 \nu - \frac{1}{2}\cosh \nu - \frac{1}{2}\Big)$ is well-defined and positive for $\nu > 0$, since we have $2t^2 - \frac{1}{2}t - \frac{1}{2} > 1$ for $t > 1$. Furthermore, since $2t^2 - \frac{1}{2}t - \frac{1}{2} > 1$ is strictly monotone increasing for $t > 1$, the function ϕ is also strictly monotone increasing on $(0, \infty)$.

Lemma 7.6. *Let x and y be isometries of $\mathbb{H}^3$. Let P be a point of $\mathbb{H}^3$, and let ν be a positive real number. Suppose that*

$$d_P(x) \leq \nu < d_P(x^2)$$

and that

$$d_P(y) \leq \nu < d_P(y^2).$$

Then

$$\min(d_P(xy) + d_P(yx), d_P(xy^{-1}) + d_P(y^{-1}x)) \leq \phi(\nu).$$

Proof. We set

$$E = d_P(xy) + d_P(yx) = \mathrm{dist}(x^{-1} \cdot P, y \cdot P) + \mathrm{dist}(x \cdot P, y^{-1} \cdot P)$$

and

$$E' = d_P(xy^{-1}) + d_P(y^{-1}x) = \mathrm{dist}(x^{-1} \cdot P, y^{-1} \cdot P) + \mathrm{dist}(x \cdot P, y \cdot P).$$

We are required to prove that

$$\min(E, E') \leq \phi(\nu). \tag{7.6.1}$$

Since $\max(d_P(x), d_P(y)) < \nu$, each of the terms $d_P(xy)$, $d_P(yx)$, $d_P(xy^{-1})$ and $d_P(y^{-1}x)$ is bounded above by 2ν. If one of these terms is bounded above by ν, then the left hand side of (7.6.1) is bounded above by $3\nu \leq \phi(\nu)$. Hence we may assume that each of these four terms is greater than ν.

Set

$$A = \frac{\cosh^2 \nu - \cosh \nu}{\sinh^2 \nu}.$$

According to Lemma 7.4 we have

$$\sum_{(u,v) \in \{\pm 1\} \times \{\pm 1\}} \cos \angle(x^u \cdot P, P, y^v \cdot P) > -2 - 2A. \tag{7.6.2}$$

The left hand side of (7.6.2) may be written as $C + C'$, where

$$C = \cos \angle(x \cdot P, P, y^{-1} \cdot P) + \cos \angle(x^{-1} \cdot P, P, y \cdot P)$$

and

$$C' = \cos \angle(x \cdot P, P, y \cdot P) + \cos \angle(x^{-1} \cdot P, P, y^{-1} \cdot P).$$

In particular we have

$$\max(C, C') \geq -1 - A. \tag{7.6.3}$$

Consider the case in which $C \geq -1 - A$. Applying Proposition 7.3 with $Q = x \cdot P$, $R = P$ and $S = y^{-1} \cdot P$, we find that

$$d(x \cdot P, y^{-1} \cdot P) \leq \max(\nu, \mathrm{arccosh}(\cosh^2 \nu - \cos(\angle(x \cdot P, P, y^{-1} \cdot P)) \sinh^2 \nu).$$

Since $d(x \cdot P, y^{-1} \cdot P) > \nu$ it follows that

$$\cosh d(x \cdot P, y^{-1} \cdot P) \leq \cosh^2 \nu - \cos(\angle(x \cdot P, P, y^{-1} \cdot P)) \sinh^2 \nu. \tag{7.6.4}$$

Similarly,

$$\cosh d(x^{-1} \cdot P, y \cdot P) \leq \cosh^2 \nu - \cos(\angle(x^{-1} \cdot P, P, y \cdot P)) \sinh^2 \nu). \tag{7.6.5}$$

