Independent Sets in Sparse Hypergraphs

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THESIS

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SUMMARY

A hypergraph $H = (V, E)$ is a set V of vertices together with a set E of edges, which are subsets of V. An independent set in H is a set $I \subset V$ which contains no edge of H. A proper coloring of H is a partition of V into independent sets. This thesis studies independent sets and colorings in sparse hypergraphs; in particular, we study hypergraphs which contain no copies of a fixed subgraph.

A graph is a hypergraph whose edges all have size 2. The independence number of a graph G, denoted $\alpha(G)$, is the size of the largest independent set in G. Suppose G has n vertices and average degree t. Ajtai, Komlós, and Szemerédi [1] showed that if G contains no triangles, then $\alpha(G) \geq \frac{n}{100t} \log t$. By analyzing their proof, we show the stronger statement that G contains at least $2^{\frac{n}{2400t} \log^2 t}$ independent sets. Using a different proof, we improve the constant to $(\log 2)/4$, not far from the best possible constant of $2 \log 2$. We also give similar results for linear hypergraphs.

The chromatic number of a hypergraph H, denoted $\chi(H)$, is the minimum number of colors needed in a proper coloring of H. Johannson [2] showed that $\chi(G) = O(\Delta/ \log \Delta)$ for triangle-free graphs G with maximum degree Δ. Building on work of Frieze and Mubayi [3; 4], we show that $\chi(H) = O((\Delta/ \log \Delta)^{1/2})$ for triangle-free, 3-uniform hypergraphs H with maximum degree Δ. We also extend this theorem to 3-uniform hypergraphs which contain a bounded number of triangles.

CHAPTER 1

INTRODUCTION

A *hypergraph* $H = (V, E)$ is a set V of vertices together with a set E of edges, which are subsets of V. When each set in E contains exactly k vertices, H is k-uniform. When H is 2-uniform, H is a *graph.* An *independent set* in H is a set of vertices which contains no edge of H. The independence number of H, $\alpha(H)$, is the size of the largest independent set in H.

Determining the independence number of a graph is a fundamental problems in graph theory. Turán's [5] basic theorem of extremal graph theory, in complementary form, states that $\alpha(G) \geq \lceil n/(t+1) \rceil$ for any graph G with average degree t. The complement of Turán's graph, $\overline{T(n,r)}$, is the disjoint union of r cliques, each with k vertices (when $n = kr$). It is easy to see that this graph has independence number r and average degree $t-1$, which shows that Turán's bound is tight.

Since $\alpha(\overline{T(n,r)})$ contains large cliques it is natural to ask whether Turán's bound on $\alpha(G)$ can be improved if we prohibit a prescribed subgraph from appearing in G. When the subgraph is a clique, this question is tightly connected to questions in Ramsey theory. For example, the Ramsey number $R(3,k)$ is the minimum n so that every graph on n vertices contains a triangle (3-vertex clique) or an independent set of size k. Determining $R(3,k)$ is equivalent to determining the minimum possible $\alpha(G)$ for a triangle-free graph G.

A major open problem, dating back to the 1940s, was to determine the order of magnitude of $R(3,t)$. Using a semi-random algorithm [1] (and, independently, a deterministic algorithm [6])

Ajtai, Komlós, and Szemerédi established that $R(3,t) = O(t^2/\log t)$. Fifteen years later, Kim [7] used the semi-random method to show that this bound is the correct order of magnitude; in other words, he proved $R(3,t) = \Omega(t^2/\log t)$.

In this thesis, we use the semi-random method to study independent sets and colorings in graphs and hypergraphs. In Chapter 3, we lower bound the number of independent sets in a triangle-free graph. We accomplish this by analyzing a semi-random algorithm for finding a large independent set in a triangle-free graph. In Chapter 4, using a different method, we improve this lower bound and extend it to families of hypergraphs.

Chapters 5 and 6 address the problem of partitioning the vertex set of a hypergraph into independent sets. A *proper coloring* of a hypergraph is a partition of its vertex set into independent sets. The *chromatic number* of a hypergraph H, denoted $\chi(H)$, is the minimum number of parts needed in a proper coloring of H. In Chapter 5, we use a semi-random algorithm to bound the chromatic number of hypergraphs which contain no triangles. In Chapter 6, we extend this bound to hypergraphs which contain a bounded number of triangles.

1.1 Early applications of the semi-random method

Over the last seventy years, the probabilistic method has proven to be one of the most invaluable tools in combinatorics. The method is typically used to prove the existence of some combinatorial structure. A probability space is defined, and, using tools from probability theory, the structure is then shown to exist with positive probability.

Semi-random algorithms can be thought of as repeated applications of the probabilistic method. A probability space is defined and some structure is shown to exist with positive probability. The structure is then removed, and the original probability space is modified deterministically. Crucially, the modified space is shown to share some properties with the original space. This permits the argument to be repeated a finite number of times, and the structures that were removed at each step can be pieced together in some way to show the existence of one large structure.

1.1.1 Independent sets in triangle-free graphs

A triangle is a graph consisting of exactly three pairwise adjacent vertices. The semi-random method was introduced by Ajtai, Komlós, and Szemerédi [1] to show the existence of large independent sets in triangle-free graphs. Starting with a triangle-free graph G, they randomly select an independent set I_1 in G. They then modify G by deleting I_1 and any vertex adjacent to some vertex in I_1. Using the triangle-free assumption, they show that the modified graph still resembles G enough to repeat the argument $T-1$ more times, giving them a large independent set $I_1 \cup \cdots \cup I_T$. More precisely, if G is triangle-free and has average degree t, then this process results in an independent set of size at least $\frac{n}{100t} \log t$, improving Turán's bound by a factor that is logarthmic in t. We will discuss their proof in more depth in Chapter 3.

1.1.2 Erdős and Hanani's conjecture

Rödl [8] later used the semi-random approach to solve a conjecture of Erdős and Hanani [9]. Fix t, k, and n with $1 \leq t < k < n$. A k-uniform hypergraph H on n vertices is called a *partial* (n, k, t)*-Steiner system* if $|E \cap E'| < t$ for all distinct $E, E' \in H$. Since each edge contains $\binom{k}{t}$ subsets of size t and each such subset can appear in at most one edge, $|H| \leq \binom{n}{t}/\binom{k}{t}$. Erdős and

Hanani conjectured, and Rödl proved, that for fixed t and k, there exist partial (n, k, t)-Steiner systems with $(1 - o(1))\binom{n}{t}/\binom{k}{t}$ edges.

After Rödl proved Erdős and Hanani's conjecture on the existence of a single hypergraph with a certain structure, Colbourn, Hoffman, Phelps, Rödl, and Winkler [10] used the same technique to lower bound the number of hypergraphs with a certain structure. A hypergraph H is a *t-wise balanced design* if every set of t vertices is contained in exactly one edge. [10] used the semi-random method to construct a t-wise balanced design by piecing together several smaller sets of edges. However, they also showed that at each step of the semi-random algorithm, there were many options for the small set of edges that would be picked for the design. Thus the algorithm could output any one of

$$n^{(1+o(1))\binom{n}{t}/(t+1)}$$

t-wise balanced designs, providing the lower bound.

In Chapter 3, we use a similar approach to count the number of independent sets in a triangle-free graph. By analyzing Ajtai, Komlós, and Szemerédi's method of constructing a large independent set, we show that triangle-free graphs contain many independent sets.

Theorem 1. *Suppose that G is a triangle-free graph on n vertices with average degree t, where t is sufficiently large. Then*

$$i(G) \geq 2^{\frac{n}{2400t}\log^2 t}.$$

1.1.3 Heilbronn's conjecture and independent sets in hypergraphs

Heilbronn [11] conjectured that any set of N points lying in the unit disk contains a triangle of area less than c/N^2, for some constant c. Erdős constructed a set of N points in the unit disk containing no triangles with area less than $\Omega(1/N^2)$, but the conjecture remained open for over thirty years until Komlós, Pintz, and Szemerédi [12] disproved it using the semi-random method.

Their counterexample used an extension of the result in [1] for 3-uniform hypergraphs. An i-cycle in a $(k+1)$-uniform hypergraph is a set of i edges whose union contains at most ki vertices. A hypergraph is *uncrowded* if it contains no 2,3, or 4-cycles. Let H be a 3-uniform hypergraph with n vertices and average degree t. [12] showed that if H is uncrowded, then (as $t \to \infty$)

$$\alpha(H) = \Omega(\frac{n}{t^{1/2}} \log^{1/2} t).$$

They used this theorem to find a large independent set in a 3-uniform hypergraph where edges correspond to triangles. This independent set provided the N points in the unit disk containing no triangle of area less than $c(\log N)/N^2$ and disproved Heilbronn's conjecture.

After [12] proved their result for 3-uniform hypergraphs, Ajtai, Komlós, Pintz, Spencer, and Szemerédi [13] extended the result to $(k+1)$-uniform hypergraphs. They showed that if H is an uncrowded, $(k+1)$-uniform hypergraph with n vertices and average degree t,

$$\alpha(H) = \Omega(\frac{n}{t^{1/k}} \log^{1/k} t).$$

Duke, Lefmann, and Rödl [14] later used this result to prove the same bound for *linear* hypergraphs, or hypergraphs which contain no 2-cycles.

By analyzing the algorithm of [13] for finding independent sets in uncrowded hypergraphs, and combining this analysis with Duke,Lefman, and Rödl's result, it is possible to lower bound the number of independent sets in linear hypergraphs. However, Asratian and Kuzjurin [15] found a simpler proof of the counting result of Colbourn, Hoffman, Pelps, Rödl, and Winkler. Their method does not require any analysis of the semi-random algorithm from Rödl. In Chapter 4, we show how this method can also be used to count independent sets in hypergraphs. For a hypergraph H, let $i(H)$ denote the number of independent sets in H.

Theorem 2. *Fix $k \geq 1$. There exists $c''_k > 0$ such that the following holds: Every $(k+1)$-uniform, linear hypergraph H on n vertices with average degree t satisfies*

$$i(H) \geq e^{c''_k \frac{n}{t^{1/k}} \ln^{1+1/k} t}.$$

While this method relieves us from analyzing the semi-random algorithm from [13], another benefit is that we substantially improve the constant in the case of triangle-free graphs.

Theorem 3. *Let G be a triangle-free graph with n vertices and average degree t. Then*

$$i(G) \geq e^{(1-n^{-1/12})\frac{1}{2}\frac{n}{t}\ln t(\frac{1}{2}\ln(t)-1)}.$$

We also prove the following corollary.

Theorem 4. *Every triangle-free graph G on n vertices satisfies*

$$i(G) \geq e^{(1-o(1))\frac{\sqrt{n \ln 2}\ln n}{4}}.$$

Finally, we give a construction which shows that the exponent in the above bound cannot be substantially improved.

Theorem 5. *For all n, there exists a triangle-free graph G on n vertices with*

$$i(G) \leq e^{(1+o(1))(1+\ln 2)\sqrt{n}\ln n}.$$

1.2 Evolution of the semi-random method

The *chromatic number* of a hypergraph is the minimum number of parts needed to partition the vertices of the hypergraph into independent sets. A greedy coloring algorithm can be used to show that for any graph G with maximum degree Δ, $\chi(G) \leq \Delta + 1$; this bound is tight for complete graphs and odd cycles. Brooks [16] extended this by showing that if G is not a complete graph or an odd cycle, then $\chi(G) \leq \Delta$. Erdős and Lovász [17] showed that any k-uniform hypergraph with maximum degree Δ has chromatic number $O(\Delta^{1/(k-1)})$.

Given the success in bounding the independence number of graphs without triangles, we may naturally ask if similar bounds hold for chromatic number. For example, if K is a tree on e edges and G contains no copy of K, then $\chi(G) \leq e$; this follows from the fact that if G contains no copy of K, then G contains a vertex of degree at most $e - 1$. Bohman, Frieze, and Mubayi [18] showed that if K is a fixed k-uniform hypertree on e edges and H is a k-uniform hypergraph containing no copy of K, then $\chi(H) \leq 2(k-1)(e-1)+1$; Loh [19] improved this to $\chi(H) \leq e$, matching the result for graphs.

When K is a cycle, the problem becomes more difficult. Catlin [20], Lawrence [21], and Borodin and Kostochka [22] made progress, showing that if a graph G with maximum degree Δ contains no triangle, then, $\chi(G) \leq \frac{3}{4}\Delta$. Nearly two decades later, the semi-random method finally brought this bound down to $O(\Delta/\log\Delta)$, which we might expect given the bound on $\alpha(G)$ by [1]. However, the method would need to evolve further before it would yield this bound.

1.2.1 Chromatic index and the list coloring conjecture

A *matching* in a hypergraph is a set of edges with pairwise empty intersections. Any matching in a k-uniform hypergraph on n vertices contains at most n/k edges. The *codegree* of a pair of vertices is the number of edges containing both of those vertices. Generalizing Rödl's proof of Erdős and Hanani's conjecture, Frankl and Rödl [23] showed that if all the vertices in a hypergraph have roughly the same degree and every pair of vertices has codegree much less than the maximum degree, then the hypergraph contains a matching with $(1-o(1))n/k$ edges.

The *chromatic index* of a hypergraph H, denoted $\chi'(H)$, is the minimum number of matchings into which the edges of the hypergraph can be partitioned. Vizing [24] showed that the chromatic index of any graph with maximum degree Δ is at most $\Delta+1$. Generalizing this result to hypergraphs, Pippenger and Spencer [25] showed that if a k-uniform hypergraph is D-regular and the maximum codegree of the hypergraph is $o(D)$, then $\chi'(H) \leq (1+o(1))D$. This strengthened Frank and Rödl's result by partitioning all of the edges into matchings of size $(1-o(1))n/k$. The proof again uses the semi-random method. In addition, they introduced the use of the Lovász Local Lemma to iterate from round to round. This has since become a crucial ingredient when using the semi-random method to prove results based on the degrees of vertices in a hypergraph.

A *proper edge coloring* of a hypergraph H is an assignment of colors to the edges of H such that any monochromatic subhypergraph of H is a matching. Thus the chromatic index of H is the minimum number of colors in a proper edge coloring of H. Suppose every edge $e \in H$ is given a list $C(e)$ of colors. The *list chromatic index* of H, denoted $\chi'_l(H)$, is the smallest r such that if every $|C(e)| \geq r$, then there is a proper edge coloring of H so that each edge is assigned a color from its list.

The list coloring conjecture asks whether $\chi'(G) = \chi'_l(G)$ for every graph G. A celebrated result of Galvin [26] is that $\chi'(G) = \chi'_l(G)$ when G is bipartite. Kahn [27] showed that for any graph G with maximum degree Δ, $\chi'_l(G) = \Delta(1 + o(1))$. Kahn used a semi-random coloring algorithm which colors a small subset of vertices at each step. When a vertex receives a color, that color is removed from the list of any adjacent vertex, and the algorithm moves to the next iteration. Kahn also showed that his theorem holds for linear k-uniform hypergraphs, giving an alternate proof of Pippenger and Spencer's result when the maximum codegree is 1.

1.2.2 Chromatic number

Kahn's algorithm positioned the semi-random method to begin solving chromatic number problems on graphs. Kim [28] modified the algorithm to prove that every graph that contains no 4-cycles or 3-cycles and has maximum degree Δ has chromatic number at most

$(1 + o(1))\Delta/\log\Delta$, which is within a factor of 2 of the best possible bound. Johannson [2] provided an ingenious variation of the algorithm to prove that every graph that contains no 3-cycles and has maximum degree Δ has chromatic number $O(\Delta/\log\Delta)$.

Frieze and Mubayi [3; 4] extended Johannson's algorithm to hypergraphs. They showed that every linear k-uniform hypergraph with maximum degree Δ has chromatic number $O((\frac{\Delta}{\log\Delta})^{\frac{1}{k-1}})$. As shown in [18], their result is tight apart from the implied constant. We weaken the hypotheses of Frieze and Mubayi's theorem for 3-uniform hypergraphs; specifically, we remove the linear restriction from their result and replace it with a list of forbidden 3-cycles.

Definition. A *triangle* in a hypergraph H is a set of three distinct edges $e, f, g \in H$ and three distinct vertices $u, v, w \in V(H)$ such that $u, v \in e$, $v, w \in f$, $w, u \in g$ and $\{u, v, w\} \cap e \cap f \cap g = \emptyset$.

For example, the three triangles in a 3-uniform hypergraph are the loose triangle $C_3 = \{abc, cde, efa\}$, $F_5 = \{abc, bcd, aed\}$, and $K_4^- = \{abc, bcd, abd\}$.

Theorem 6. *Suppose H is a 3-uniform triangle-free hypergraph with maximum degree Δ. Then*

$$\chi(H) = O((\frac{\Delta}{\log\Delta})^{\frac{1}{2}}).$$

In Chapter 5, we will in fact prove a stronger version of this theorem, which also generalizes Johansson's result for graphs.

Alon, Krivelivich, and Sudakov [29] extended Johansson's result to graphs with a bounded number of triangles. They used this to show that if $H - v$ is bipartite for some $v \in V(H)$ and G contains no copy of H, then $\chi(G) = O(\Delta/\log\Delta)$. They prove their result by randomly partitioning the vertices of a graph with few triangles into triangle-free subgraphs. They then color each subgraph with Johansson's theorem. Since the number of triangles is bounded, they can do this with a small enough partition so that when they combine the colorings, the total number of colors remains small.

Using similar ideas, we give an extension of Theorem 6 to 3-uniform hypergraphs with a bounded number of triangles.

Theorem 7. *Let G be a 3-uniform hypergraph with maximum degree Δ. Let $\mathcal{C}_3^3$ denote the set of 3-uniform triangles. Suppose that for all $u \in V(G)$ and all $H \in \mathcal{C}_3^3$, u is in at most $\Delta^{(|V(H)|-1)/2}/f$ copies of H. Then*

$$\chi(G) = O((\frac{\Delta}{\log f})^{1/2}).$$

1.3 Ramsey numbers

Let K_n denote the complete graph on n vertices. The *Ramsey number* $R(k,t)$ is the smallest n so that in any red-blue coloring of the edges of K_n, there exists a red K_k or a blue K_t. Recall that [1] showed that any triangle-free (in other words, K_3-free) graph with n vertices and average degree d has independence number at least $\frac{n}{100d}\log d$. This result implies that any red-blue coloring of the edges of K_n either contains a red triangle or a blue K_t, where $t = \Omega(\sqrt{n\log n})$. Also using the semi-random method, Kim [30] constructed a triangle-free graph on n vertices with independence number $O(\sqrt{n\log n})$. Thus the semi-random method has completely determined the order of magnitude of the Ramsey-number $R(3,t)$ to be $\Theta(\sqrt{n\log n})$.