Adding (7.6.4) and (7.6.5), and using the definition of C, we obtain

$$\begin{aligned} \cosh d(x \cdot P, y^{-1} \cdot P) + \cosh d(x^{-1} \cdot P, y \cdot P) &\leq 2\cosh^2 \nu - C \sinh^2 \nu \\ &\leq 2\cosh^2 \nu + (1 + A) \sinh^2 \nu \\ &= 4\cosh^2 \nu - \cosh \nu - 1. \end{aligned} \tag{7.6.6}$$

On the other hand, since cosh is convex, we have

$$\begin{aligned}\cosh(E/2) &= \cosh(\frac{1}{2}(d(x^{-1}\cdot P, y\cdot P) + d(x\cdot P, y^{-1}\cdot P)))\\ &\leq \frac{1}{2}(\cosh d(x\cdot P, y^{-1}\cdot P) + \cosh d(x^{-1}\cdot P, y\cdot P)),\end{aligned}$$

which with (7.6.6) gives

$$\cosh(E/2) \leq 2\cosh^2\nu - \frac{1}{2}\cosh\nu - \frac{1}{2}.$$

This implies (7.6.1).

If $C' \geq -1 - A$, the same argument shows that

$$\cosh(E'/2) \leq 2\cosh^2\nu - \frac{1}{2}\cosh\nu - \frac{1}{2},$$

which again implies (7.6.1).

Thus, in view of (7.6.3), the conclusion is seen to hold in all cases. □

8. Proof of the main theorem

Lemma 8.1. *Let Γ be the fundamental group of an orientable hyperbolic 3-manifold. Then the centralizer of every non-trivial element of Γ is abelian. Furthermore, if t and u are elements of Γ and if t commutes with utu^{-1}, then t commutes with u.*

Proof. Up to isomorphism, we may identify Γ with a torsion-free discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$. Any non-trivial element x of Γ is either loxodromic or parabolic. In these respective cases we let A_x denote the axis of x or its fixed point on the sphere at infinity.

If $1 \neq x \in \Gamma$, any element of the centralizer of x must leave A_x invariant. Since Γ is discrete and torsion-free, the stabilizer of A_x in Γ is abelian. This proves the first assertion.

In proving the second assertion we may assume that t and u are non-trivial. We have $u\cdot A_t = A_{utu^{-1}}$. On the other hand, since utu^{-1} commutes with t, we have $A_{utu^{-1}} = A_t$. Hence $u\cdot A_t = A_t$, so that u leaves A_t invariant and therefore commutes with t. □

Corollary 8.2. *Let Γ be the fundamental group of an orientable hyperbolic 3-manifold, and let x and y be elements of Γ.*

(1) If x and y do not commute, then $xyx^{-1}y^{-1}$ and $yx^{-1}y^{-1}x$ do not commute.
(2) If x^m and y^n commute, for some non-zero integers m and n, then x and y commute.

Proof. To prove (1), suppose that $xyx^{-1}y^{-1}$ commutes with $yx^{-1}y^{-1}x$. Apply Lemma 8.1 with $t = xyx^{-1}y^{-1}$ and $u = x^{-1}$ to deduce that $yx^{-1}y^{-1}$ commutes with x^{-1}. Then apply the lemma again, with $t = yx^{-1}y^{-1}$ and $u = y$, to deduce that x commutes with y, a contradiction.

In proving (2) , we may assume that $x \neq 1$ and $y \neq 1$. Since Γ is torsion-free it then follows from Lemma 8.1 that for any $m \neq 0$ the centralizer C of x^m is abelian. We have $x \in C$, and if x^m commutes with y^n for some $n \neq 0$ then $y^n \in C$. Since C is abelian it follows that x

commutes with y^n. But Lemma 8.1 also implies that the centralizer C' of y^n is abelian, and since C' contains x and y we conclude that x and y commute, as required. □

Proposition 8.3. *Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold containing an incompressible surface F and let T denote the dual Γ-tree of F. Let x and y be non-commuting elements of Γ, and let P be a point of $\mathbb{H}^3$.*

(1) If x and y are both T-hyperbolic, and F is not a fiber or a semi-fiber, then

$$\max(d_P(x), d_P(y)) \geq \frac{1}{2}\log 2 = 0.346\ldots.$$

(2) If x^2 fixes at least one edge of T, then

$$\max(d_P(x), d_P(y)) \geq \frac{1}{2}\log \alpha = 0.304\ldots,$$

where $\alpha = 1.839\ldots$ is the unique real root of the polynomial $Q(t) = t^3 - t^2 - t - 3$.