Let C_3^r denote the loose 3-cycle. The *hypergraph Ramsey number* $R(C_3^r, K_t^r)$ is the smallest n so that in any red-blue coloring of the edges of the complete r-uniform hypergraph K_n^r, there exists a red C_3^r or a blue K_t^r. Kostochka, Mubayi, and Verstraete [31] proved that there exist constants a and b such that

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(C_3^3, K_t^3) \leq bt^{3/2}.$$

They conjectured that the upper bound could be improved to $o(t^{3/2})$. Using Theorem 7, we prove a weaker form of this conjecture, where the hypergraph C_3^r is replaced by the family of triangles $\mathcal{C}_3^3$. Namely, we show that

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(\mathcal{C}_3^3, K_t^3) \leq b\frac{t^{3/2}}{(\log t)^{1/2}}.$$

One approach to prove the full conjecture would be to extend Theorem 7 to C_3-free hypergraph. In fact, one may wonder whether Theorem 7 holds for F_5-free or K_4^--free hypergraphs. In Chapter 7, we partially answer this question in the negative by constructing K_4^--free hypergraphs with chromatic number at least $\sqrt{\Delta}/4$.

1.4 Notation

Let H be a k-uniform hypergraph. The independence number of H is denoted by $\alpha(H)$. The number of independence sets in H is denoted by $i(H)$. The chromatic number of H is denoted by $\chi(H)$. For a hypergraph H, let $n(H), e(H)$ and $t(H)$ denote the number of vertices, edges, and average degree of H.

For $u \in V(H)$, $d(u)$ denotes the degree of u (number of edges containing u). For $u, v \in V(H)$, $d(u,v)$ denotes the number of edges containing both u and v. Chapter 6 introduces additional notation for rank k hypergraphs, where each edge contains at most k vertices.

Let X be a random variable. Let $\mathbf{E}[X]$ denote the expectation of X and $\mathbf{Var}[X]$ denote the variance of X. When A is an event in a probability space, $\mathbf{I}[A]$ denotes the $0-1$ indicator random variable which is 1 if and only if A holds.

For functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$, we write $f = O(g)$ if there exists a constant c such that $|f| < c|g|$. We write $f = \Omega(g)$ if $g = O(f)$. If $f = O(g)$ and $f = \Omega(g)$, then $f = \Theta(g)$. If $\lim_{n\rightarrow\infty} f(n)/g(n) = 0$, then we write $f(n) = o(g(n))$.

CHAPTER 2

PROBABILISTIC TOOLS

The probabilistic method allows theorems from probability theory to be used to show the existence of combinatorial structures. In this chapter, we state theorems that will be used throughout the thesis. We also prove Corollary 11, which will be used in Chapter 5.

2.1 Union bound, linearity, and Markov's inequality

Let $A_1 \cup \cdots \cup A_n$ be a events in a probability space. The *union bound* states that

$$\Pr[A_1 \cup \cdots \cup A_n] \leq \Pr[A_1] + \cdots + \Pr[A_n].$$

If $X_1, \ldots, X_n$ are random variables, *linearity of expectation* states that

$$\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n].$$

Suppose X is a nonnegative random variable and $a > 0$. Markov's inequality asserts

$$\Pr[X \geq a] \leq \mathbf{E}[X]/a.$$

2.2 Lovász local lemma

Local Lemma ([17]). *Consider a set $\mathcal{E}$ of (typically bad) events such that for each $A \in \mathcal{E}$,*

- $\Pr[A] \leq p < 1$, *and*

- *A is mutually independent of a set of all but at most d of the other events.*

If $4pd \leq 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.

Asymmetric Local Lemma ([32]). *Consider a set $\mathcal{E} = \{A_1, \ldots, A_n\}$ of (typically bad) events that such each A_i is mutually independent of $\mathcal{E} - (\mathcal{D}_i \cup A_i)$, for some $\mathcal{D}_i \subset \mathcal{E}$. If for each $1 \leq i \leq n$*

- $\Pr[A_i] \leq 1/4$, *and*

- $\sum_{A_j \in \mathcal{D}_i} \Pr[A_j] \leq 1/4$,

then with positive probability, none of the events in $\mathcal{E}$ occur.

2.3 Concentration inequalities

2.3.1 Sums of independent random variables

Theorem 8 (Hoeffding [33]). *Suppose that $X = X_1 + \cdots + X_m$, where the X_i are independent random variables satisfying $|X_i| \leq a_i$ for all i. Then for any $t > 0$,*

$$\Pr[X \geq \mathbf{E}[X] + t] \leq e^{-\frac{2t^2}{\sum_{i=1}^m a_i^2}},$$

and

$$\Pr[X \leq \mathbf{E}[X] - t] \leq e^{-\frac{2t^2}{\sum_{i=1}^m a_i^2}}.$$

We will also use the following theorem, which is Theorem 2.7 from [34].

Theorem 9. *Suppose that* $X = X_1 + \cdots + X_m$*, where the* X_i *are independent random variables satisfying* $X_i \leq \mathbf{E}[X_i] + b$ *for all* i*. Then for any* $t > 0$*,*

$$\Pr[X \geq \mathbf{E}[X] + t] \leq e^{-\frac{t^2}{2\,\mathbf{Var}[X] + bt}}.$$

2.3.2 Functions of independent random variables

McDiarmid [35] proved the following generalization of Theorem 8.

Theorem 10 (McDiarmid [35])**.** *For* $i = 1, \ldots, n$*, let* Ω_i *be a probability space, and let* $\Omega = \prod_{i=1}^{n} \Omega_i$*. For each* i*, let* $X_i : \Omega_i \to \mathcal{A}_i$ *be a random variable, and let* $f : \prod \mathcal{A}_i \to \mathbb{R}$*. Suppose that* f *satisfies* $|f(x) - f(x')| \leq c_i$ *whenever the vectors* x *and* x' *differ only in the* i^{th} *coordinate. Let* $Y : \Omega \to \mathbb{R}$ *be the random variable* $f(X_1, \ldots, X_n)$*. Then for any* $t > 0$*,*

$$\Pr[|Y - \mathbf{E}[Y]| > t) \leq 2e^{-2t^2/\sum_{i=1}^{n} c_i^2}$$

Note that in the above theorem, we may view $\prod \mathcal{A}_k$ as a probability space induced by the random variables $X_1, \ldots, X_n$. We will use the following corollary, which resembles Theorem 7.2 from [36].

Corollary 11. *Let* $X_1, \ldots, X_n$ *be independent random variables, with* X_i *taking values in a set* $\mathcal{B}_i$ *for each* i*. Let* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *be events, where each* $\mathcal{A}_i \subset \mathcal{B}_i$*. Set* $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$*. Suppose that the (measurable) function* $f : \prod \mathcal{B}_k \to \mathbb{R}$ *is non-negative and satisfies* $|f(x) - f(x')| \leq d_i$ *for* *any two vectors* $x, x' \in \mathcal{A}$ *differing only in the* i^{th} *coordinate. Let* Y *be the random variable* $f(X_1, \ldots, X_n)$*. Then*

$$\Pr\left[Y > \mathbf{E}[Y]/\Pr[\mathcal{A}] + t\right] \leq e^{-2t^2/\sum_{i=1}^n d_i^2} + \Pr[\bar{\mathcal{A}}].$$

Proof. Define $g : \mathcal{A} \to \mathbb{R}$ by $g(x) := f(x)$ (in other words, $g = f|\mathcal{A}$). For each i, let $Z_i : X_i^{-1}(\mathcal{A}_i) \to \mathcal{A}_i$ be the random variable with $Z_i(s) = X_i(s)$ for all $s \in X_i^{-1}(\mathcal{A}_i)$. Let W be the random variable $g(Z_1, \ldots, Z_n)$. Since the X_i are independent, the Z_i are also independent, so we will be able to apply Theorem 10 to bound $\Pr[W > \mathbf{E}[W] + t]$.

By total probability and the non-negativity of f,

$$\mathbf{E}[Y] = \mathbf{E}[Y|\mathcal{A}]\Pr[\mathcal{A}] + \mathbf{E}[Y|\bar{\mathcal{A}}]\Pr[\bar{\mathcal{A}}] \geq \mathbf{E}[Y|\mathcal{A}]\Pr[\mathcal{A}]$$

so

$$\mathbf{E}[W] = \mathbf{E}[Y|\mathcal{A}] \leq \mathbf{E}[Y]/\Pr[\mathcal{A}].$$

Combining this with Theorem 10 implies

$$\begin{aligned}
\Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t\right] &= \Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t|\mathcal{A}\right]\Pr[\mathcal{A}] + \Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t|\bar{\mathcal{A}}\right]\Pr[\bar{\mathcal{A}}] \\
&\leq \Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t|\mathcal{A}\right] + \Pr[\bar{\mathcal{A}}] \\
&\leq \Pr\left[Y > \mathbf{E}[Y|\mathcal{A}] + t|\mathcal{A}\right] + \Pr[\bar{\mathcal{A}}] \\
&= \Pr\left[W > \mathbf{E}[W] + t\right] + \Pr[\bar{\mathcal{A}}] \\
&\leq e^{-2t^2/\sum_{i=1}^{n} d_i^2} + \Pr[\bar{\mathcal{A}}].
\end{aligned}$$

□

Kim and Vu [37] proved the following theorem.

Theorem 12 (Kim-Vu [37])**.** *Suppose F is a hypergraph such that $W = V(F)$ and $|f| \leq s$ for all $f \in F$. Let*

$$Z = \sum_{f \in F} \prod_{i \in f} z_i,$$

where the z_i, $i \in W$ are independent random variables taking values in $[0,1]$. For $A \subset W$ with $|A| \leq s$, let

$$Z_A = \sum_{f \in F : f \supset A} \prod_{i \in f - A} z_i.$$

Let $M_A = \mathbf{E}[Z_A]$ and $M_j = \max_{A:|A| \geq j} M_A$ for $j \geq 0$. Then there exists positive constants $a = a(s)$ and $b = b(s)$ such that for any $\lambda > 0$,

$$\Pr[|Z - \mathbf{E}[Z]| \geq a\lambda^s \sqrt{M_0 M_1}] \leq b|W|^{s-1} e^{-\lambda}.$$

The next theorem is due to Alon, Kim, and Spencer [38].

Theorem 13. *Let* $X_1, \ldots, X_n$ *be independent random variables with*

$$\Pr[X_i = 0] = 1 - p_i \text{ and } \Pr[X_i = 1] = p_i.$$

For $Y = Y(X_1, \ldots, X_n)$*, suppose that*

$$|Y(X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n) - Y(X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)| \leq c_i$$

for all $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$*,* $i = 1, \ldots, n$*. Then for*

$$\sigma^2 = \sum_{i=1}^{n} p_i(1 - p_i)c_i^2$$

and a positive constant α *with* $\alpha \max_i c_i < 2\sigma^2$*,*

$$\Pr[|Y - \mathbf{E}[Y]| > \alpha) \leq 2e^{-\frac{\alpha^2}{4\sigma^2}}.$$

CHAPTER 3

COUNTING INDEPENDENT SETS IN GRAPHS

Throughout this chapter, we suppose that G is a graph with n vertices and average degree t. Using two different methods [6; 1], Ajtai, Komlós, and Szemerédi showed that if G contains no K_3, then

$$\alpha(G) \geq \frac{n}{100t} \log t.$$

Shortly after, Shearer [39] improved this to $\alpha(G) \geq (1 - o(1))\frac{n}{t} \log t$. Random graphs [39] show that for infinitely many t and n with $t = t(n) \to \infty$ as $n \to \infty$, there are n-vertex triangle-free graphs with average degree t and independence number

$$(2 - o(1))\frac{n}{t} \log t.$$

Consequently, the results of [6; 1; 39] cannot be improved apart from the multiplicative constant.

In this chapter, our goal is to take the method of Ajtai, Komlós, and Szemerédi [1] further by not only finding an independent set of the size guaranteed by their result, but by showing that many of the vertex subsets of approximately that size are independent sets.

Recall that $i(G)$ denotes the number of independent sets in G. In [40], Alon showed that if G is a d-regular graph, then $i(G) \leq 2^{(1/2+o(d))n}$. He also conjectured that

$$i(G) \leq (2^{d+1} - 1)^{n/2d}.$$

Kahn [41] proved this conjecture for d-regular bipartite graphs. Galvin [42] obtained a similar bound for any d-regular graph G, namely

$$i(G) \leq 2^{n/2(1+1/d+c/d\sqrt{\log d/d})}$$

for some constant c . Finally, Zhao [43] recently resolved Alon's conjecture.

In this chapter, we consider lower bounds for $i(G)$. This problem is fundamental in extremal graph theory, indeed, the Erdős-Stone theorem [44] gives a lower bound for $i(G)$ that is the correct order of magnitude *provided* n/t *is a constant*. More recently, the problem in the range $t = \Theta(n)$ has been investigated by Razborov [45], Nikiforov [46], and Reiher [47]. For example, the results of Razborov and Nikiforov determine $g(\rho, 3)$, the minimum triangle density of an n-vertex graph with edge density $\frac{1}{2} < \rho < 1$. Looking at the complementary graph, this gives tight lower bounds on the number of independent sets of size three in a graph with density $1 - \rho \sim \frac{t}{n}$.

Lower bounds for $i(G)$ appear not to have been studied with the same intensity when t is much smaller than n, in particular, when $t \to \infty$ and $t/n \to 0$. Let us make some easy observations that are relevant for our work here. We assume that $\alpha := \alpha(G) \leq n/4$. Since every subset of an independent set is also independent, Turan's theorem implies

$$i(G) \geq 2^{\alpha} \geq 2^{n/(t+1)}.$$

In Section 3.1, we will improve this to

$$i(G) \geq 2^{\frac{1}{250}\frac{n}{t}\log t}. \tag{3.1}$$

Our proof uses the standard probabilistic argument which establishes the order of magnitude given by Turán's bound on $\alpha(G)$. This result is certainly not new, and we present it only to serve as a warm-up for our main result in Section 7.1. Let us observe below that the result is essentially tight.

As no subset of size more than $\alpha(G)$ is independent, an easy upper bound on $i(G)$ (using $\alpha \leq n/4$) is

$$i(G) \leq \sum_{i=0}^{\alpha} \binom{n}{i} \leq 2\binom{n}{\alpha}. \tag{3.2}$$

Since $\alpha(\overline{T(kr,r)}) = r = n/(t+1)$ (recall that $n = kr$ and $t = k-1$), this bound implies that as $n \to \infty$

$$i(\overline{T(kr,r)}) \leq 2\binom{kr}{r} \leq 2(ek)^r = 2^{1+r\log ek} = 2^{(1+o(1))\frac{n}{t}\log t}.$$

Thus, apart from the constant, the exponent in (3.1) cannot be improved.

Our main result addresses the case where G contains no triangles. As in the case of the independence number, prohibiting triangles improves the bound in (3.1).

Theorem 14. *(Main Result) Suppose that G is a triangle-free graph on n vertices with average degree t, where t is sufficiently large. Then*

$$i(G) \geq 2^{\frac{n}{2400t}\log^2 t}. \tag{3.3}$$

Suitable modifications of Random graphs provide constructions of n-vertex triangle-free graphs G with average degree $t = t(n) \to \infty$ as $n \to \infty$, and $\alpha(G) = O((n/t)\log t)$. Plugging this into (3.2), we see that Theorem 14 is tight (apart from the constant) for infinitely many t.

Since all subsets of the neighborhood of a vertex of maximum degree are independent, $i(G) > 2^t$. Combining this with (3.3) we get

$$i(G) > \max\{2^t, 2^{\frac{n}{2400t}\log^2 t}\} > 2^{cn^{1/2}\log n}$$

for some constant $c > 0$. In Chapter 4, we show that this is tight up to the constant in the exponent.

3.1 General case

In this section, we give the simple proof of (3.1). Our purpose in doing this is to familiarize the reader with the general approach to the proof of Theorem 14 in the next section.

Proposition 15. *If G is a graph on n vertices with average degree t, where $2 \leq t \leq \frac{n}{800}$, then $i(G) \geq 2^{\frac{1}{250}\frac{n}{t}\log t}$.*

Proof. Set $k = \lfloor \frac{1}{100}\frac{n}{t} \rfloor$. Pick a k-set uniformly at random from all k-sets in $V(G)$. Let H be the subgraph induced by the k vertices. Then

$$\mathbf{E}[e(H)] = \frac{1}{2}nt\frac{\binom{n-2}{k-2}}{\binom{n}{k}} = \frac{1}{2}nt\frac{k(k-1)}{n(n-1)} < \frac{1}{2}\frac{tk^2}{n}.$$

Recall that Markov's inequality states that if X is a positive random variable and $a > 0$, then $\Pr[X \geq a] \leq \mathbf{E}[X]/a$; hence $\Pr[e(H) \geq 2\frac{1}{2}\frac{tk^2}{n}] \leq 1/2$. So for at least half of the choices for H, $e(H) \leq \frac{tk^2}{n}$. Therefore, the number of choices of H for which $e(H) \leq \frac{tk^2}{n}$ is at least

$$\frac{1}{2}\binom{n}{k} \geq \frac{1}{2}(\frac{n}{k})^k > 2^{\frac{k}{2}\log n/k} = 2^{\frac{k}{2}(\log n - \log k)}. \tag{3.4}$$

Now, if $e(H) < \frac{tk^2}{n} = \frac{1}{100}k$, then at most $\frac{1}{50}k$ of the vertices in H have degree at least one. This in turn implies that H contains an independent set I of size at least $\frac{49}{50}k$. The set I can be obtained from any H which contains it; the number of ways to pick the $\frac{1}{50}k$ vertices of $H - I$ is at most

$$\binom{n}{k/50} \leq (\frac{50ne}{k})^{k/50} = 2^{\frac{k}{50}\log 50ne - \frac{k}{50}\log k} \leq 2^{\frac{k}{50}(\log n - \log k) + \frac{k}{50}\log 100t}. \tag{3.5}$$

Combining this with (3.4) and using $\frac{20}{23}\frac{1}{100}\frac{n}{t} < k \leq \frac{1}{100}\frac{n}{t}$ for $t \leq \frac{n}{800}$,

$$i(G) \geq 2^{k(\frac{1}{2}-\frac{1}{50})(\log n - \log k) - \frac{k}{50}\log 100t} \geq 2^{k\frac{24}{50}\log 100t - \frac{k}{50}\log 100t} = 2^{k\frac{23}{50}\log 100t} > 2^{\frac{1}{250}\frac{n}{t}\log t}.$$

□

3.2 Triangle-free graphs

In this section we prove our main result, Theorem 14. We begin with some modifications of a lemma from [1] (see the proof of Lemma 4 in [1]).