(3) If x is T-elliptic and $\mathrm{Fix}(x) \cap \mathrm{Fix}(yxy^{-1}) \neq \emptyset$, then

$$\max(d_P(x), d_P(y)) \geq \log \gamma = 0.593\ldots,$$

where $\gamma = 1.8105\ldots$ is the unique real root of the polynomial $R(t) = t^4 - t^3 - t - 3$.

(4) If x is T-elliptic then

$$\max(d_P(x), d_P(y)) \geq 0.286.$$

Proof. To prove (1) we observe that if F is not a fiber or a semi-fiber, then the action of Γ on T is non-trivial by Proposition 4.4 and linewise faithful by Proposition 4.6. Hence by Proposition 3.6, if x and y are both T-hyperbolic then they are semi-independent. The assertion therefore follows from Corollary 5.3.

To prove (2), we first observe that since x and y do not commute, it follows from Assertion (2) of Corollary 8.2 that x^2 and y do not commute. It now follows from Lemma 8.1 that x^2 and yx^2y^{-1} do not commute. If x^2 fixes at least one edge of T, then yx^2y^{-1} also fixes at least one edge of T, and hence by Proposition 4.12, x^2 and yx^2y^{-1} are independent in Γ. It therefore follows from [2, Theorem 4.1] (which is in turn a consequence of results proved in [4], [1] and [6]) that

$$\frac{1}{1 + \exp d_P(x^2)} + \frac{1}{1 + \exp d_P(yx^2y^{-1})} \leq \frac{1}{2}.$$

If we set $D = \max(d_P(x), d_P(y))$, we have $d_P(x^2) \leq 2D$ and $d_P(yx^2y^{-1}) \leq 4D$. Hence

$$\frac{1}{1 + \exp(2D)} + \frac{1}{1 + \exp(4D)} \leq \frac{1}{2}.$$

If we now set $u = \exp(2D)$ we obtain $u^3 - u^2 - u - 3 \geq 0$. But the polynomial $Q(t) = t^3 - t^2 - t - 3$ increases monotonically for $t \geq 1$. Hence $\exp(2D) = u \geq \alpha$, and the conclusion follows.

To prove (3) we note that, by Proposition 4.4 and Proposition 4.11, the elements x and yxy^{-1} of Γ are independent. Hence by [2, Theorem 4.1] we have

$$\frac{1}{1+\exp d_P(x)} + \frac{1}{1+\exp d_P(yxy^{-1})} \leq \frac{1}{2}.$$

If we set $D = \max(d_P(x), d_P(y))$, we have $d_P(x) \leq 2D$ and $d_P(yxy^{-1}) \leq 3D$. Hence

$$\frac{1}{1+\exp D} + \frac{1}{1+\exp(3D)} \leq \frac{1}{2}.$$

If we now set $v = \exp D$ we obtain $v^4 - v^3 - v - 3 \geq 0$. But the polynomial $R(t) = t^4 - t^3 - t - 3$ increases monotonically for $t \geq 1$. Hence $\exp D = v \geq \gamma$, and the conclusion follows.

To prove (4) we first consider the special case in which the inequality

$$\max(d_P(x), d_P(y)) < \min(d_P(x^2), d_P(y^2)) \tag{8.3.1}$$

holds.