Lemma 16 (Ajtai-Komlós-Szemerédi [1]). *Suppose that G is a triangle-free graph on n vertices with average degree t, and let $k \leq n/100t$. Let H be the subgraph consisting of k vertices chosen uniformly at random from all the k-sets contained in $\{v \in V(G) : deg(v) \leq 10t\}$. Let M be the subgraph of G consisting of vertices adjacent to no vertex in H. Let n' and t' denote the number of vertices and average degree of M. Then the random variables H and M satisfy:*

$$\mathbf{E}[n(M)] > n(1 - \frac{k}{n-t})^{t+1} > \frac{9n}{10} \tag{3.6}$$

$$\mathbf{E}[e(M)] > \frac{nt}{2}(1 - \frac{k}{n-20t})^{20t+1} > \frac{nt}{10} \tag{3.7}$$

$$\mathbf{E}[e(H)] = \frac{1}{2}nt\frac{k(k-1)}{n(n-1)} \leq \frac{tk^2}{n} \tag{3.8}$$

$$\mathbf{Var}[n(M)] < \frac{2nk(t+1)(10t+1)}{n-k-20t-2} < nt \tag{3.9}$$

$$\mathbf{Var}[e(M)] < 2400kt^4 < 40nt^3 \tag{3.10}$$

$$\mathbf{Var}[e(H)] \leq \frac{tk^2(10k+n)}{n^2} \tag{3.11}$$

Further, if $e(M) < (1+\delta)\,\mathbf{E}[e(M)]$ and $n(M) > (1-\delta)\,\mathbf{E}[n(M)]$, then $n'/t' > \nu n/t$, where $\delta = 800\sqrt{t/n}$ and $\nu = 1 - 1/t - c_{10}\sqrt{t/n}$ for some positive constant c_{10}.

Remark. Ajtai-Komlós-Szemerédi state their lemma for $k = n/100t$ and prove the first inequalities in (3.6), (3.7), (3.9), and (3.10) for all k. They prove the second inequalities in (3.6), (3.7), (3.9), and (3.10) for $k = n/100t$, but it is easily observed that they continue to hold for $k < n/100t$.

The next lemma is implied by the computation in the proof of Lemma 4 from [1]. However, the last statement of Lemma 17 is crucial to our proof of Theorem 14, so we make the computations in [1] explicit.

Lemma 17. *Suppose G is a triangle-free graph on $n > 2^{50}$ vertices with average degree $t \leq 2\sqrt{n}\log n$ and $k \leq n/100t$. Then G contains a subgraph H with $n(H) = k$ and $e(H) \leq k/50$. Moreover, if M is the subgraph of G consisting of vertices adjacent to no vertex in H, then*

1. *$n(M) > n(G)/2$ and*
2. *$n(M)/t(M) > \nu n/t$, where $\nu = 1 - 1/t - c_{10}\sqrt{t/n}$.*

Further, if the vertices in H are chosen uniformly at random from all the k-sets contained in $\{v \in V(G) : deg(v) \leq 10t\}$, then at least half of the choices for H satisfy $e(H) \leq k/50$, along with conditions 1 and 2.

Proof. Recall that for a random variable X and $a > 0$, Chebyshev's inequality states that $\Pr[|X - \mathbf{E}[X]| \geq a] \leq \mathbf{Var}[X]/a^2$. Thus, with $a = k/50 - \mathbf{E}[e(H)]$, Lemma 16 implies

$$\begin{aligned}
\Pr[e(H) \geq k/50] &\leq \frac{tk^2(10k+n)}{n^2(k/50 - \frac{k^2 t}{2n})^2} \\
&= \frac{t(10k+n)}{n^2(1/50 - \frac{kt}{2n})^2} \\
&\leq \frac{t(n/10t+n)}{n^2(1/50 - 1/200)^2} \\
&= \frac{1/10+t}{n(3/200)^2} \\
&< 5000\frac{t}{n} \\
&\leq 5000\frac{2\log n}{\sqrt{n}} \\
&\leq 1/1000.
\end{aligned}$$

So with probability at most 1/1000, the condition $e(H) \leq k/50$ fails.

Set $\delta = 800\sqrt{t/n}$. Again by Lemma 16 and Chebyshev, with $a = \delta\,\mathbf{E}[n(M)]$,

$$\Pr[n(M) \leq n/2] \leq \Pr[n(M) \leq (1-\delta)\,\mathbf{E}[n(M)]] \leq \frac{nt}{(\frac{9n}{10})^2 800^2 \frac{t}{n}} \leq 1/1000.$$

Thus the probability that condition (1) fails is at most 1/1000.

With $a = \delta\,\mathbf{E}[e(M)]$,

$$\Pr[e(M) \geq (1+\delta)\,\mathbf{E}[e(M)]] \leq \frac{40nt^3}{\frac{nt}{10}800^2\frac{t}{n}} = 1/160.$$

Since $\Pr[e(M) \geq (1+\delta)\,\mathbf{E}[e(M)]$ or $n(M) \leq n/2] < 1/160+1/1000$, the last assertion of Lemma 16 implies that the probability of condition (2) failing is at most $1/160 + 1/1000$. Therefore, the probability that condition $e(H) \leq k/50$ fails or condition (1) fails or condition (2) fails is at most $1/1000 + 1/1000 + 1/160 + 1/1000 < 1/2$. □

Our proof of Theorem 14 is achieved by analyzing Algorithm 2 below. The algorithm is a slight modification of the algorithm from [1] that yields an independent set of size $\frac{1}{100}\frac{n}{t}\log t$. Recall that c_{10} is the constant that appear in Lemma 16.

Algorithm 1: Independent set algorithm

Input: Triangle-free graph G with n vertices, average degree t
Output: Independent set I

$M_o = G$;
$R = \lfloor(\log t)/2\rfloor$;
for $i \leftarrow 0$ **to** R **do**
    n_i = number of vertices in M_i;
    t_i = average degree in M_i;
    $\nu_i = 1 - 1/t_{i-1} - c_{10}\sqrt{t_{i-1}/n_{i-1}}$;
    if $i = 0$ or $\nu_i > 1 - 1/\log t$ **then**
        Apply Lemma 17 with $G = M_i$ and $k = \lfloor \frac{1}{200}\frac{n}{t} \rfloor$;
        $M_{i+1}, H_{i+1} = M, H$ from Lemma 17;
    else
        I = Independent set in M_{i-1} of size $\lceil n_{i-1}/(t_{i-1}+1)\rceil$;
        return I;
    end
end
$H = H_1 \cup \cdots \cup H_R$;
I = Independent set in H of size $\lceil \frac{48}{50} kR \rceil$;
return I;

Theorem 18. *Suppose Algorithm 2 is run on a triangle-free graph G with n vertices and average degree t, where $2^{100} < t < \sqrt{n}\log n$ and $n > (3c_{10})^{12}$. If Algorithm 2 terminates at line 12, then $|I| > \frac{1}{2}\frac{n}{t}\log^2 t$. Otherwise, for each iteration $i = 0, ..., R-1$, Algorithm 2 successfully applies Lemma 17 to the graph M_i to obtain a graph H_{i+1} with $k = \lfloor \frac{1}{200}\frac{n}{t} \rfloor$ vertices. Moreover, the graph H in line 15 is the disjoint union of the H_i, and the independent set I in line 16 consists of $\frac{48}{50}kR$ vertices from H.*

Proof. We break our proof into two cases, depending on whether Algorithm 2 terminates at line 10 or 12.

Line 10: We need to show that Lemma 17 can be applied at every iteration and that the graph H in line 15 contains an independent set of size at least $\frac{48}{50}kR \geq \frac{1}{500}\frac{n}{t}\log t$.

If $i = 0$, then $k \leq n/100t$, $t \leq \sqrt{n}\log n$, and $n > t > 2^{50}$, so we may apply Lemma 17 to obtain graphs M_1 and H_1, where $|V(H_1)| = k$. Suppose that $i > 0$ and that Lemma 17 was successfully applied at all previous iterations. Using 1 of Lemma 17, $i < R$, and $R = \lfloor(\log t)/2\rfloor < (\log n)/2$,

$$n_i \geq n/2^i > n/2^R > \sqrt{n} > \sqrt{t} > 2^{50}. \tag{3.12}$$

By the condition in line 7, $\nu_i > 1 - 1/\log t$ for each iteration i. So by 2 of Lemma 17, $\frac{n_i}{t_i} \geq \frac{n}{t}\nu_1\nu_2\ldots\nu_i > \frac{n}{t}(1-1/\log t)^R$. Thus

$$\frac{n_i}{t_i} > \frac{n}{t}(1-1/\log t)^R > \frac{n}{t}(1-\frac{R}{\log t}) \geq \frac{1}{2}\frac{n}{t}.$$

In particular,

$$k \leq \frac{1}{200}\frac{n}{t} \leq \frac{1}{100}\frac{n_i}{t_i}, \tag{3.13}$$

and also, $t < \sqrt{n}\log n$ and $n_i < n$ yield

$$t_i < 2n_i\frac{t}{n} \leq 2n_i\frac{\log n}{\sqrt{n}} \leq 2n_i\frac{\log n_i}{\sqrt{n_i}} = 2\sqrt{n_i}\log n_i. \tag{3.14}$$

The inequalities (3.12), (3.13), and (3.14) ensure that we may again apply Lemma 17 with M_i, M_{i+1}, H_{i+1}, and k playing the roles of G, M, H, and k, respectively. Applying Lemma 17 R times yields a collection of sparse graphs $H_1, H_2, \ldots, H_R$, each with k vertices. Each H_i contains at most $\frac{2}{50}k$ vertices of degree at least one, so each H_i contains an independent set of size at least $\frac{48}{50}k$. By definition of M_i, these independent sets may be combined into one independent set of size at least $\frac{48}{50}kR$.

Line 12: Suppose that the algorithm terminates at line 12 during iteration $i+1$. Then n_i, t_i (and n_{i+1}, t_{i+1}) have been defined and $\nu_{i+1} = 1 - 1/t_i - c_{10}\sqrt{t_i/n_i}$. If $t_i \leq (\frac{3}{2})^{2/3}$, then $1/t_i > 1/\log^3 t$. Assume $t_i > (\frac{3}{2})^{2/3}$. Then

$$\frac{2}{3}t_i^{-1/3} > 1/t_i. \tag{3.15}$$

By (3.12), $n_i > \sqrt{n}$, so for $n > (3c_{10})^{12}$,

$$n_i^3 > n^{3/2} > (3c_{10})^6 n > (3c_{10})^6 t_i.$$

This implies

$$\frac{1}{3}t_i^{1/3} > c_{10}\sqrt{t_i/n_i}. \tag{3.16}$$

Combining (3.15) and (3.16) yields

$$1 - t_i^{-1/3} < 1 - 1/t_i - c_{10}\sqrt{\frac{t_i}{n_i}} = \nu_{i+1} \le 1 - 1/\log t.$$

Thus $1/t_i > 1/\log^3 t$. Since $t \ge 2^{100}$ (which implies that $t > (\log^5 t + \log^2 t)^2$) and $i < R \le \frac{\log t}{2}$,

$$\frac{n_i}{t_i + 1} > \frac{n}{2^i}\frac{1}{(\log^3 t + 1)} \ge \frac{n}{\sqrt{t}}\frac{1}{(\log^3 t + 1)} = \frac{n}{t}\frac{\sqrt{t}}{(\log^3 t + 1)} > \frac{n}{t}\log^2 t.$$

Turan's theorem now implies that M_i contains an independent set of size at least $\frac{n_i}{t_i+1} > \frac{1}{2}\frac{n}{t}\log^2 t$. □

We now complete the proof of Theorem 14 by obtaining a lower bound on the number of outcomes given by line 10 of Algorithm 2.

Proof of Theorem 14. Recall that we are to show that if G is a triangle-free graph on n vertices with average degree t sufficiently large, then $i(G) \ge 2^{\frac{n}{2400t}\log^2 t}$. Assume $t > \max\{(3c_{10})^{12}, 2^{100}\}$. Then $n > t > (3c_{10})^{12}$. Also, if $t > \sqrt{n}\log n$, then G has a vertex whose neighborhood contains at least $t > \frac{n}{\sqrt{n}}\log n > \frac{n}{t}\log^2 n$ vertices. Since G is triangle-free, this neighborhood forms an independent set, which contains at least $2^{\frac{n}{t}\log^2 n} > 2^{\frac{n}{2400t}\log^2 t}$ subsets, which are also independent. Thus we may assume that $t \le \sqrt{n}\log n$; in particular, G satisfies the hypotheses of Theorem 18.

If the algorithm terminates at line 12, then G contains an independent set of size at least $\frac{n}{2t}\log^2 t$; so G contains at least $2^{\frac{n}{2t}\log^2 t} > 2^{\frac{n}{2400t}\log^2 t}$ independent sets, and we are done. Thus we may assume that Algorithm 2 terminates at line 10. Consequently, at each iteration i, the algorithm applies Lemma 17 to pick a sparse graph with $k = \lfloor \frac{1}{2}\frac{n}{100t} \rfloor$ vertices. The vertices in this graph are chosen from

$$L_i = \{v \in V(M_i) : deg(v) \leq 10t_i\}.$$

Note that

$$n_i t_i = \sum_{v \in L_i} deg(v) + \sum_{v \in V(M_i) - L_i} deg(v) \geq \sum_{v \in V(M_i) - L_i} deg(v) \geq (n_i - |L_i|)10t_i.$$

This, together with 1 in Lemma 17, implies $|L_i| \geq \frac{9}{10}n_i > \frac{9}{10}n_{i-1}/2 \geq \frac{9}{10}n/2^i$. At least half of the k-sets in L_i satisfy the conditions of Lemma 17, so the number of choices for H_i is at least

$$\frac{1}{2}\binom{|L_i|}{k} \geq \frac{1}{2}\binom{.9n/2^i}{k}.$$

Therefore, the number of choices for the sequence $H_1, \ldots, H_R$ is at least

$$\begin{aligned}
\prod_{i=0}^{R-1} \frac{1}{2}\binom{|L_i|}{k} \geq \frac{1}{2^R}\prod_{i=0}^{R-1}\binom{.9n/2^i}{k} &\geq \frac{1}{2^R}(\frac{.9n}{k})^{kR}2^{-kR^2/2} \\
&= 2^{kR\log .9n - kR\log k - kR^2/2 - R} \\
&= 2^{kR(\log n - \log k) - kR^2/2 + kR\log .9 - R} \\
&> 2^{kR(\log n - \log k) - \frac{kR}{2}\frac{\log t}{2} - kR - R} \\
&> 2^{kR(\log n - \log k) - \frac{kR}{4}\log t - 2kR}. \qquad (3.17)
\end{aligned}$$

Recall that Algorithm 2 obtains an independent set I of size $\lceil \frac{48}{50}kR \rceil$ from the graph $H = H_1 \cup \cdots \cup H_R$. For a fixed I, the number of graphs H that yield I is at most the number of possibilities for $H - I$. This is at most $\binom{n}{|V(H-I)|}$, which is at most

$$\begin{aligned}
\binom{n}{\frac{2}{50}kR} \leq (\frac{50ne}{2kR})^{\frac{2}{50}kR} &= 2^{\frac{2}{50}kR\log n + \frac{2}{50}kR\log 50e - \frac{2}{50}kR\log 2kR} \\
&< 2^{\frac{2}{50}kR(\log n - \log k) + \frac{2}{50}kR\log 50e} \\
&< 2^{\frac{2}{50}kR(\log n - \log k) + kR}. \qquad (3.18)
\end{aligned}$$

For a fixed H, the number of partitions $H_1 \cup \cdots \cup H_R = H$ is at most the number of partitions of kR elements into R sets of size k, which is less than

$$\binom{kR}{k}^R \leq (Re)^{kR} = 2^{kR\log Re} < 2^{kR\log R + 2kR} < 2^{kR\frac{1}{4}R + 2kR} \leq 2^{kR\frac{1}{8}\log t + 2kR}. \qquad (3.19)$$

Since each H yields an independent set, the total number of independent sets that can be returned at line 10 of the algorithm is at least

$$\frac{\#\text{ of ways to obtain } H}{(\#\text{ of } H \text{ that yield a fixed } I)(\#\text{ of partitions that yield } H)}.$$

Since $\frac{n}{268t} \leq k \leq \frac{n}{200t}$ and $R > (\log t)/3$, (3.17), (3.18), and (3.19) imply that this is at least

$$\begin{aligned} 2^{\frac{48}{50}kR(\log n - \log k) - \frac{5}{8}kR\log t - 5kR} &\geq 2^{\frac{48}{50}kR\log 200t - \frac{5}{8}kR\log t - 5kR} \\ &> 2^{\frac{134}{400}kR\log t} \\ &> 2^{\frac{134}{1200}k\log^2 t} \\ &> 2^{\frac{1}{2400}\frac{n}{t}\log^2 t}. \end{aligned}$$

□

CHAPTER 4

COUNTING INDEPENDENT SETS IN HYPERGRAPHS

In [1], Ajtai, Komlós, and Szemerédi gave a semi-random algorithm for finding an independent set of size at least $\frac{n}{100t}\ln t$ in In Chapter 1, we analyzed this algorithm to show that any triangle-free graph G with n vertices and average degree t satisfies

$$i(G) \geq 2^{\frac{1}{2400}\frac{n}{t}\log_2^2 t}. \tag{4.1}$$

As a consequence, we proved that every triangle-free graph has at least $2^{\Omega(\sqrt{n}\ln n)}$.

In this chapter, we give a simpler proof of (4.1), which substantially improves the constant in the exponent and avoids any analysis of the algorithm of Ajtai, Komlós, and Szemerédi. Further, we show that our bound is not far from optimal, by constructing a triangle-free graph with at most $2^{O(\sqrt{n}\ln n)}$ independent sets. The construction is based on recent results by Bohman-Keevash [48] and Fiz Pontiveros-Griffiths-Morris [49] analyzing the triangle-free process.

Theorem 19. *Let G be a triangle-free graph with n vertices and average degree t. Then*

$$i(G) \geq \max\{e^{(1-n^{-1/12})\frac{1}{2}\frac{n}{t}\ln t(\frac{1}{2}\ln(t)-1)}, 2^t\}.$$

Consequently, for every triangle-free graph H on n vertices,

$$i(H) \geq e^{(1-o(1))\frac{\sqrt{n \ln 2} \ln n}{4}}.$$

Theorem 20. *For all n, there exists a triangle-free graph G on n vertices with*

$$i(G) \leq e^{(1+o(1))2\sqrt{\ln 2}\sqrt{n} \ln n}.$$

Using random graphs, one can show that for $t < n^{1/3}$, there is a triangle-free graph G with independence number at most $(2n/t)\ln t$. Consequently,

$$i(G) \leq \sum_{i=1}^{\alpha(G)} \binom{n}{i} \leq 2\binom{n}{\alpha(G)} \leq 2\left(\frac{te}{2\ln t}\right)^{\frac{2n}{t}\ln t} < 2e^{\ln(te)\frac{2n}{t}\ln t} = e^{(1+o(1))\frac{2n}{t}\ln^2 t},$$

so the constant in the exponent of Theorem 19 is within a factor of 8 of the best possible constant.

4.1 Linear hypergraphs

Fix $k \geq 1$. Using the semi-random method, Ajtai, Komlós, Pintz, Spencer, and Szemerédi [13] showed that there exists c_k such that every $(k+1)$-uniform hypergraph H with n vertices, average degree t, and girth 5 satisfies $\alpha(H) \geq c_k \frac{n}{t^{1/k}} \ln^{1/k} t$. Duke, Lefmann, and Rödl [14] (using the result of [13]) showed that there exists c'_k such that every linear $(k+1)$-uniform hypergraph H with n vertices and average degree t satisfies

$$\alpha(H) \geq c'_k \frac{n}{t^{1/k}} \ln^{1/k} t.$$

This leads to our second theorem.