In the subcase where x^2 fixes at least one edge of T, the assertion follows from assertion (2), which we have already proved. Likewise, in the subcase where $\mathrm{Fix}(x) \cap \mathrm{Fix}(yxy^{-1}) \neq \emptyset$, the assertion follows from assertion (3). We therefore need only address the subcase in which $\mathrm{Fix}(x) \cap \mathrm{Fix}(yxy^{-1}) = \emptyset$ and x^2 fixes no edge of T. Note that in this subcase, the hypotheses of Proposition 6.4 hold with the elements x and yxy^{-1} of Γ playing the respective roles of x and y in the latter proposition.

In this subcase we shall assume that $\max(d_P(x), d_P(y)) < 0.286$ and deduce a contradiction. In view of (8.3.1), we may choose a real number $\nu < 0.286$ such that

$$\max(d_P(x), d_P(y)) < \nu < \min(d_P(x^2), d_P(y^2)).$$

In particular the hypotheses of Lemma 7.6 hold with this choice of ν, and it follows from that lemma that $\min(d_P(xy) + d_P(yx), d_P(xy^{-1}) + d_P(y^{-1}x)) \leq \phi(\nu)$, where ϕ is the function defined in 7.5. In particular we have

$$\min(d_P(xy), d_P(yx), d_P(xy^{-1}), d_P(y^{-1}x)) \leq \frac{1}{2}\phi(\nu).$$

If $d_P(xy) \leq \frac{1}{2}\phi(\nu)$ then

$$d_P(y^{-1}xy) \leq d_P(y^{-1}) + d_P(xy) = d_P(y) + d_P(xy) \leq \frac{1}{2}\phi(\nu) + \nu.$$

Similarly, if $d_P(xy^{-1}) \leq \frac{1}{2}\phi(\nu)$ then $d_P(y^{-1}xy) \leq \frac{1}{2}\phi(\nu) + \nu$, and if $d_P(yx) \leq \frac{1}{2}\phi(\nu)$ or $d_P(y^{-1}x)) \leq \frac{1}{2}\phi(\nu)$ then $d_P(yxy^{-1}) \leq \frac{1}{2}\phi(\nu) + \nu$. Hence, after possibly interchanging the roles of y and y^{-1}, we may assume that

$$d_P(yxy^{-1}) \leq \frac{1}{2}\phi(\nu) + \nu.$$

In view of the monotonicity of ϕ (see 7.5), we have

$$d_P(yxy^{-1}) \leq \frac{1}{2}\phi(0.286) + 0.286 < 0.8227. \tag{8.3.2}$$

If we set $D_x = \exp(2d_P(x))$ and $D_{yxy^{-1}} = \exp(2d_P(yxy^{-1}))$, we have $D_x \leq \exp(2 \cdot 0.286) < 1.772$ and $D_y \leq \exp(2 \cdot 0.8227) < 5.1831$. Applying Proposition 6.4 with the elements x and yxy^{-1} of Γ playing the respective roles of x and y in the latter proposition, and noting that the function $g(t) = (\sqrt{8t+1} - 3)/(t-1)$ is strictly decreasing on the interval $(1, \infty)$, we find that

$$\begin{aligned} 2 &\geq g(D_x) + g(D_{yxy^{-1}}) \\ &> g(1.772) + g(5.1831) \\ &= 2.0007\ldots, \end{aligned}$$

which is the required contradiction in this case.

We now turn to the general case of (4), in which the inequality 8.3.1 is not assumed to hold. Again we argue by contradiction, assuming that $\max(d_P(x), d_P(y)) < 0.286$. Since x and y are loxodromic, the quantities $d_P(x^n)$ and $d_P(y^n)$ tend to ∞ with n. Hence there is a largest integer n_1 such that $d_P(x^{n_1}) < 0.286$, and there is a largest integer n_2 such that $d_P(y^{n_2}) < 0.286$. If we set $x_0 = x^{n_1}$ and $y_0 = y^{n_2}$, it follows that $d_P(x_0^2) \geq 0.286$ and that $d_P(y_0^2) \geq 0.286$. Hence

$$\min(d_P(x_0^2), d_P(y_0^2)) \geq 0.286 > \max(d_P(x_0), d_P(y_0)).$$

The element $x_0 = x^{n_1}$ of Γ is T-elliptic since x is T-elliptic, and since x and y do not commute it follows from Assertion (2) of Corollary 8.2 that $x_0 = x^{n_1}$ and $y_0 = y^{n_2}$ do not commute. By the special case of (4) already proved, with x_0 and y_0 playing the roles of x and y, it now follows that $\max(d_P(x_0), d_P(y_0)) \geq 0.286$. This is a contradiction. □