Theorem 21. *Fix $k \geq 1$. There exists $c''_k > 0$ such that the following holds: For every $(k+1)$-uniform, linear hypergraph H on n vertices with average degree t,*

$$i(H) \geq e^{c''_k \frac{n}{t^{1/k}} \ln^{1+1/k} t}. \tag{4.2}$$

In [13], Ajtai, Komlós, Pintz, Spencer, and Szemerédi observed that, for infinitely many t and n, there exists a $(k+1)$-uniform, linear hypergraph H with n vertices, average degree t, and independence number at most $b'_k \frac{n}{t^{1/k}} \ln^{1/k} t$. For this hypergraph,

$$i(H) \leq e^{b''_k \frac{n}{t^{1/k}} \ln^{1+1/k} t},$$

so (4.2) is tight up to the constant in the exponent.

4.2 Hereditary properties

In [10], Colbourn, Hoffman, Phelps, Rödl, and Winkler counted the number of partial $S(t, t+1, n)$ Steiner systems by analyzing a semi-random algorithm; Using the same techniques, Grable and Phelps [50] extended their result to partial $S(t, k, n)$ Steiner systems. Asratian and

Kuzjurin [15] gave a simpler proof of the bound in [50], which avoids any algorithm analysis. Theorems 19 and 21 both follow from a more general result (Theorem 22 below), which is based on this simpler proof. Since our proof avoids any analysis of how the independent sets are obtained, we are able to extend the bound in [51] from triangle-free graphs to a more general hypergraph setting. Recall that a *hereditary property* $\mathcal{P}$ of hypergraphs is any set of hypergraphs which is closed under vertex-deletion.

Theorem 22. *Fix $k \geq 1$ and $\epsilon \in (0, \frac{4}{k+1})$. Let $\mathcal{P}$ be any hereditary hypergraph property. Suppose there exists a non-decreasing function f so that every $(k+1)$-uniform hypergraph $H \in \mathcal{P}$ with n vertices and average degree at most t satisfies*

$$\alpha(H) \geq \frac{n}{t^{1/k}} f(t).$$

Then there exists $n_0 = n_0(\epsilon)$ such that every $(k+1)$-uniform hypergraph $H \in \mathcal{P}$ with $n \geq n_0$ vertices and average degree at most $t < n^k$ satisfies

$$i(H) \geq e^{\alpha' \frac{n}{t^{1/k}} \ln t},$$

where

$$\alpha' = \begin{cases} (1 - n^{-\epsilon/21}) \frac{1}{k+1} f(t^{\frac{1}{k+1}}), & \text{if } H \text{ is linear} \\ (1 - n^{-\epsilon/21}) \frac{1-\epsilon}{k(2k+1)} f(t^{\frac{2k+\epsilon}{2k+1}}), & \text{otherwise.} \end{cases}$$

4.3 Lower bounds

Theorems 19 and 21 follow from the linear case of Theorem 22. We will prove Theorem 22 for linear hypergraphs and afterward describe the changes needed for non-linear hypergraphs.

We first state a version of the Chernoff bound and two claims, which contain the main differences between the linear and non-linear cases. The proofs of the claims will follow the proof of the theorem.

Chernoff Bound (Chernoff bound [32])**.** *Suppose X is the sum of n independent variables, each equal to* 1 *with probability p and 0 otherwise. Then for any $0 \leq t \leq np$,*

$$\Pr(|X - np| > t) < 2e^{-t^2/3np}.$$

Setup. *Fix $k \geq 1$ and $\epsilon \in (0, \frac{4}{k+1})$. Let H be a $(k+1)$-uniform hypergraph with n vertices, average degree at most $t < n^k$, and maximum degree at most $nt^{\epsilon/8}$. Select each vertex of H independently with probability p. Let m' denote the sum of vertex degrees in the subgraph induced by the selected vertices.*

The next two claims come under the assumption of the setup.

Claim 23. *If H is linear and $p = t^{\frac{-1}{k+1}}$, Then for all $n > n_0(\epsilon)$,*

$$\Pr\left[m' > ntp^{k+1} + \frac{ntp^{k+1}}{n^{\epsilon/20}}\right] < n^{-2}.$$

Claim 24. *If* $p = t^{\frac{\epsilon-1}{k(2k+1)}}$*, then for all* $n > n_0(\epsilon)$*,*

$$\Pr\left[m' > ntp^{k+1} + \frac{ntp^{k+1}}{n^{\epsilon/20}}\right] < n^{-2}.$$

Proof of Theorem 22. Fix $k \geq 1$ and $\epsilon \in (0, \frac{4}{k+1})$. Let $H \in \mathcal{P}$ be a $(k+1)$-uniform, linear hypergraph with n vertices and average degree at most $t < n^k$. We assume $n \geq n_0$, where n_0 is chosen implicitly so that several inequalities throughout the proof are satisfied. We consider two cases. In Case 1, we require that the maximum degree of H is at most $tn^{\epsilon/8}$, while Case 2 requires the maximum degree of H to be at least $tn^{\epsilon/8}$.

Case 1: The maximum degree of H is at most $tn^{\epsilon/8}$.

Select each vertex of H independently with probability $p = t^{-\frac{1}{k+1}}$. Let H' denote the subgraph of H induced by the selected vertices. Let n' denote the the number of vertices in H'. Since $t < n^k$ and $\epsilon < \frac{4}{k+1}$,

$$np = nt^{-\frac{1}{k+1}} > n^{1-k/(k+1)} = n^{\frac{1}{k+1}} > n^{\epsilon/4}.$$

By the Chernoff bound,

$$\Pr[|n' - np| > \frac{np}{n^{\epsilon/20}}] \leq 2e^{-np/3n^{\epsilon/20}} < n^{-2}. \tag{4.3}$$

Let m' denote the sum of vertex degrees in H'. By linearity of expectation,

$$\mathbf{E}[m'] = ntp^{k+1}.$$

Set $\lambda = n^{-\epsilon/20}$. By Claim 23,

$$\Pr[m' > (1+\lambda)ntp^{k+1}] < n^{-2}. \tag{4.4}$$

Therefore, by the union bound, with probability at least $1 - 2n^{-2} > 1 - 1/n$, H' satisfies both

$$m' \leq (1+\lambda)ntp^{k+1}$$

and

$$n' \geq (1-\lambda)np.$$

Let $t' = (1+3\lambda)tp^k$. Then with probability at least $1 - 1/n$, H' has average degree at most

$$m'/n' \leq \frac{(1+\lambda)ntp^{k+1}}{(1-\lambda)np} \leq (1+3\lambda)tp^k = t'.$$

Since $\mathcal{P}$ is hereditary, $H' \in \mathcal{P}$. Thus, with probability at least $1 - 1/n$, H' has an independent set of size at least

$$\begin{aligned}
\frac{n'}{t'^{1/k}} f(t') \geq \frac{(1-\lambda)np}{((1+3\lambda)tp^k)^{1/k}} f((1+3\lambda)tp^k) &= \frac{(1-\lambda)n}{(1+3\lambda)^{1/k} t^{1/k}} f((1+3\lambda)tp^k) \\
&\geq \frac{(1-\lambda)n}{(1+3\lambda)t^{1/k}} f((1+3\lambda)tp^k) \\
&> (1-6\lambda)\frac{n}{t^{1/k}} f((1+3\lambda)tp^k) \\
&\geq (1-6\lambda)\frac{n}{t^{1/k}} f(tp^k),
\end{aligned}$$

where we used that f is non-decreasing in the last inequality.

Let $g = (1-6\lambda)\frac{n}{t^{1/k}} f(tp^k)$. Suppose I is an independent set in H with at least g vertices. Then

$$\Pr[I \subset V(H')] = p^{|I|} \leq p^g.$$

Let N denote the number of independent sets in H with at least g vertices, and let the random variable N' denote the number of independent sets in H' with at least g vertices. By Markov's inequality,

$$1 - 1/n < \Pr[N' \geq 1] \leq \mathbf{E}[N'] \leq Np^g = Ne^{-g \ln p}$$

Thus

$$N > (1-1/n)e^{-g\ln p} = (1-1/n)e^{(1-6\lambda)\frac{1}{k+1}\frac{n}{t^{1/k}}f(t^{\frac{1}{k+1}})\ln t} \tag{4.5}$$
$$> (1-1/n)e^{(1-n^{-\epsilon/21})\frac{1}{k+1}\frac{n}{t^{1/k}}f(t^{\frac{1}{k+1}})\ln t}.$$

Case 2: The maximum degree of H is more than $tn^{\epsilon/8}$.

Let

$$K = \{u \in V(H) : \deg(u) > tn^{\epsilon/8}/2\}.$$

Let H' denote the subgraph of H induced by $V(H) - K$, and let $n' = |V(H')|$. Since

$$\begin{aligned}
\frac{1}{n'}\sum_{v\in V(H')}\deg_{H'}(v) - \frac{1}{n}\sum_{v\in V(H')}\deg_H(v) &\le (\frac{1}{n'}-\frac{1}{n})\sum_{v\in V(H')}\deg_H(v)\\
&\le (\frac{1}{n'}-\frac{1}{n})n'tn^{\epsilon/8}/2\\
&= (n-n')\frac{tn^{\epsilon/8}}{2n}\\
&\le \frac{1}{n}\sum_{v\in K}\deg_H(v),
\end{aligned}$$

the average degree of H' is at most

$$\frac{1}{n}\sum_{v\in K}\deg_H(v) + \frac{1}{n}\sum_{v\in V(H')}\deg_H(v) = \frac{1}{n}\sum_{v\in V(H)}\deg_H(v) \le t.$$

Also, because

$$tn \geq \sum_{u \in V(H)} \deg_H(u) \geq \sum_{u \in K} \deg_H(u) > |K| t n^{\epsilon/8}/2,$$

$|K| < 2n^{1-\epsilon/8}$, and so $n' > n(1 - 2n^{-\epsilon/8}) > n/2^{8/\epsilon}$. Thus H' has maximum degree at most $tn^{\epsilon/8}/2 < tn'^{\epsilon/8}$. Further, since H has maximum degree at least $tn^{\epsilon/8}$ and at most n^k, $t < n^{k-\epsilon/8}$. Hence $t < n^{k-\epsilon/8} < n'^k$. Thus Case 1 implies that

$$i(H') \geq (1 - 1/n')e^{(1-6\lambda)\frac{1}{k+1}\frac{n'}{t^{1/k}} f(t^{\frac{1}{k+1}}) \ln t} > (1 - 2/n) e^{(1-6\lambda)(1-n^{-\epsilon/8})\frac{1}{k+1}\frac{n}{t^{1/k}} f(t^{\frac{1}{k+1}}) \ln t},$$

where $\lambda = n'^{-\epsilon/20}$. We conclude that

$$i(H) \geq i(H') \geq e^{(1-n^{-\epsilon/21})\frac{1}{k+1}\frac{n}{t^{1/k}} f(t^{\frac{1}{k+1}}) \ln t}. \tag{4.6}$$

□

The proof of Theorem 22 when H is non-linear is similar. We set $p = t^{\frac{\epsilon-1}{k(2k+1)}}$. Since we still have $np > n^{\epsilon/4}$, (4.3) still holds. We then use Claim 24 instead of Claim 23 to prove (4.4). The proof then proceeds in the same way until we get to (4.5), where, using the different value of p, we instead obtain

$$N > (1 - 1/n) e^{(1-6\lambda)\frac{1-\epsilon}{k(2k+1)}\frac{n}{t^{1/k}} f(t^{\frac{2k+\epsilon}{2k+1}}) \ln t}.$$

Finally, (4.6) becomes

$$e^{(1-n^{-\epsilon/21})\frac{1-\epsilon}{k(2k+1)}\frac{n}{t^{1/k}}f(t^{\frac{2k+\epsilon}{2k+1}})\ln t}.$$

We now prove Theorem 19 and Theorem 21.

Proof of Theorem 19. Shearer [39] showed that every triangle-free graph with n vertices and average degree t has independence number at least $\frac{n}{t}(\ln(t) - 1)$. Since being triangle-free is hereditary and graphs are 2-uniform, linear hypergraphs, we may apply Theorem 22 (with $f(t) = \ln(t) - 1$) to conclude that for $\epsilon = 21/12 \in (0, 2)$, there exists n_0 such that every triangle-free graph G with $n \geq n_0$ vertices and average degree at most t satisfies

$$i(G) \geq e^{(1-n^{-\epsilon/21})\frac{1}{2}\frac{n}{t}\ln t(\frac{1}{2}\ln(t)-1)} > e^{(1-n^{-1/12})\frac{1}{2}\frac{n}{t}\ln t(\frac{1}{2}\ln(t)-1)}.$$

Suppose G is a triangle-free graph with $n < n_0$ vertices and average degree t. Choose an integer r so that $rn \geq n_0$. Let G' be the disjoint union of r copies of G. Then $i(G') = i(G)^r$, so by the previous paragraph,

$$\begin{aligned} i(G) = i(G')^{1/r} &\geq (e^{(1-(rn)^{-1/12})\frac{1}{2}\frac{rn}{t}\ln t(\frac{1}{2}\ln(t)-1)})^{1/r} \\ &\geq e^{(1-n^{-1/12})\frac{1}{2}\frac{n}{t}\ln t(\frac{1}{2}\ln(t)-1)}. \end{aligned}$$

This completes the proof of the first bound in Theorem 19. For the second part, consider a triangle-free graph G having average degree t. G contains a vertex u with degree at least t.

The neighborhood of u is an independent set, which contains 2^t independent sets. Therefore, every triangle-free graph has at least

$$\max\{2^t, e^{(1-n^{-1/12})\frac{1}{2}\frac{n}{t}\ln t(\frac{1}{2}\ln(t)-1)}\}$$

independent sets. This is minimized when $t = (\frac{1}{4}+o(1))\sqrt{n/\ln 2}\ln n$, so every triangle-free graph on n vertices has at least

$$2^{(1-o(1))\cdot\frac{\sqrt{n}\ln n}{4\sqrt{\ln 2}}} = e^{(1-o(1))\cdot\frac{\sqrt{n\ln 2}\ln n}{4}}$$

independent sets. □

Proof of Theorem 21. Duke, Lefmann, and Rödl [14] showed that every $(k+1)$-uniform linear hypergraph with n vertices and average degree at most t has independence number at least $c'_k \frac{n}{t^{1/k}} \ln^{1/k} t$. Since linearity is a hereditary property, we may apply Theorem 22 (with $f(t) = c'_k \ln^{1/k} t$) to conclude that for $\epsilon = \frac{3}{k+1} \in (0, \frac{4}{k+1})$, there exists n_0 such that every $(k+1)$-uniform linear hypergraph H with $n \geq n_0$ vertices satisfies

$$i(H) \geq e^{(1-n^{-1/(7(k+1))})\frac{c'_k}{k+1}\frac{1}{(k+1)^{1/k}}\frac{n}{t^{1/k}}\ln^{1+1/k} t} > e^{c''_k \frac{n}{t^{1/k}}\ln^{1+1/k} t}.$$

If H is a $(k+1)$-uniform linear hypergraph with $n < n_0$ vertices, then we proceed in the same way as in the proof of Theorem 19. □

It only remains to prove the claims stated at the beginning of this section. We first prove Claim 23.

Proof of Claim 23. Apply Theorem 12 with $F = H$ and $\Pr[z_i = 1] = p = t^{-\frac{1}{k+1}}$. Note first that

$$\mathbf{E}[Z_\emptyset] \leq ntp^{k+1} = nt^{1-1} = n.$$

Since the maximum degree of H is at most $tn^{\epsilon/8}$,

$$\mathbf{E}[Z_{\{u\}}] \leq tn^{\epsilon/8}p^k = n^{\epsilon/8}t^{\frac{1}{k+1}}$$

for any $u \in V(G)$. By linearity, for any $A \subset V(G)$ with $|A| \geq 2$,

$$\mathbf{E}[Z_A] \leq p^{k+1-|A|} \leq 1.$$

Since $t \leq n^k$ and $\epsilon < \frac{4}{k+1}$, $n \geq n^{\epsilon/8}t^{\frac{1}{k+1}}$. Further, $n^{\epsilon/8}t^{\frac{1}{k+1}} \geq 1$. Therefore $M_0 \leq n$ and $M_1 \leq n^{\epsilon/8}t^{1/(k+1)}$. Theorem 12 therefore implies that there exist constants $a = a(k)$ and $b = b(k)$ such that

$$\Pr[|m' - \mathbf{E}[m']| > a((k+3)\ln n)^{k+1}\sqrt{ntp^{k+1}tn^{\epsilon/8}p^k}] \leq bn^k e^{-(k+3)\ln n}.$$

Since $t \le n^k$ and $\epsilon < \frac{4}{k+1}$,

$$\sqrt{ntp^{k+1}tn^{\epsilon/8}p^k} = \frac{ntp^{k+1}}{n^{1/2-\epsilon/16}p^{1/2}} \le \frac{ntp^{k+1}}{n^{\epsilon/16}}.$$

Thus, since $\mathbf{E}[m'] \le ntp^{k+1}$,

$$\begin{aligned}\Pr[m' > ntp^{k+1} + \frac{ntp^{k+1}}{n^{\epsilon/20}}] &< \Pr[m' > \mathbf{E}[m'] + a((k+3)\ln n)^{k+1}\frac{ntp^{k+1}}{n^{\epsilon/16}}] \\ &\le bn^k e^{-(k+3)\ln n} \\ &< n^{-2}.\end{aligned}$$

□

Proof of Claim 24. Recall that $p = t^{\frac{\epsilon-1}{k(2k+1)}}$. The random variable m' is determined by the n independent, indicator random variables $\mathbf{I}[v \in V(H')]$. Each of these affects m' by at most $\deg(v) \le tn^{\epsilon/8}$. Set $\alpha = \frac{ntp^{k+1}}{n^{\epsilon/16}}$ and $\sigma^2 = n^{1+\epsilon/4}p(1-p)t^2$. Note that $\alpha tn^{\epsilon/8} \le 2\sigma^2$. Also, because $t \le n^k$,

$$\frac{\alpha^2}{4\sigma^2} = \frac{np^{2k+1}}{16n^{\epsilon/4+\epsilon/8}(1-p)} \ge \frac{np^{2k+1}}{16n^{\epsilon/4+\epsilon/8}} = \frac{nt^{\frac{\epsilon-1}{k}}}{16n^{3\epsilon/8}} \ge \frac{n^{\epsilon}}{16n^{3\epsilon/8}} = n^{5\epsilon/8}/16.$$

Since $\mathbf{E}[m'] \le ntp^{k+1}$, Theorem 13 implies

$$\Pr[m' > ntp^{k+1} + \frac{ntp^{k+1}}{n^{\epsilon/20}}] < \Pr[m' > \mathbf{E}[m'] + \frac{ntp^{k+1}}{n^{\epsilon/16}}] \le 2e^{-n^{5\epsilon/8}/16} < n^{-2}.$$

□

4.4 Upper bound for triangle-free graphs

In this section we prove Theorem 20. We use the results of Bohman-Keevash [48] and Fiz Pontiveros-Griffiths-Morris [49] on the triangle-free graph process: Let G be the maximal graph in which the triangle-free process terminates.