Remark 8.4. Conclusions (2) and (3) of Proposition 8.3 could be improved by using Lemma 7.6, but this would not affect our main result in this paper.

Proposition 8.5. *Let M be a hyperbolic 3-manifold such that either $H_1(M; \mathbb{Q}) \neq 0$ or M is closed and contains a semi-fiber. Then 0.292 is a Margulis number for M.*

Proof. We write $M = \mathbb{H}^3/\Gamma$. We suppose that x and y are elements of Γ and that P is a point of $\mathbb{H}^3$ such that $\max(d_P(x), d_P(y)) < 0.292$. We must show that x and y commute.

We first consider the special case in which the inequality

$$\min(d_P(x^2), d_P(y^2)) \geq 0.292 \tag{8.5.1}$$

holds. In this case the hypotheses of Lemma 7.6 hold with $\nu = 0.292$, and it follows from that lemma that $\min(d_P(xy) + d_P(yx), d_P(xy^{-1}) + d_P(y^{-1}x)) \leq \phi(0.292)$, where ϕ is defined by 7.5. After possibly interchanging the roles of y and y^{-1}, we may therefore assume that

$$d_P(xy) + d_P(yx) \leq \phi(0.292).$$

It follows that

$$\max(d_P(xyx^{-1}y^{-1}), d_P(x^{-1}y^{-1}xy)) \leq \phi(0.292). \tag{8.5.2}$$

We claim that at least one of the subgroups $\langle xyx^{-1}y^{-1}, yx^{-1}y^{-1}x\rangle$, $\langle x^2, yx^2y^{-1}\rangle$ or $\langle y^2, x^{-1}y^2x\rangle$ has infinite index in Γ. If $H_1(M;\mathbb{Q}) \neq 0$, it is immediate that $\langle xyx^{-1}y^{-1}, yx^{-1}y^{-1}x\rangle$ has infinite index. If M is closed and contains a semi-fiber F, then the image of the inclusion homomorphism $\pi_1(F) \to \pi_1(M)$ is a normal subgroup N of $\pi_1(N)$, and $D = \pi_1(M)/N$ is an infinite dihedral group. Hence the commutator subgroup D' of D is infinite cyclic. If the images $\bar{x}$ and $\bar{y}$ of x and y in D belong to D', then $\bar{x}$ and $\bar{y}$ commute; thus in this case $\langle xyx^{-1}y^{-1}, yx^{-1}y^{-1}x\rangle$ is contained in N, and therefore has infinite index in Γ. If $\bar{x}$ does not belong to D' then $\bar{x}$ has order 2 in D and hence $x^2 \in N$; thus $\langle x^2, yx^2y^{-1}\rangle$ is contained in N, and therefore has infinite index in Γ. Similarly, if $\bar{y} \notin D'$, then $\langle y^2, x^{-1}y^2x\rangle$ has infinite index in Γ.

Since M is a hyperbolic 3-manifold, the manifold-with-boundary obtained from M by removing a standard open cusp neighborhood for each $\mathbb{Z}\times\mathbb{Z}$-cusp satisfies the hypothesis of [13, Theorem VI.4.1]. It therefore follows from the latter theorem that every two-generator subgroup of infinite index in $\Gamma = \pi_1(M)$ is either free of rank at most 2 or free abelian of rank 2. Hence at least one of the subgroups $\langle xyx^{-1}y^{-1}, yx^{-1}y^{-1}x\rangle$, $\langle x^2, yx^2y^{-1}\rangle$ or $\langle y^2, x^{-1}y^2x\rangle$ is either free of rank at most 2 or free abelian of rank 2.