Theorem 25 (Bohman-Keevash, Fiz Pontiveros-Griffiths-Morris)**.** *With high probability, every vertex of G has degree $d \leq (1+o(1))\sqrt{\frac{1}{2}n\ln n}$, and independence number $\alpha \leq (1+o(1))\sqrt{2n\ln n}$.*

Let $r > 0$ be a real parameter to be optimized later. Construct the graph G' from G as follows:

4.4.0.0.1 Construction of G':

We take the strong graph product of G and $\bar{K}_r$, the empty graph on r vertices: Replace each vertex v of G by a copy C_v of $\bar{K}_r$. Introduce a complete bipartite graph between all the vertices of C_v and C_u if and only if $\{u, v\} \in E(G)$. We obtain the graph G'. Notice that $|V(G')| = N = nr$.

Define the function $f : V(G') \to V(G)$, such that given any $i \in C_u \subset V(G')$, $f(i) = u$. For a set $S \subset V(G')$, define $f(S) = \bigcup_{i \in S}\{f(i)\}$.

Claim 26. *For every $S \subset V(G')$, S is independent only if $f(S)$ is independent in G. Further $|S| \leq r|f(S)|$.*

Proof. Given an independent set $I \subset G'$, consider $i, j \in I$. Clearly, if $f(i) \neq f(j)$, then $f(i), f(j)$ are not adjacent in G, by the construction. Further, if $f(i) = f(j)$, then i, j must belong to some copy of $\bar{K_r}$ in G'. □

Proof of Theorem 20. We shall show that G' is the required graph. By Claim 26,

$$\begin{aligned} i(G') &\leq \sum_{I \subset G : I \text{ ind. set}} 2^{r|I|} \\ &\leq \alpha \binom{n}{\alpha} 2^{r\alpha} \\ &\leq e^{\ln \alpha + \alpha \ln(ne/\alpha) + r\alpha \ln 2}. \end{aligned}$$

To finish the proof, note that

$$\begin{aligned} \ln \alpha + \alpha \ln(ne/\alpha) + r\alpha \ln 2 &= (\frac{\ln n}{2} + r \ln 2 + o(1))\alpha \\ &\leq (\frac{\ln(N/r)}{2} + r \ln 2 + o(1))\sqrt{2(N/r)\ln(N/r)} \end{aligned}$$

where the last line was obtained by substituting the value of α in terms of N and r. Now maximizing the above expression with respect to r, we get that when $r = \frac{1}{2}\log_2 n$,

$$i(G') \leq e^{(1+o(1))2\sqrt{N \ln 2} \ln N}.$$

□

CHAPTER 5

COLORING TRIANGLE-FREE HYPERGRAPHS

Johansson [2] showed that if a graph G contains no 3-cycles, then $\chi(G) = O(\Delta/\log\Delta)$. Recall that a hypergraph is *linear* (or contains no 2-cycles) if any two of its edges intersect in at most one vertex. A *triangle* in a linear hypergraph is a set of three pairwise intersecting edges with no common point. In [3], Frieze and Mubayi showed that if H is a 3-uniform, linear, triangle-free hypergraph, then $\chi(H) = O(\sqrt{\Delta}/\sqrt{\log\Delta})$. They subsequently removed the triangle-free condition and generalized their result from 3 to k, showing that $\chi(H) = O((\Delta/\log\Delta)^{1/(k-1)})$ for any k-uniform, linear hypergraph H. As shown in [18], these results are tight apart from the implied constants.

In this chapter, we generalize the results of Johansson and Frieze and Mubayi to rank 3 hypergraphs. We also remove the linear condition from Frieze and Mubayi's theorem and replace it with the weaker condition that the hypergraph contains no triangle.

Definition. A *triangle* in a hypergraph H is a set of three distinct edges $e, f, g \in H$ and three distinct vertices $u, v, w \in V(H)$ such that $u, v \in e$, $v, w \in f$, $w, u \in g$ and $\{u, v, w\} \cap e \cap f \cap g = \emptyset$.

Given a set $C(v)$ of colors for every vertex $v \in V(H)$, a *proper list coloring* of H is a proper coloring where every vertex v receives a color from $C(v)$. The list chromatic number of H, $\chi_l(H)$, is the minimum l so that if $|C(v)| \geq l$ for all v, then H has a proper list coloring. It is not hard to see that $\chi(H) \leq \chi_l(H)$. As in [28] and [2], our main theorem can be stated in terms of list chromatic number.

Theorem 27. *Suppose H is a rank 3, triangle-free hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. Then*

$$\chi_l(H) \leq c_1 \max\{(\frac{\Delta}{\log \Delta})^{\frac{1}{2}}, \frac{\Delta_2}{\log \Delta_2}\},$$

for some constant c_1.

We prove Theorem 27 by using a semi-random algorithm to properly color the hypergraph. Our algorithm is similar to the algorithm in [3], however, several new ideas are developed to deal with the non-linear case. At each iteration, we randomly color a few of the vertices. When a vertex in a 3-edge is colored c, we add a c-colored 2-edge between the remaining two vertices to record the fact that those two vertices cannot both be colored c in the future. [3] assumed the hypergraph was linear, which implied that at most one such 2-edge could be added between two vertices. Here we maintain a 2-graph for every color and allow two vertices to share an edge in multiple graphs. This allows us to extend our algorithm to rank 3 hypergraphs: for each 2-edge in the original hypergraph, we simply add a copy of that 2-edge to every color's 2-graph. After several iterations, we color the remaining vertices with the asymmetric version of the local lemma. This prevents the 3-edges from becoming monochromatic, while also enforcing the constraints from the 2-graphs.

5.1 Application to hypergraph Ramsey numbers

Let $\mathcal{C}_3^r$ be the collection of r-uniform hypergraph triangles. Notice that for graphs, $\mathcal{C}_3^2$ consists of only the 3-vertex cycle, and for triple systems, $\mathcal{C}_3^3 = \{C_3, F_5, K_4^-\}$. The hypergraph Ramsey number $R(\mathcal{C}_3^r, K_t^r)$ is the smallest n so that in every red-blue coloring of the edges of the complete r-uniform hypergraph K_n^r, there exists a red triangle or a blue K_t^r. Ajtai-Komlós-Szemerédi [52] and Kim [30] proved that $R(\mathcal{C}_3^2, K_t^2) = \Theta(t^2/\log t)$.

In [31], Kostochka, Mubayi, and Verstraëte proved a version of this result for $r = 3$. In this setting, $R(C_3, K_t^3)$ is the smallest n so that in every red-blue coloring of the edges of the complete 3-uniform hypergraph K_n^3, there exists a red C_3 or a blue K_t^3. [31] showed that there exist constants a, b such that

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(C_3, K_t^3) \leq bt^{3/2},$$

and they conjectured that the upper bound could be reduced to $o(t^{3/2})$. In proving the upper bound, they showed that if H is a 3-uniform, C_3-free hypergraph with n vertices and average degree $d \geq n^{2/3}/12$, then $\alpha(H) \geq d/18$. Our result implies that any 3-uniform, $\mathcal{C}_3$-free hypergraph with n vertices and average degree d satisfies $\alpha = \Omega(\frac{n}{d^{1/2}} \log^{1/2} d)$. Setting $d = n^{2/3} \log^{1/3} n$, we obtain $R(\mathcal{C}_3^3, K_t^3) = O(t^{3/2}/\sqrt{\log t})$. Since the C_3-free construction given in [31] is also F_5 and K_4^- free, this implies that for some constants a and b,

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(\mathcal{C}_3^3, K_t^3) \leq b\frac{t^{3/2}}{(\log t)^{1/2}}.$$

5.2 Independent set algorithm

The main idea of our coloring algorithm is to use simple graphs to track which hyperedges contain vertices that have already been colored. To familiarize the reader with this idea, we first give a simple algorithm for finding an independent set in a 3-uniform hypergraph. The algorithm takes a 3-uniform hypergraph H with n vertices and maximum degree Δ and outputs an independent set with at least $\frac{n}{2\Delta^{1/2}}$ vertices.

Algorithm 2: Independent set algorithm

Input: 3-uniform hypergraph H with n vertices, maximum degree Δ
Output: Independent set I

$V(G) = V(H)$, $E(G) = \emptyset$;
$I = \emptyset$;
while $|V(H)| \geq n/2$ **do**
  $u =$ vertex of minimum degree in G;
  $I = I \cup \{u\}$;
  $E(G) = E(G) \cup \{vw : uvw \in E(H)\}$;
  $V(G) = V(H) - \{u\} - N_G(u)$;
  $V(H) = V(H) - \{u\} - N_G(u)$;
end
return I;

We first show that I is an independent set. Suppose $u, v, w \in I$ and $uvw \in H$. Since the algorithm adds one vertex per iteration to I, we may assume that the vertices were added to I in the order u, v, w. When u was added to I, the edge vw was added to G. So when v was added to I, $w \in N_G(v)$, and so w was removed from G and H. Thus w could not have been added to I.

Observe now that at most Δ edges are added to G at every iteration, so after iteration i, $|E(G)| \leq i\Delta$. Since the chosen vertex has minimum degree in G, the degree of the vertex chosen at iteration $i+1$ is at most

$$2i\Delta/n(G) \leq 4i\Delta/n.$$

Therefore at most $4i\Delta/n$ vertices are deleted at iteration $i+1$, and so

$$(4\Delta/n)\sum_{i=0}^{T-1} i < (2\Delta/n)T^2$$

vertices have been deleted after T iterations. Since the algorithm runs until $n/2$ vertices have been deleted, the algorithm runs for at least $\frac{n}{2\sqrt{\Delta}}$ iterations. Since the algorithms adds one vertex to I at every iteration, the algorithm outputs an independent set with at least $\frac{n}{2\sqrt{\Delta}}$ vertices.

5.3 Coloring algorithm

The input to our algorithm is a rank 3 hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. Let H denote the input hypergraph restricted to its size 3 edges, and let G denote the input hypergraph restricted to its size 2 edges. At the beginning, each vertex u has a list

$C(u)$ of acceptable colors. We assume $|C(u)| = C$ for all vertices u. For each vertex u and color c, we set

$$p_u^0(c) = \begin{cases} 1/C, & \text{if } c \in C(u) \\ 0, & \text{if } c \notin C(u). \end{cases}$$

We define a parameter $\hat{p}$, which will serve as an upper bound on the weights $p_u^i(c)$. Set $W^0(u) = \{p_u^0(c) : c \in \cup_v C(v)\}$. We start with the hypergraph $H^0 = H$ and the collection $\{W^0(u)\}_u$. For each color c, we also construct a graph G_c^0, which is initially a copy of the 2-graph G. Finally, we assign to each vertex an empty set $B^0(u)$.

At the $(i+1)^{th}$ step, $i = 0, 1, \ldots, T-1$, our input to the algorithm is a quadruple, $(H^i, \{G_c^i\}_c, \{W_u^i\}_u, \{B^i(u)\}_u)$. We generate a small random set of colors at each vertex u as follows: For each color c, we choose c with probability $\theta p_u^i(c)$. Let

$$\gamma_u^i(c) = \begin{cases} 1, & \text{if } c \text{ is chosen at } u, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the $\gamma_u^i(c)$ are independent random variables.

Consider a vertex u. We define the set of colors lost at u as

$$L^i(u) = \{c : \exists e \in E(H^i) \cup E(G_c^i) \text{ such that } u \in e \text{ and } \gamma_v^i(c) = 1 \; \forall v \in e - u\}.$$

We say a color c *survives* at u if $c \notin B^i(u)$ and $c \notin L^i(u)$. For $c \notin B^i(u)$, we define

$$q_u^i(c) := \Pr[c \text{ survives at } u] = \Pr[\bigcap_{\substack{\{v,w\}: \\ uvw \in H^i}} (\gamma_v^i(c) = 0 \cup \gamma_w^i(c) = 0) \bigcap_{v: uv \in G_c^i} \gamma_v^i(c) = 0]. \quad (5.1)$$

In other words, if $c \notin B^i(u)$, then $q_u^i(c) = \Pr[c \notin L^i(u)]$. Note that at the $(i+1)^{th}$ step, $q_u^i(c)$ is a fixed number, which can be computed given H^i, G_c^i, and all of the $p_v^i(c)$; it does not depend on the random variables $\gamma_u^i(c)$. In the analysis below, we will use the bound

$$\begin{aligned} q_u^i(c) &= 1 - \Pr[\bigcup_{uvw \in H^i} (\gamma_v^i(c) = 1 \cap \gamma_w^i(c) = 1) \bigcup_{uv \in G_c^i} \gamma_v(c) = 1] \\ &\geq 1 - \sum_{uvw \in H^i} \theta^2 p_v^i(c) p_w^i(c) - \sum_{uv \in G_c^i} \theta p_v^i(c). \end{aligned} \quad (5.2)$$

Let $\mathbf{I}[X]$ denote the $0, 1$ indicator variable for the event X. Define $p_u^{i+1}(c)$ as:

- If $p_u^i(c)/q_u^i(c) < \hat{p}$ and $c \notin B^i(u)$, then

$$p_u^{i+1}(c) = p_u^i(c) \frac{\mathbf{I}[c \text{ survives at } u]}{\Pr[c \text{ survives at } u]} = \begin{cases} p_u^i(c)/q_u^i(c), & \text{if } c \text{ survives at } u, \\ 0, & \text{else.} \end{cases} \quad (5.3)$$

- If $p_u^i(c)/q_u^i(c) \geq \hat{p}$ or $c \in B^i(u)$, then we toss a biased coin with $\Pr[Head] = p_u^i(c)/\hat{p}$. We then set

$$\eta_u^i(c) = \mathbf{I}[Head],$$

and

$$p_u^{i+1}(c) = p_u^i(c)\frac{\mathbf{I}[Head]}{\Pr[Head]} = \begin{cases} \hat{p}, & \text{if } \eta_u^i(c) = 1 \\ 0, & \text{else.} \end{cases} \tag{5.4}$$

Crucially, (5.3) and (5.4) imply

$$\mathbf{E}[p_u^{i+1}(c)] = p_u^i(c). \tag{5.5}$$

Color u with c if c survives at u and $\gamma_u^i(c) = 1$ (if there are multiple such c, pick one arbitrarily). Let U^{i+1} denote the set of uncolored vertices in H after the iteration i. Let H^{i+1} be the hypergraph induced from H by U^{i+1}, let $B^{i+1}(u) = \{c : p_u^{i+1}(c) = \hat{p}\}$, and let $W_u^{i+1} = \{p_u^{i+1}(c)\}$. To form G_c^{i+1}, start with G_c^i, and for each triple $u, v, w \in U^i$ with $u, v \in U^{i+1}$, $uv \notin G_c^i$, and w colored c, add an edge uv to G_c^{i+1}. Then delete any vertex from G_c^{i+1} that is not in U^{i+1}.

Observe that if uvw is an edge in H^i and u and v are both colored c in the current round, then $p_w^{i+1}(c) \in \{0, \hat{p}\}$; in particular, c is never considered for w in a future round. Similarly, if $vw \in G_c^i$ and v is colored with c in the current round, then c is never considered for w in the future. Thus the algorithm always maintains a proper partial coloring of H.

After T iterations, some vertices will remain uncolored. We color these in one final step, which is described in Section 5.4.5.

5.3.1 Parameters and notation

We summarize all of the variables used in the algorithm and its analysis in the two tables below. The first table contains descriptions of the independent variables in our algorithm. We set them for one family of hypergraphs in Section 5.5, when we prove that our algorithm works for triangle-free hypergraphs. The values of the remaining parameters are defined in the second table.

Our algorithm requires that the parameter ω_0 satisfy the following properties:

- For any edge $uvw \in H^i$ and any color c,

$$\Pr[c \notin L^i(u) \cup L^i(v) \cup L^i(w)] \leq q^i_u(c) q^i_v(c) q^i_w(c)(1 + 1/\omega_0). \tag{5.6}$$

- For any color c and any pair u, v with an edge $uvw \in H^i$ for some w,

$$\Pr[c \notin L^i(u) \cup L^i(v)] \leq q^i_u(c) q^i_v(c)(1 + 1/\omega_0). \tag{5.7}$$

- For any color c and any edge $uv \in G^i_c$,

$$\Pr[c \notin L^i(u) \cup L^i(v)] \leq q^i_u(c) q^i_v(c)(1 + 1/\omega_0). \tag{5.8}$$

	Description
Δ	Maximum degree of 3-graph
Δ_2	Maximum degree of 2-graph
δ	Maximum codegree
ω	Color bound, tending to ∞ with Δ
ϵ	Small constant
ω_0	Error term depending on H
$\hat{p}$	Threshold probability

	Value	Description
C	$\sqrt{\Delta}/\sqrt{\omega}$	Number of colors
T	$(5\omega/\epsilon)\log\omega$	Number of iterations
θ	ϵ/ω	Activation probability
m	21	Used to control codegrees

We will use the following notation:

$$
\begin{aligned}
N_H^i(u) &= \{v \in V(H^i) - u : \exists e \in H^i \text{ with } u, v \in e\} \\
N_H^i(u,v) &= \{w \in V(H^i) - \{u,v\} : \{u,v,w\} \in H^i\} \\
N_c^i(u) &= \{v \in V(G_c^i) - u : \exists e \in G_c^i \text{ with } u, v \in e\} \\
N^i(u) &= N_H^i(u) \cup \cup_c N_c^i(u) \\
N_G^0(u) &= \{v \in V(G) : uv \in E(G)\} \\
d_H^i(u) &= |\{e \in H^i : u \in e\}| \\
d_H^i(u,v) &= |\{e \in H^i : u, v \in e\}| \\
d_{G_c}^i(u) &= |\{v \in G_c^i : uv \in G_c\}|.
\end{aligned}
$$

At the beginning of iteration i of the algorithm, we also define the following parameters:

$$w(p_u^i) = \sum_{c \in C(u)} p_u^i(c)$$

$$f_u^i(c) = \sum_{v: uv \in G_c^i} p_v^i(c)$$

$$f_u^i = \sum_{c \in C(u)} \sum_{v: uv \in G_c^i} p_u^i(c) p_v^i(c)$$

$$e_{uvw}^i = \sum_{c \in C(u)} p_u^i(c) p_v^i(c) p_w^i(c)$$

$$e_u^i = \sum_{\{v,w\}: uvw \in H^i} e_{uvw}^i$$

$$e_u^i(c) = \sum_{\{v,w\}: uvw \in H^i} p_v^i(c) p_w^i(c)$$

$$h_u^i = -\sum_{c \in C(u)} p_u^i(c) \log p_u^i(c), \text{ where } x \log x := 0 \text{ if } x = 0 \ .$$

Our analysis assumes that the parameters of the algorithm satisfy the following relations. All asymptotic notation assumes $\Delta \to \infty$.

(R1) $\theta \log(\hat{p}C) \geq 149$

(R2) $\omega = \Delta^{o(1)}$.

(R3) $\omega_0 > \omega^4$

(R4) $\epsilon \leq 1/72$

(R5) $\delta \leq \Delta^{6/10}$

(R6) $\Delta_2 \leq \sqrt{\Delta}\sqrt{\omega}$

(R7) $\Delta^{-1/2} \leq \hat{p} \leq \Delta^{-11/24}$.

The analysis in Section 5.4 only requires that (5.6), (5.7), (5.8), and (R1)-(R7) hold; the parameters ω, ϵ, $\hat{p}$, and ω_0 depend on the structure of the hypergraph. For instance, we will use the following bounds when applying the analysis to triangle-free hypergraphs:

$$\epsilon = 1/72 \qquad \omega = (1/24)(\epsilon/150)\log\Delta \qquad \hat{p} = \Delta^{-11/24} \qquad \omega_0 = 1/19\theta\hat{p}.$$

5.4 Analysis of algorithm

Theorem 28. *If* (5.6), (5.7), (5.8), *and (R1)-(R7) hold and* $|C(u)| \leq C$ *for all vertices* u, *then the algorithm produces a proper list coloring of* $H \cup G$.

Proof. By Lemma 29, our algorithm proceeds for T iterations, coloring most of the vertices. Since Lemmas 29, 30 and 32 hold after iteration T, we may color the remaining vertices as described in Section 5.4.5. □

Lemma 29 (Main Lemma)**.** *If* (5.6), (5.7), (5.8), *and (R1)-(R7) hold, then for each* $i = 0, 1, \ldots T$, *the following properties hold:*

(P1) $|1 - w(p_u^i)| \leq i/(T\log C)$.

(P2) $e_u^i \leq (1-\theta/3)^i\omega + i/\omega^2$

(P3) $f_u^i \leq 16(1-\theta/4)^i\omega$

(P4) $h_u^i \geq h_u^0 - 37\epsilon\sum_{j=0}^{i-1}(1-\theta/4)^j$

(P5) $d_H^i(u) \leq (1-\theta/3)^i\Delta$

(P6) $d^i_{G_c}(u) \leq 3i\theta\Delta^{5/4}\hat{p}$.

The proof of the Main Lemma relies on the next three lemmas.

Lemma 30. *For any* $i = 0, 1, \ldots T-1$, *if* (5.6), (5.7), (5.8), *and (R1)-(R7) hold and* $|B^i(u)| \leq \epsilon/\hat{p}$ *for all* $u \in U^i$, *then there is an assignment of colors to the vertices in* U^i *so that the following properties hold:*

(Q1) $|w(p^{i+1}_u) - w(p^i_u)| \leq 1/(T \log C)$

(Q2) $e^{i+1}_{uvw} \leq e^i_{uvw} + 1/(\Delta\omega^2)$

(Q3) $f^{i+1}_u \leq f^i_u(1-\theta/2) + 3\theta e^i_u + 1/\omega^2$

(Q4) $h^i_u - h^{i+1}_u \leq 2\theta(f^i_u + e^i_u) + 1/\omega^2$

(Q5) $d^{i+1}_H(u) \leq (1-\theta/2)d^i_H(u) + \Delta^{19/20}$

(Q6) $d^{i+1}_{G_c}(u) \leq d^i_{G_c}(u) + 2\theta\Delta^{5/4}\hat{p}$.

Lemma 31. *If (Q1)-(Q6) hold for* i *and (P1)-(P6) hold for* i*, then (P1)-(P6) hold for* $i+1$.

Lemma 32. *If (P1)-(P6) hold for* $i+1$ *and (R1) holds, then* $|B^{i+1}(u)| \leq \epsilon/\hat{p}$.

5.4.1 Proof of Main Lemma

The proof relies on Lemmas 30, 31 and 32. Assuming these lemmas, we proceed inductively as follows: properties (P1)-(P6) hold for $i = 0$ ((P3) holds by (R6)). Assume (P1)-(P6) hold for i. By Lemma 32, $|B^i(u)| \leq \epsilon/\hat{p}$, so by Lemma 30, (Q1)-(Q6) hold for i. Thus Lemma 31 implies (P1)-(P6) hold for $i+1$.

5.4.2 Proof of Lemma 31

Proof of (P1). By (P1) (for i) and (Q1),

$$\begin{aligned}|1-w(p_u^{i+1})| &= |1-w(p_u^i)+w(p_u^i)-w(p_u^{i+1})| \\ &\leq |1-w(p_u^i)|+|w(p_u^{i+1})-w(p_u^i)| \\ &\leq (i+1)/(T\log C).\end{aligned}$$

Proof of (P5). Recall that (see [53], pg. 434)

$$(1-p)^n \to e^{-pn} \text{ if } p^2 n \to 0 \text{ as } n \to \infty.$$

Since $\theta^2 T = o(1)$,

$$(\theta\Delta/6)(1-\theta/3)^T \to (\theta\Delta/6)e^{-T\theta/3} = (\theta\Delta/6)e^{-\frac{5}{3}\log\omega} > \Delta^{19/20}. \tag{5.9}$$

Using (P5) (for i),

$$\begin{aligned}d_H^{i+1}(u) \overset{(Q5)}{\leq} (1-\theta/2)d_H^i(u)+\Delta^{19/20} &\overset{(P5)}{\leq} (1-\theta/2)(1-\theta/3)^i\Delta+\Delta^{19/20} \\ &= (1-\theta/3)^{i+1}\Delta-\frac{\theta}{6}(1-\theta/3)^i\Delta+\Delta^{19/20} \\ &\leq (1-\theta/3)^{i+1}\Delta-\frac{\theta}{6}(1-\theta/3)^T\Delta+\Delta^{19/20} \\ &\overset{(5.9)}{<} (1-\theta/3)^{i+1}\Delta.\end{aligned}$$

Proof of (P2). By (Q2),

$$e^{i+1}_{uvw} \le e^0_{uvw} + (i+1)/\Delta\omega^2 \le C(1/C^3) + (i+1)/\Delta\omega^2 = \omega/\Delta + (i+1)/\Delta\omega^2.$$

So by (P5) (for $i+1$),

$$e^{i+1}_u = \sum_{uvw} e^{i+1}_{uvw} \le (1-\theta/3)^{i+1}\Delta(\omega/\Delta + (i+1)/\Delta\omega^2) \le (1-\theta/3)^{i+1}\omega + (i+1)/\omega^2.$$

Proof of (P3). Since $\theta^2 T = o(1)$,

$$\theta\omega(1-\theta/4)^T \to \theta\omega e^{-\theta T/4} = \epsilon e^{-\frac{5}{4}\log\omega} = \epsilon\omega^{-5/4} > 3(\theta T + 1/3)\omega^2. \tag{5.10}$$

Using this with (P3) and (P2) (for i),

$$\begin{aligned}
f^{i+1}_u &\overset{(Q3)}{\le} f^i_u(1-\theta/2) + 3\theta e^i_u + 1/\omega^2 \\
&\overset{(P3)}{\le} 16(1-\theta/4)^i\omega(1-\theta/2) + 3\theta e^i_u + 1/\omega^2 \\
&\overset{(P2)}{\le} 16(1-\theta/4)^i\omega(1-\theta/2) + 3\theta\omega(1-\theta/3)^i + 3\theta T/\omega^2 + 1/\omega^2 \\
&= 16(1-\theta/4)^i\omega(1-\theta/4-\theta/4) + \theta\omega(1-\theta/3)^i + 3(\theta T + 1/3)/\omega^2 \\
&= 16(1-\theta/4)^{i+1}\omega - 4\theta\omega(1-\theta/4)^i + \theta\omega(1-\theta/3)^i + 3(\theta T + 1/3)/\omega^2 \\
&< 16(1-\theta/4)^{i+1}\omega - \theta\omega(1-\theta/4)^i + 3(\theta T + 1/3)/\omega^2 \\
&\overset{(5.10)}{<} 16(1-\theta/4)^{i+1}\omega.
\end{aligned}$$

Proof of (P4). Again using $\theta^2 T = o(1)$,

$$\omega(1-\theta/4)^i \geq \omega(1-\theta/4)^T \to \omega e^{-\theta T/4} = \omega^{-1/4} > T/\omega^2. \tag{5.11}$$

Since $T = (5\omega/\epsilon)\log\omega$, this implies

$$\epsilon(1-\theta/4)^i > (5\log\omega)/\omega^2 > 1/\omega^2. \tag{5.12}$$

Therefore, using $\epsilon = \omega\theta$ and (P4) (for i),

$$\begin{aligned}
h_u^{i+1} &\overset{\text{(Q4)}}{\geq} h_u^i - 2\theta(f_u^i + e_u^i) - 1/\omega^2 \\
&\overset{\text{(P3)}}{\geq} h_u^i - 2\theta(16(1-\theta/4)^i\omega + e_u^i) - 1/\omega^2 \\
&\overset{\text{(P2)}}{\geq} h_u^i - 2\theta(16(1-\theta/4)^i\omega + (1-\theta/3)^i\omega + T/\omega^2) - 1/\omega^2 \\
&\geq h_u^i - 2\theta(17(1-\theta/4)^i\omega + T/\omega^2) - 1/\omega^2 \\
&\overset{(5.11)}{>} h_u^i - 2\theta(18(1-\theta/4)^i\omega) - 1/\omega^2 \\
&= h_u^i - 36\epsilon(1-\theta/4)^i - 1/\omega^2 \\
&\overset{(5.12)}{\geq} h_u^i - 37\epsilon(1-\theta/4)^i \\
&\overset{\text{(P4)}}{\geq} h_u^0 - 37\epsilon\sum_{j=0}^{i-1}(1-\theta/4)^j - 37\epsilon(1-\theta/4)^i \\
&= h_u^0 - 37\epsilon\sum_{j=0}^{i}(1-\theta/4)^j.
\end{aligned}$$

Proof of (P6). By (R6) and (R7), $\Delta_2 < \theta\Delta^{5/4}\hat{p}$. Using this with (Q6),

$$d_{G_c}^{i+1}(u) \overset{\text{(Q6)}}{\leq} \Delta_2 + 2(i+1)\theta\Delta^{5/4}\hat{p} \leq 3(i+1)\theta\Delta^{5/4}\hat{p}.$$

5.4.3 Proof of Lemma 32

First,

$$\begin{aligned}
|B^{i+1}(u)|\hat{p}\log(\hat{p}C) = \sum_{c\in B^{i+1}(u)} \hat{p}\log(\hat{p}C) &= \sum_{c\in B^{i+1}(u)} p_u^{i+1}(c)\log(p_u^{i+1}(c)C) \\
&\leq \sum_{c\in C(u)} p_u^{i+1}(c)\log(p_u^{i+1}(c)C) \\
&= \sum_{c\in C(u)} p_u^{i+1}(c)\log p_u^{i+1}(c) + \sum_{c\in C(u)} p_u^{i+1}(c)\log C \\
&= -h_u^{i+1} + \log C \sum_{c\in C(u)} p_u^{i+1}(c). \qquad (5.13)
\end{aligned}$$

Using $p_u^0(c) = 1/C$ for all $c \in C(u)$,

$$\begin{aligned} h_u^0 &= -\sum_{c \in C(u)} p_u^0(c) \log p_u^0(c) \\ &= \log C \sum_{c \in C(u)} p_u^0(c) \\ &= \log C \sum_{c \in C(u)} (p_u^0(c) - p_u^{i+1}(c)) + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &= \log C (1 - w(p_u^{i+1})) + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &\overset{\text{(P1)}}{\geq} -1 + \log C \sum_{c \in C(u)} p_u^{i+1}(c). \end{aligned}$$

Using $\sum_{j=0}^{i}(1-\theta/4)^j \leq 4/\theta$, the above inequality, and inequality (5.13),

$$\begin{aligned} h_u^{i+1} &\overset{\text{(P4)}}{\geq} h_u^0 - 37\epsilon \sum_{j=0}^{i} (1-\theta/4)^j \geq h_u^0 - 148\epsilon/\theta \geq \log C \sum_{c \in C(u)} p_u^{i+1}(c) - 149\epsilon/\theta \\ &\overset{(5.13)}{\geq} h_u^{i+1} + |B^{i+1}(u)| \hat{p} \log(\hat{p}C) - 149\epsilon/\theta. \end{aligned}$$

So

$$|B^{i+1}(u)| \leq \frac{149\epsilon}{\theta \hat{p} \log(\hat{p}C)} \overset{\text{(R1)}}{\leq} \epsilon/\hat{p}.$$

5.4.4 Proof of Lemma 30

Throughout this section, we drop the notation $i+1$ and i, and use, for instance, $p'_u(c)$ and $p_u(c)$ to denote values in iterations $i+1$ and i, respectively.

We are going to apply the Local Lemma. Our probability space is determined by coin flips at each vertex which determine the random variables $\gamma_u(c)$ and $\eta_u(c)$. Recall that

$$N(u) = N^i(u) = N^i_H(u) \cup \cup_c N^i_c(u),$$

where

$$N^i_c(u) = \{v \in V(G^i_c) - u : \exists e \in G^i_c \text{ with } u, v \in e\}.$$

The random variable $p'_u(c)$ is determined by the coin flips in $N(u) + u$. Thus an event "(Qk) fails to hold for u" (or "(Q6) fails to hold for uvw") does not depend on the coin flips outside of $N(N(u))$ (or $N(N(u)) \cup N(N(v)) \cup N(N(w))$). Consequently, if $v \notin \Gamma(u) := N(N(N(N(N(N(u))))))$ (sixth neighborhood), then the events "(Qk) fails to hold for u (or ubc)" and "(Ql) fails to hold for v (or vwx)" are mutually independent. Since $|N(u)| \leq 2\Delta + \Delta_2$, (R6) implies

$$|\Gamma(u)| \leq (2\Delta + \Delta_2)^6 < (3\Delta)^6.$$

To apply the Local Lemma, it therefore suffices to show that the probability that each (Qk) fails is less than $(3\Delta)^{-6}/4$. We prove this for (Q1), (Q2), (Q4), and (Q6) first, and then move on to (Q3) and (Q5).

Proof of (Q1). By (5.5), $\mathbf{E}[p'_u(c)] = p_u(c)$ for each color c. By linearity of expectation,

$$\mathbf{E}[w(p'_u)] = w(p_u).$$

By (R7), $C\hat{p}^2 \leq \Delta^{-10/24}$. Since $w(p'_u)$ is the sum of C independent non-negative random variables, each bounded by $\hat{p}$, Theorem 8 implies

$$\Pr[|w(p'_u) - w(p_u)| \geq 1/(T \log C)] \leq 2e^{-2/(C\hat{p}^2(T\log C)^2)} < 2e^{-7\log\Delta}.$$

Proof of (Q2). Suppose $uvw \in H$. We first prove

$$\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] \leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0). \tag{5.14}$$

Assume that $p'_u(c)$, $p'_v(c)$, and $p'_w(c)$ are determined by (5.3). If $c \in L(u) \cup L(v) \cup L(w)$, then $p'_u(c)p'_v(c)p'_w(c) = 0$, so by (5.6),

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] &\leq \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\frac{p_w(c)}{q_w(c)}\Pr[c \notin L(u) \cup L(v) \cup L(w)] \\ &\leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0).\end{aligned}$$

Suppose $p'_u(c)$ and $p'_v(c)$ are determined by (5.3), and $p'_w(c)$ is determined by (5.4). Then $p'_w(c)$ is independent of $p'_u(c)$ and $p'_v(c)$, so by (5.7),

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] &= \mathbf{E}[p'_u(c)p'_v(c)]\,\mathbf{E}[p'_w(c)] \\ &\leq \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[c \notin L(u) \cup L(v)]p_w(c) \\ &\leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0).\end{aligned}$$

If at least two of $p'_u(c)$, $p'_v(c)$, and $p'_w(c)$ are determined by (5.4), then all three are independent of each other, and

$$\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] = p_u(c)p_v(c)p_w(c),$$

finishing the proof of (5.14).

By definition, $e^0_{uvw} \leq C/C^3 = \omega/\Delta$. By (R3), $\omega^3 + T < \omega_0/2$. So by (Q2) (for i),

$$e_{uvw}/\omega_0 \overset{(Q2)}{\leq} (e^0_{uvw} + \frac{i}{\Delta\omega^2})\frac{1}{\omega_0} \leq (\frac{\omega}{\Delta} + \frac{T}{\Delta\omega^2})\frac{1}{\omega_0} = \frac{\omega^3 + T}{\omega_0}\frac{1}{\Delta\omega^2} < 1/(2\Delta\omega^2).$$

So by (5.14),

$$\begin{aligned}
\mathbf{E}[e'_{uvw}] = \sum_{c \in C(u)} \mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] &\leq \sum_{c \in C(u)} p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0) \\
&= e_{uvw}(1 + 1/\omega_0) \\
&< e_{uvw} + 1/(2\Delta\omega^2).
\end{aligned}$$

Now e'_{uvw} is the sum of C independent random variables, each bounded by $\hat{p}^3$. By (R7), $\Delta^2 C\hat{p}^6 \leq \Delta^{-6/24}$. Thus Theorem 8 yields

$$\begin{aligned}
\Pr[e'_{uvw} \geq e_{uww} + 1/(\Delta\omega^2)] &\leq \Pr[e'_{uvw} \geq e_{uvw} + 1/(2\Delta\omega^2) + 1/(2\Delta\omega^2)] \\
&\leq \Pr[e'_{uvw} \geq \mathbf{E}[e'_{uvw}] + 1/(2\Delta\omega^2)] \\
&< e^{-2/(4\Delta^2\omega^4 C\hat{p}^6)} \\
&< e^{-7\log\Delta}.
\end{aligned}$$

Proof of (Q4). By (5.3) and (5.4), $p'_u(c) = p_u(c)\,\mathbf{I}[A]/\Pr[A]$ for some event A. Thus, using $x\log x = 0$ for $x \in \{0,1\}$,

$$\begin{aligned}
\mathbf{E}[p'_u(c)\log p'_u(c)] &= \mathbf{E}[p_u(c)\,\mathbf{I}[A]/\Pr[A]\log(p_u(c)\,\mathbf{I}[A]/\Pr[A])] \\
&= \mathbf{E}[p_u(c)\,\mathbf{I}[A]/\Pr[A]\log p_u(c) + p_u(c)\,\mathbf{I}[A]/\Pr[A]\log(\mathbf{I}[A]/\Pr[A])] \\
&= \frac{p_u(c)\log p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]] + \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]\log(\mathbf{I}[A]/\Pr[A])] \\
&= p_u(c)\log p_u(c) + \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]\log\mathbf{I}[A]] - \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[\mathbf{I}[A]\log\Pr[A]] \\
&= p_u(c)\log p_u(c) + \frac{p_u(c)}{\Pr[A]}\,\mathbf{E}[0] - p_u(c)\log\Pr[A] \\
&= p_u(c)\log p_u(c) - p_u(c)\log\Pr[A].
\end{aligned}$$