We shall now assume that x and y do not commute, and deduce a contradiction. It follows from Assertion (1) of Corollary 8.2 that the elements $xyx^{-1}y^{-1}$ and $yx^{-1}y^{-1}x$ do not commute. On the other hand, it follows from Assertion (2) of Corollary 8.2 that x^2 does not commute with y, and it therefore follows from the second assertion of Lemma 8.1 that x^2 does not commute with yx^2y^{-1} or with $y^{-1}x^2y$. Thus the subgroups $\langle xyx^{-1}y^{-1}, yx^{-1}y^{-1}x\rangle$, $\langle x^2, yx^2y^{-1}\rangle$ or $\langle y^2, x^{-1}y^2x\rangle$ are all non-abelian. Hence at least one of these subgroups is free of rank 2; that is, at least one of the pairs $(xyx^{-1}y^{-1}, yx^{-1}y^{-1}x)$, (x^2, yx^2y^{-1}) or $(y^2, x^{-1}y^2x)$ is independent.

If $xyx^{-1}y^{-1}$ and $yx^{-1}y^{-1}x$ are independent, it follows from [2, Theorem 4.1] that

$$\frac{1}{1+\exp d_P(xyx^{-1}y^{-1})} + \frac{1}{1+\exp d_P(yx^{-1}y^{-1}x)} \leq \frac{1}{2}. \tag{8.5.3}$$

On the other hand, by (8.5.2) we have

$$\frac{1}{1+\exp d_P(xyx^{-1}y^{-1})} + \frac{1}{1+\exp d_P(yx^{-1}y^{-1}x)} \geq \frac{2}{1+\exp\phi(0.292)} = 0.5009\ldots,$$

which contradicts (8.5.3).

Now suppose that x^2 and yx^2y^{-1} are independent. We have $d_P(x^2) \leq 2d_P(x) \leq 2\cdot 0.292$ and $d_P(yx^2y^{-1}) \leq 2d_P(x) + 2d_P(y) \leq 4\cdot 0.292$. From [2, Theorem 4.1] we find that

$$\begin{aligned} \frac{1}{2} &\geq \frac{1}{1+\exp d_P(x^2)} + \frac{1}{1+\exp d_P(yx^2y^{-1})} \\ &\geq \frac{1}{1+\exp(2\cdot 0.292)} + \frac{1}{1+\exp(4\cdot 0.292)} \\ &= 0.595\ldots, \end{aligned}$$

a contradiction. We obtain a contradiction in the same way if y^2 and $x^{-1}y^2x$ are independent. This completes the proof of the proposition in the special case where (8.5.1) holds.

We now turn to the general case, in which the inequality 8.5.1 is not assumed to hold. Since x and y are loxodromic, the quantities $d_P(x^n)$ and $d_P(y^n)$ tend to ∞ with n. Hence there is a largest integer n_1 such that $d_P(x^{n_1}) < 0.292$, and there is a largest integer n_2 such that $d_P(y^{n_2}) < 0.292$. If we set $x_0 = x^{n_1}$ and $y_0 = y^{n_2}$, it follows that $d_P(x_0^2) \geq 0.292$ and that $d_P(y_0^2) \geq 0.292$. We may now apply the special case of the proposition that has already been proved, with x_0 and y_0 in the roles of x and y, to deduce that x_0 and y_0 commute. It then follows that x and y commute. □

Proof of Theorem 1.1. The second assertion is Proposition 8.5. In proving the first assertion we may assume that $H_1(M;\mathbb{Q}) = 0$ and that M is not a closed manifold containing a semi-fiber. The condition $H_1(M;\mathbb{Q}) = 0$ implies that M is closed and not fibered. The proof of the first assertion is thus reduced to the case where M is closed and contains an incompressible surface F which is not a fiber or a semi-fiber. In this case the result follows immediately from assertions (1) and (4) of Proposition 8.3. □

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