Recall that

$$q_u(c) = \Pr[\bigcap_{\substack{\{v,w\}:\\ uvw\in H}} (\gamma_v(c) = 0 \cup \gamma_w(c) = 0) \bigcap_{v:uv\in G_c} \gamma_v(c) = 0]$$

Also, $1 - rx \geq (1-x)^r$ for $r, x \in (0,1)$. Finally, $\mathbf{I}[\gamma_v(c) = 0 \cup \gamma_w(c) = 0]$ and $\mathbf{I}[\gamma_v(c) = 0]$ are increasing functions of the indicators $\mathbf{I}[\gamma_v(c) = 0]$, so by the FKG inequality,

$$\begin{aligned}
q_u(c) &= \mathbf{E}\left[\prod_{uvw\in H} \mathbf{I}[\gamma_v(c) = 0 \cup \gamma_w(c) = 0] \prod_{uv\in G_c} \mathbf{I}[\gamma_v(c) = 0]\right] \\
&\overset{\text{FKG}}{\geq} \prod_{uvw\in H} \mathbf{E}[\mathbf{I}[\gamma_v(c) = 0 \cup \gamma_w(c) = 0]] \prod_{uv\in G_c} \mathbf{E}[\mathbf{I}[\gamma_v(c) = 0]] \\
&= \prod_{uvw\in H} \Pr[\gamma_v(c) = 0 \cup \gamma_w(c) = 0] \prod_{uv\in G_c} \Pr[\gamma_v(c) = 0] \\
&= \prod_{uvw\in H} (1 - \theta^2 p_v(c) p_w(c)) \prod_{uv\in G_c} (1 - \theta p_v(c)) \\
&\geq \prod_{uvw\in H} (1-\theta)^{\theta p_v(c) p_w(c)} \prod_{uv\in G_c} (1-\theta)^{p_v(c)}.
\end{aligned}$$

By the algorithm, $\Pr[A] \geq q_u(c)$. Also, $\log(1-x) \geq -x - x^2$ for $x \in [0, 1/3]$. Combining these inequalities with the previous inequality, we obtain

$$\begin{aligned}
\log \Pr[A] \geq \log q_u(c) \geq \log\Big(&\prod_{uvw \in H} (1-\theta)^{\theta p_v(c) p_w(c)} \prod_{uv \in G_c} (1-\theta)^{p_v(c)}\Big) \\
&= \sum_{uvw \in H} \theta p_v(c) p_w(c) \log(1-\theta) + \sum_{uv \in G_c} p_v(c) \log(1-\theta) \\
&\geq \sum_{uvw \in H} \theta p_v(c) p_w(c)(-\theta - \theta^2) + \sum_{uv \in G_c} p_v(c)(-\theta - \theta^2) \\
&= (-\theta^2 - \theta^3) \sum_{uvw \in H} p_v(c) p_w(c) + (-\theta - \theta^2) \sum_{uv \in G_c} p_v(c) \\
&= -(\theta^2 + \theta^3) e_u(c) - (\theta + \theta^2) f_u(c).
\end{aligned}$$

Therefore, using the definition of h_u and $\theta < 1/3$,

$$\begin{aligned}
\mathbf{E}[h_u - h'_u] &= h_u + \sum_{c \in C(u)} \mathbf{E}[p'_u(c) \log p'_u(c)] \\
&= h_u + \sum_{c \in C(u)} p_u(c) \log p_u(c) - \sum_{c \in C(u)} p_u(c) \log \Pr[A]) \\
&= -\sum_{c \in C(u)} p_u(c) \log \Pr[A] \\
&\leq \sum_{c \in C(u)} p_u(c)((\theta + \theta^2) f_u(c) + (\theta^2 + \theta^3) e_u(c)) \\
&= (\theta + \theta^2) f_u + (\theta^2 + \theta^3) e_u \\
&< 2\theta(f_u + e_u).
\end{aligned}$$

The terms in $\sum_c -p'_u(c)\log p'_u(c)$ are independent and, since $-x\log x$ is increasing for $0 < x \leq \hat{p}$, bounded by $-\hat{p}\log\hat{p}$. Also $x^2\log^2 x$ is increasing, so by (R7), $C(-\hat{p}\log\hat{p})^2 \leq \Delta^{-10/24-o(1)}$. Thus, by Theorem 8,

$$\Pr[h_u - h'_u \geq 2\theta(f_u + e_u) + 1/\omega^2] < e^{-2/(\omega^4 C(-\hat{p}\log\hat{p})^2)} < e^{-7\log\Delta}.$$

Proof of (Q6). Fix $c \in C(u)$. For each $v \in N_H(u)$, set

$$X_v = d_H(u,v)\gamma_v(c),$$

and set

$$X = \sum_{v\in N_H(u)} X_v.$$

Then

$$\mathbf{E}[X] = \sum_{v\in N_H(u)} d_H(u,v)p_v(c)\theta \leq \hat{p}\theta \sum_{v\in N_H(u)} d_H(u,v) \leq 2\Delta\hat{p}\theta.$$

Since the X_v are independent from each other (because the $\gamma_v(c)$ are independent), and $x(1-x)$ is increasing for $x < 1/2$,

$$\begin{aligned}
\mathbf{Var}[X] = \sum_{v \in N_H(u)} \mathbf{Var}[X_v] &= \sum_{v \in N_H(u)} (\mathbf{E}[X_v^2] - \mathbf{E}[X_v]^2) \\
&= \sum_{v \in N_H(u)} (d_H(u,v)^2 p_v(c)\theta - d_H(u,v)^2 p_v(c)^2 \theta^2) \\
&\leq \sum_{v \in N_H(u)} d_H(u,v)^2 \hat{p}\theta(1-\hat{p}\theta) \\
&= \hat{p}\theta(1-\hat{p}\theta) \sum_{v \in N_H(u)} d_H(u,v)^2 \\
&\leq \hat{p}\theta(1-\hat{p}\theta)\delta \sum_{v \in N_H(u)} d_H(u,v) \\
&= \hat{p}\theta(1-\hat{p}\theta) 2\Delta\delta \\
&< \hat{p}\theta 2\Delta\delta.
\end{aligned}$$

If $uv \notin G_c$ and $uv \in G'_c$, then there exists an edge $uvw \in H$ such that $\gamma_w(c) = 1$. Hence

$$d'_{G_c}(u) - d_{G_c}(u) \leq \sum_{uvw \in H} (\gamma_v(c) + \gamma_w(c)) = \sum_{v \in N_H(u)} d_H(u,v)\gamma_v(c) = X.$$

Since $\hat{p} \geq \Delta^{-1/2}$ and $\delta \leq \Delta^{6/10}$ ((R7) and (R5)), $\Delta^{5/4}\hat{p}/\delta \geq \Delta^{3/20}$. Applying Theorem 9 (with $b = \delta$),

$$\begin{aligned}
\Pr[d'_{G_c}(u) - d_{G_c}(u) \geq 2\Delta^{5/4}\hat{p}\theta] &\leq \Pr[X \geq \Delta^{5/4}\hat{p}\theta + \Delta^{5/4}\hat{p}\theta] \\
&\leq \Pr[X \geq \mathbf{E}[X] + \Delta^{5/4}\hat{p}\theta] \\
&\leq e^{-\Delta^{10/4}\hat{p}^2\theta^2/(4\hat{p}\theta\Delta\delta + \delta\Delta^{5/4}\hat{p}\theta)} \\
&< e^{-\Delta^{10/4}\hat{p}^2\theta^2/5\delta\Delta^{5/4}\hat{p}\theta} \\
&= e^{-\Delta^{5/4}\hat{p}\theta/5\delta} \\
&< e^{-7\log\Delta}.
\end{aligned}$$

We now prove (Q3) and (Q5). The following two claims will be used in both proofs.

Claim 33. *For any $v \in U$ and $c \in C(v)$,*

$$\Pr[v \notin U' | c \notin L(v)] \geq \Pr[v \notin U'] \geq 3\theta/4,$$

and if $uv \in G_c$, then

$$\Pr[v \notin U' | c \notin L(u)] \geq \Pr[v \notin U'] - \theta\hat{p} \geq 5\theta/8,$$

$$\Pr[v \notin U' | c \notin L(u) \cup L(v)] \geq \Pr[v \notin U'] - \theta\hat{p} \geq 5\theta/8.$$

Proof of claim. The vertex v is colored (i.e., $v \notin U'$) if and only if for some color $d \notin B(v)$, $\gamma_v(d) = 1$ and $d \notin L(v)$. Let R_d denote the event that $\gamma_v(d) = 1$ and $d \notin L(v)$. If $c \in B(v)$, then v cannot be colored c, so the event $v \notin U'$ is independent of the events $c \notin L(v)$ and $c \notin L(u)$; hence

$$\Pr[v \notin U'] = \Pr[v \notin U' | c \notin L(v)] = \Pr[v \notin U' | c \notin L(u)] = \Pr[v \notin U' | c \notin L(u) \cup L(v)].$$

Otherwise,

$$\begin{aligned}
\Pr[v \notin U' | c \notin L(v)] &= \frac{\Pr[v \notin U', c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[\cup_{d \in C(v)-B(v)} R_d, c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[(\cup_{d \in C(v)-B(v)-c} R_d \cup R_c), c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[(\cup_{d \in C(v)-B(v)-c} R_d \cup \gamma_v(c) = 1), c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[(\cup_{d \in C(v)-B(v)-c} R_d \cup \gamma_v(c) = 1)] \Pr[c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \Pr[(\cup_{d \in C(v)-B(v)-c} R_d) \cup (\gamma_v(c) = 1)] \\
&\geq \Pr[(\cup_{d \in C(v)-B(v)-c} R_d) \cup R_c] \\
&= \Pr[v \notin U'].
\end{aligned}$$

Suppose $uv \in G_c$. If $c \notin L(u)$, then $\gamma_w(c) = 0$ for all $w \in N_{G_c}(u)$, so in particular, $\gamma_v(c) = 0$. Consequently,

$$\Pr[R_c | c \notin L(u) \cup L(v)] = \Pr[\gamma_v(c) = 1 \cap c \notin L(v) | c \notin L(u) \cup L(v)] = 0.$$

So by the independence of colors and the inequality

$$\Pr[\cup_{d \in C(v)-B(v)} R_d] \leq \Pr[\cup_{d \in C(v)-B(v)-c} R_d] + \Pr[R_c],$$

we obtain

$$\begin{aligned}
\Pr[v \notin U' | c \notin L(u) \cup L(v)] &= \Pr[\cup_{d \in C(v)-B(v)} R_d | c \notin L(u) \cup L(v)] \\
&= \Pr[\cup_{d \in C(v)-B(v)-c} R_d] \\
&\geq \Pr[\cup_{d \in C(v)-B(v)} R_d] - \Pr[R_c] \\
&\geq \Pr[v \notin U'] - \theta \hat{p}.
\end{aligned}$$

Since we only used the condition $c \notin L(u)$, this also implies

$$\Pr[v \notin U' | c \notin L(u)] \geq \Pr[v \notin U'] - \theta \hat{p}.$$

To finish the proof of the claim, we now show $\Pr[v \notin U'] \geq 3\theta/4$. First,

$$\begin{aligned}
\Pr[v \notin U'] &= \Pr[\cup_{d \in C(v)-B(v)} R_d] \\
&\geq \sum_{d \in C(v)-B(v)} \Pr[R_d] - \sum_{d,d' \in C(v)-B(v)} \Pr[R_d] \Pr[R_{d'}] \\
&= \sum_{d \in C(v)-B(v)} \theta p_v(d) q_v(d) - \sum_{d,d' \in C(v)-B(v)} \theta^2 p_v(d) p_v(d') q_v(d) q_v(d') \\
&\geq \theta \sum_{d \in C(v)} p_v(d) q_v(d) - \theta \sum_{d \in B(v)} p_v(d) q_v(d) - \theta^2 \sum_{d,d' \in C(v)-B(v)} p_v(d) p_v(d') \\
&\geq \theta \sum_{d \in C(v)} p_v(d) q_v(d) - \theta |B(v)| \hat{p} - \theta^2 \sum_{d,d' \in C(v)-B(v)} p_v(d) p_v(d').
\end{aligned}$$

By (5.2),

$$\begin{aligned}
q_v(d) &\geq 1 - \sum_{uvw \in H} \theta^2 p_u(d) p_w(d) - \sum_{uv \in G_d} \theta p_u(d) \\
&= 1 - \theta^2 \sum_{uvw \in H} p_u(d) p_w(d) - \theta \sum_{uv \in G_d} p_u(d) \\
&= 1 - \theta^2 e_v(d) - \theta f_v(d).
\end{aligned}$$

Since $\sum_{d \in C(v)} p_v(d) \leq \sqrt{2}$ (by (P1) and $1/\log C = o(1)$),

$$\theta^2 \sum_{d,d' \in C(v)-B(v)} p_v(d) p_v(d') \leq \frac{1}{2} \theta^2 \sum_{d \in C(v)} \sum_{d' \in C(v)-d} p_v(d) p_v(d') \leq \frac{1}{2} \theta^2 (\sum_{d \in C} p_v(d))^2 \leq \theta^2.$$

By our lemma's assumption, $|B(v)| \leq \epsilon/\hat{p}$. By (P3), $f_v < 16\omega$, so $\theta f_v < 16\epsilon$. By (P2), $e_v \leq \omega + T/\omega^2$, so by definition of T and θ, $\theta^2 e_v \leq \epsilon/3$. Using these three inequalities, $\sum_{d \in C(v)} p_v(d) \geq (1-\epsilon/3)$, and (R4), we finally obtain

$$\begin{aligned}
\Pr[v \notin U'] &\geq \theta \sum_{d \in C(v)} p_v(d)(1 - \theta^2 e_v(d) - \theta f_v(d)) - \theta|B(v)|\hat{p} - \theta^2 \\
&= \theta \sum_{d \in C(v)} p_v(d) - \theta^3 \sum_{d \in C(v)} p_v(d) e_v(d) - \theta^2 \sum_{d \in C(v)} p_v(d) f_v(d) - \theta|B(v)|\hat{p} - \theta^2 \\
&\geq \theta \sum_{d \in C(v)} p_v(d) - \theta^3 \sum_{d \in C(v)} p_v(d) e_v(d) - \theta^2 \sum_{d \in C(v)} p_v(d) f_v(d) - \theta\epsilon - \theta^2 \\
&= \theta \sum_{d \in C(v)} p_v(d) - \theta^3 e_v - \theta^2 f_v - \theta\epsilon - \theta^2 \\
&\geq \theta(1 - \epsilon/3) - \theta\epsilon/3 - 16\theta\epsilon - \theta\epsilon - \theta\epsilon/3 \\
&= \theta(1 - 18\epsilon) \\
&\geq 3\theta/4.
\end{aligned}$$

□

Recall that m is a fixed constant.

Claim 34. *For each $l = 0, \ldots, m-2$, let*

$$N^0(u,l) = \{v \in N_H^0(u) - N_G^0(u) : \Delta^{l/2m} < d_H^0(u,v) \leq \Delta^{(l+1)/2m}\},$$

and for $l = m-1$, let

$$N^0(u,l) = \{v \in N_H^0(u) : d_H^0(u,v) > \Delta^{l/2m}\} \cup N_G^0(u).$$

For each l and color c, let $\mathcal{A}_{c,l}$ be the event that $\gamma_v(c) = 1$ for at most $\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u,l)$. Let $\mathcal{A}$ denote the event that $\mathcal{A}_{c,l}$ holds for all l and c. Then

$$\Pr[\bar{\mathcal{A}}] \leq e^{-10\log\Delta}.$$

Proof of claim. Suppose $l < m-1$. Since each $v \in N^0(u,l)$ contributes at least $\Delta^{l/2m}$ edges to $d_H^0(u)$, and each edge is counted at most twice,

$$|N^0(u,l)| \leq 2\Delta/\Delta^{l/2m} = 2\Delta^{1-l/2m}.$$

If $l = m-1$,

$$|N^0(u,l)| \leq 2\Delta/\Delta^{l/2m} + \Delta_2 = 2\Delta^{1-l/2m} + \Delta_2 \overset{\text{(R6)}}{<} 3\Delta^{1-l/2m}.$$

Thus $|N^0(u,l)| < 3\Delta^{1-l/2m}$ for each l.

Since $\Pr[\gamma_v(c) = 1] \le \hat{p}\theta$ and $3e\theta < 1/e$,

$$\begin{aligned}
\Pr[\bar{\mathcal{A}}_{c,l}] \le \binom{|N^0(u,l)|}{\Delta^{1-l/2m}\hat{p}}(\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} &\le \binom{3\Delta^{1-l/2m}}{\Delta^{1-l/2m}\hat{p}}(\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} \\
&\le (\frac{3e}{\hat{p}})^{\Delta^{1-l/2m}\hat{p}}(\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} \\
&= (3e\theta)^{\Delta^{1-l/2m}\hat{p}} \\
&< e^{-\Delta^{1-l/2m}\hat{p}} \\
&\overset{(R7)}{\le} e^{-\Delta^{(m+1)/2m}\Delta^{-1/2}} \\
&= e^{-\Delta^{1/2m}}.
\end{aligned}$$

So by the union bound,

$$\Pr[\bar{\mathcal{A}}] \le Cme^{-\Delta^{1/2m}} \le e^{-10\log\Delta}.$$

□

Proof of (Q3). Observe that

$$
\begin{aligned}
f'_u &= \sum_{c\in C(u)} \sum_{v:uv\in G'_c} p'_u(c)p'_v(c) \\
&= \sum_{c\in C(u)} \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[uv \in G'_c] + \sum_{c\in C(u)} \sum_{\substack{v:uv\notin G_c,\\ uv\in G'_c}} p'_u(c)p'_v(c) \\
&\le \sum_{c\in C(u)} \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[v \in U'] \\
&+ \sum_{c\in C(u)} \sum_{\substack{\{v,w\}:\\ uvw\in H}} (p'_u(c)p'_v(c)\,\mathbf{I}[\gamma_w(c) = 1] + p'_u(c)p'_w(c)\,\mathbf{I}[\gamma_v(c) = 1]) \\
&= D_1 + D_2,
\end{aligned}
$$

where

$$D_1 = \sum_{c\in C(u)} \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[v \in U'],$$

and

$$D_2 = \sum_{c\in C(u)} \sum_{\substack{\{v,w\}:\\ uvw\in H}} (p'_u(c)p'_v(c)\,\mathbf{I}[\gamma_w(c) = 1] + p'_u(c)p'_w(c)\,\mathbf{I}[\gamma_v(c) = 1]).$$

5.4.4.1 Bound on D_1

To bound D_1, we first prove that for $uv \in G_c$,

$$\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] \le p_u(c)p_v(c)(1 - 9\theta/16). \tag{5.15}$$

Note that by (R3),

$$1/\omega_0 < 1/\omega^4 = o(\theta). \tag{5.16}$$

First assume that $p'_u(c)$ and $p'_v(c)$ are determined by (5.3). If $c \in L(u) \cup L(v)$, then $p'_u(c)p'_v(c) = 0$, so using (5.8), Claim 33, and then (5.16),

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &= \mathbf{E}[p'_u(c)p'_v(c)|v \in U']\Pr[v \in U'] \\
&\le \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[c \notin L(u) \cup L(v)|v \in U']\Pr[v \in U'] \\
&= \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[v \in U'|c \notin L(u) \cup L(v)]\Pr[c \notin L(u) \cup L(v)] \\
&\overset{(5.8)}{\le} p_u(c)p_v(c)(1 + 1/\omega_0)\Pr[v \in U'|c \notin L(u) \cup L(v)] \\
&\overset{\text{C.33}}{\le} p_u(c)p_v(c)(1 + 1/\omega_0)(1 - 5\theta/8) \\
&\overset{(5.16)}{\le} p_u(c)p_v(c)(1 - 9\theta/16).
\end{aligned}$$

Suppose $p'_u(c)$ is determined by (5.3) and $p'_v(c)$ is determined by (5.4). Then $p'_u(c)$ and $p'_v(c)$ are independent of each other, and $p'_v(c)$ is independent of the event $v \in U'$, so

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &= \mathbf{E}[p'_u(c)p'_v(c)|v \in U']\,\Pr[v \in U'] \\
&= \mathbf{E}[p'_u(c)|v \in U']\,\mathbf{E}[p'_v(c)]\,\Pr[v \in U'] \\
&\overset{(5.5)}{\leq} \mathbf{E}[p'_u(c)|v \in U']p_v(c)\,\Pr[v \in U'] \\
&\leq \frac{p_u(c)}{q_u(c)}\Pr[c \notin L(u)|v \in U']\,\Pr[v \in U']p_v(c) \\
&= \frac{p_u(c)}{q_u(c)}\Pr[v \in U'|c \notin L(u)]\,\Pr[c \notin L(u)]p_v(c) \\
&= p_u(c)p_v(c)\,\Pr[v \in U'|c \notin L(u)] \\
&\overset{\text{C.33}}{\leq} p_u(c)p_v(c)(1 + 1/\omega_0)(1 - 5\theta/8) \\
&\overset{(5.16)}{\leq} p_u(c)p_v(c)(1 - 9\theta/16).
\end{aligned}$$

Similarly, if $p'_u(c)$ is determined by (5.4) and $p'_v(c)$ is determined by (5.3),

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &\leq p_u(c)p_v(c)\,\Pr[v \in U'|c \notin L(v)] \\
&\overset{\text{C.33}}{\leq} p_u(c)p_v(c)(1 + 1/\omega_0)(1 - 5\theta/8) \\
&\overset{(5.16)}{\leq} p_u(c)p_v(c)(1 - 9\theta/16).
\end{aligned}$$

If $p'_u(c)$ and $p'_v(c)$ are both determined by (5.4),

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &= \mathbf{E}[p'_u(c)p'_v(c)]\,\Pr[v \in U'] \\ &= \mathbf{E}[p'_u(c)]\,\mathbf{E}[p'_v(c)]\,\Pr[v \in U'] \\ &\overset{(C.33)}{\leq} p_u(c)p_v(c)(1 - 3\theta/4) \\ &< p_u(c)p_v(c)(1 - 9\theta/16),\end{aligned}$$

concluding the proof of (5.15).

By (5.15),

$$\begin{aligned}\mathbf{E}[D_1] &= \sum_{c \in C(u)} \sum_{v : uv \in G_c} \mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] \\ &\leq \sum_{c \in C(u)} \sum_{v : uv \in G_c} p_u(c)p_v(c)(1 - 9\theta/16) \\ &= f_u(1 - 9\theta/16).\end{aligned}$$

For $c \in C(u)$, let

$$T_c = \{\gamma_v(c) : v \in N(N(u))\} \cup \{\eta_v(c) : v \in N(N(u))\}.$$

Then each T_c is a (vector valued) random variable, and the set of random variables $\{T_c : c \in C(u)\}$ are mutually independent and determine the variable D_1. We will now apply Corollary 11 with parameters:

- $\mathcal{B}_c = \{0,1\}^{2|N(N(u))|}$, for each $c \in C(u)$
- Independent random variables $T_c : \{c\} \to \{0,1\}^{2|N(N(u))|}$, for each $c \in C(u)$
- Events $\mathcal{A}_c = \cap_{l=1}^{m} \mathcal{A}_{c,l}$, for each $c \in C(u)$ (where $\mathcal{A}_{c,l}$ is from Claim 34)
- $\mathcal{A} = \prod_{c \in C(u)} \mathcal{A}_c$, for each $c \in C(u)$ (this is the same $\mathcal{A}$ as in Claim 34)
- D_1 (which is non-negative) in the role of Y, so that $D_1 : \prod_{c \in C(u)} \mathcal{B}_c \to \mathbb{R}$.
- $d_c = d_{G_c(u)} \hat{p}^2 + 2m\hat{p}^3 \Delta^{1+1/2m}$, for each $c \in C(u)$.

Fix $c \in C(u)$. Let $x, x' \in \mathcal{A}$ such that x and x' differ only in coordinate c. Our goal is to show that $|D_1(x) - D_1(x')| \leq d_c$. Note first that

$$\begin{aligned} D_1 &= \sum_{v:uv \in G_c} p'_u(c) p'_v(c)\, \mathbf{I}[v \in U'] + \sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} \mathbf{I}[v \in U'] \sum_{\substack{d \in C(u)-c: \\ uv \in G_d}} p'_u(d) p'_v(d) \\ &:= D_{1,1} + \sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} D_{1,2}^v. \end{aligned}$$

Since $0 \leq D_{1,1} \leq d_{G_c}(u)\hat{p}^2$,

$$|D_{1,1}(x) - D_{1,1}(x')| \leq d_{G_c}(u)\hat{p}^2. \tag{5.17}$$

Fix l. Let $v \in N^0(u,l)$. If $l \leq m-2$, then $uv \in G_d$ only if there exists a vertex w such that $uvw \in H^0$ and w received color d in a previous round. Thus uv is in at most $d_H^0(u,v) \leq \Delta^{(l+1)/2m}$ graphs G_d, which implies

$$|D_{1,2}^v(x) - D_{1,2}^v(x')| \leq \Delta^{(l+1)/2m} \hat{p}^2, \quad l \leq m-2. \tag{5.18}$$

If $l = m-1$, then uv is in at most C graphs G_d, so

$$|D^v_{1,2}(x) - D^v_{1,2}(x')| \leq C\hat{p}^2 \leq \Delta^{1/2}\hat{p}^2, \quad l = m-1. \tag{5.19}$$

Note that x and x' are vectors of length $|C(u)|$. The c^{th} coordinate of each vector is a vector of length $2|N(N(u))|$, and the entries in these vectors correspond to the values assigned to the random variables $\gamma_v(c)$ and $\eta_v(c)$ for all $v \in N(N(u))$. Let $x_c(\gamma, v)$ and $x'_c(\gamma, v)$ denote the values of $\gamma_v(c)$ corresponding to x and x', respectively.

Suppose $x_c(\gamma, v) = x'_c(\gamma, v) = 0$. Then v cannot be colored c during the current iteration. Thus changing the value of $\gamma_w(c)$ for any $w \in N(N(u)) - v$ does not affect whether or not v is colored. Therefore $\mathbf{I}[v \in U'](x) = \mathbf{I}[v \in U'](x')$. In addition, $p'_u(d)p'_v(d)(x) = p'_u(d)p'_v(d)(x')$ for any $d \in C(u) - c$. Thus $D^v_{1,2}(x) = D^v_{1,2}(x')$ if $x_c(\gamma, v) = x'_c(\gamma, v) = 0$.

By definition of $\mathcal{A}_{c,l}$, $x_c(\gamma, v) = 1$ for at most $\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u, l)$. Therefore $D^v_{1,2}(x) \neq D^v_{1,2}(x')$ for at most $2\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u, l)$. So by (5.19) and (5.18),

$$\begin{aligned}\sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} |D^v_{1,2}(x) - D^v_{1,2}(x')| &\leq 2\Delta^{1+1/2m}\hat{p}^3 + \sum_{l=0}^{m-2} (2\Delta^{1-l/2m}\hat{p})(\Delta^{(l+1)/2m}\hat{p}^2) \\ &= 2m\Delta^{1+1/2m}\hat{p}^3.\end{aligned}$$

Combining this with (5.17),

$$|D_1(x) - D_1(x')| \leq d_{G_c}(u)\hat{p}^2 + 2m\Delta^{1+1/2m}\hat{p}^3 = d_c.$$

Since $\sum_{c\in C(u)} d_{G_c}(u) \leq \Delta + \Delta_2 < 2\Delta$ and, by (P6), $d_{G_c}(u) \leq 3T\theta\Delta^{5/4}\hat{p}$,

$$\begin{aligned}
&\sum_{c\in C(u)} (d_{G_c}(u)\hat{p}^2 + 2m\hat{p}^3\Delta^{1+1/2m})^2 \\
&\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + \sum_{c\in C(u)} (\hat{p}^4 d_{G_c}(u)^2 + d_{G_c}(u)4m\hat{p}^5\Delta^{1+1/2m}) \\
&\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + 8m\hat{p}^5\Delta^{2+1/2m} + \hat{p}^4 \sum_{c\in C(u)} d_{G_c}(u)^2 \\
&\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + 8m\hat{p}^5\Delta^{2+1/2m} + 3\hat{p}^5T\theta\Delta^{5/4} \sum_{c\in C(u)} d_{G_c}(u) \\
&\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + 8m\hat{p}^5\Delta^{2+1/2m} + 6T\theta\hat{p}^5\Delta^{9/4}.
\end{aligned}$$

By (R7), $\hat{p}^5\Delta^{9/4} \leq \Delta^{-1/24}$, $\hat{p}^5\Delta^{2+1/2m} \leq \Delta^{-15/56}$, and $C\hat{p}^6\Delta^{2+1/m} \leq \Delta^{-17/84}$. Together with Claim 34, Corollary 11 now implies

$$\begin{aligned}
\Pr[D_1 > f_u(1-\theta/2) + 1/2\omega^2] &\leq \Pr[D_1 > f_u(1-9\theta/16)/\Pr[\mathcal{A}] + 1/2\omega^2] \\
&\leq \Pr[D_1 > \mathbf{E}[D_1]/\Pr[\mathcal{A}] + 1/2\omega^2] \\
&\overset{\text{C.11}}{\leq} e^{-1/4\omega^4(6T\theta\hat{p}^5\Delta^{9/4} + 8m\hat{p}^5\Delta^{2+1/2m} + 4Cm^2\hat{p}^6\Delta^{2+1/m})} + \Pr[\bar{\mathcal{A}}] \\
&\leq e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}] \\
&\overset{\text{C.34}}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta} \\
&< 2e^{-7\log\Delta}.
\end{aligned}$$

5.4.4.2 Bound on D_2

We now bound D_2. We first prove that for any edge uvw,

$$\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c) = 1] \leq p_u(c)p_v(c)(1 + 1/\omega_0). \tag{5.20}$$

Assume that both $p'_u(c)$ and $p'_v(c)$ are determined by (5.3). Since the function $\mathbf{I}[c \notin L(u) \cup L(v)]$ is decreasing and $\mathbf{I}[\gamma_w(c) = 1]$ is increasing, the FKG inequality implies

$$\begin{aligned}
\Pr[c \notin L(u) \cup L(v)|\gamma_w(c) = 1] &= \frac{\Pr[c \notin L(u) \cup L(v), \gamma_w(c) = 1]}{\Pr[\gamma_w(c) = 1]} \\
&\leq \frac{\Pr[c \notin L(u) \cup L(v)] \Pr[\gamma_w(c) = 1]}{\Pr[\gamma_w(c) = 1]} \\
&= \Pr[c \notin L(u) \cup L(v)]. \qquad (5.21)
\end{aligned}$$

Similarly,

$$\Pr[c \notin L(u)|\gamma_w(c) = 1] \leq \Pr[c \notin L(u)]. \tag{5.22}$$

If $c \in L(u)$ or $c \in L(v)$, then $p'_u(c)p'_v(c) = 0$, so by (5.8),

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c) = 1] &\leq \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)} \Pr[c \notin L(u) \cup L(v)|\gamma_w(c) = 1] \\
&\overset{(5.21)}{\leq} \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)} \Pr[c \notin L(u) \cup L(v)] \\
&\overset{(5.8)}{\leq} p_u(c)p_v(c)(1 + 1/\omega_0).
\end{aligned}$$

Suppose $p'_u(c)$ is determined by (5.3) and $p'_v(c)$ is determined by (5.4). Then $p'_u(c)$ and $p'_v(c)$ are independent of each other, and $p'_v(c)$ is independent of the event $\gamma_w(c) = 1$, so

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c) = 1] &= \mathbf{E}[p'_u(c)|\gamma_w(c) = 1]\,\mathbf{E}[p'_v(c)] \\
&\overset{(5.5)}{=} \mathbf{E}[p'_u(c)|\gamma_w(c) = 1]p_v(c) \\
&\leq \frac{p_u(c)}{q_u(c)} \Pr[c \notin L(u)|\gamma_w(c) = 1]p_v(c) \\
&\overset{(5.22)}{\leq} \frac{p_u(c)}{q_u(c)} \Pr[c \notin L(u)]p_v(c) \\
&= p_u(c)p_v(c) \\
&< p_u(c)p_v(c)(1 + 1/\omega_0).
\end{aligned}$$

If $p'_u(c)$ and $p'_v(c)$ are both determined by (5.4), then

$$\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c) = 1] = \mathbf{E}[p'_u(c)p'_v(c)] = \mathbf{E}[p'_u(c)]\,\mathbf{E}[p'_v(c)] \overset{(5.5)}{=} p_u(c)p_v(c),$$

which establishes (5.20).

Now, by (5.20),

$$\begin{aligned}
\mathbf{E}[D_2] &= \sum_{c\in C(u)} \sum_{uvw} (\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[\gamma_w(c)=1]] + \mathbf{E}[p'_u(c)p'_w(c)\,\mathbf{I}[\gamma_v(c)=1]]) \\
&= \sum_{c\in C(u)} \sum_{uvw} \mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c)=1]\Pr[\gamma_w(c)=1] \\
&+ \sum_{c\in C(u)} \sum_{uvw} \mathbf{E}[p'_u(c)p'_w(c)|\gamma_v(c)=1]\Pr[\gamma_v(c)=1] \\
&\leq (1+1/\omega_0) \sum_{c\in C(u)} \sum_{uvw} (p_u(c)p_v(c)\Pr[\gamma_w(c)=1] + p_u(c)p_w(c)\Pr[\gamma_v(c)=1]) \\
&= (1+1/\omega_0) \sum_{c\in C(u)} \sum_{uvw} (p_u(c)p_v(c)\theta p_w(c) + p_u(c)p_w(c)\theta p_v(c)) \\
&= (1+1/\omega_0)2\theta e_u.
\end{aligned}$$

We prove concentration of D_2 using the same setup that we used for D_1. Again, for $c \in C(u)$, let

$$T_c = \{\gamma_v(c) : v \in N(N(u))\} \cup \{\eta_v(c) : v \in N(N(u))\}.$$

Then D_2 is determined by the set of random variables $\{T_c : c \in C(u)\}$. Observe that

$$\begin{aligned}
D_2 &= \sum_{c\in C(u)} \sum_{l=0}^{m-1} \sum_{v\in N_H(u)\cap N^0(u,l)} \mathbf{I}[\gamma_v(c)=1] \sum_{w\in N_H(u,v)} p'_u(c)p'_w(c) \\
&:= \sum_{c\in C(u)} \sum_{l=0}^{m-1} \sum_{v\in N_H(u)\cap N^0(u,l)} D_2^{v,c}.
\end{aligned}$$

Fix $c \in C(u)$. Let $x, x' \in \mathcal{A}$ (from Claim 34) such that x and x' differ only in coordinate c. Fix l and $v \in N_H(u) \cap N^0(u,l)$. Note that for $d \in C(u) - c$,

$$D_2^{v,d}(x) = D_2^{v,d}(x').$$

By definition of $\mathcal{A}_{c,l}$ (from Claim 34), $\mathbf{I}[\gamma_v(c) = 1](x) = 1$ or $\mathbf{I}[\gamma_v(c) = 1](x') = 1$ for at most $2\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u,l)$. Furthermore,

$$\sum_{w \in N_H(u,v)} p'_u(d)p'_w(d) \leq d_H(u,v)\hat{p}^2.$$

Thus

$$\begin{aligned}
|D_2(x) - D_2(x')| &\leq \sum_{l=0}^{m-1} \sum_{v \in N_H(u) \cap N^0(u,l)} |D_2^{v,c}(x) - D_2^{v,c}(x')| \\
&\leq \sum_{l=0}^{m-1} \sum_{\substack{v \in N_H(u) \cap N^0(u,l): \\ \mathbf{I}[\gamma_v(c)=1](x)=1 \text{ or} \\ \mathbf{I}[\gamma_v(c)=1](x')=1}} d_H(u,v)\hat{p}^2 \\
&\leq \sum_{l=0}^{m-2} (2\Delta^{1-l/2m}\hat{p})\Delta^{(l+1)/2m}\hat{p}^2 + (2\Delta^{1-(m-1)/2m}\hat{p})\delta\hat{p}^2 \\
&< 2m\Delta^{1+1/2m}\hat{p}^3 + 2\delta\Delta^{1/2+1/2m}\hat{p}^3.
\end{aligned}$$

Recall that $\hat{p} \leq \Delta^{-11/24}$ and $\delta \leq \Delta^{6/10}$ ((R7) and (R5)). Thus

$$C(2m\Delta^{1+1/2m}\hat{p}^3 + 2\delta\Delta^{1/2+1/2m}\hat{p}^3)^2 \leq \Delta^{1/2}(3\Delta^{-211/840})^2 \leq 9\Delta^{-1/420}.$$

By Corollary 11 and Claim 34,

$$\begin{aligned}\Pr[D_2 > 3\theta e_u + 1/2\omega^2] &\leq \Pr[D_2 > (1+1/\omega_0)2\theta e_u/\Pr[\mathcal{A}] + 1/2\omega^2] \\ &\overset{\text{C.11}}{\leq} e^{-2/(4\omega^4 C(2m\Delta^{1+1/2m}\hat{p}^3 + 2\delta\Delta^{1/2+1/2m}\hat{p}^3)^2)} + \Pr[\bar{\mathcal{A}}] \\ &\leq e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}] \\ &\overset{\text{C.34}}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta} \\ &\leq 2e^{-7\log\Delta}.\end{aligned}$$

Therefore, with probability at least $1 - 2\Delta^{-5}$,

$$\begin{aligned}f'_u &\leq f_u(1-\theta/2) + 1/2\omega^2 + 3\theta e_u + 1/2\omega^2 \\ &\leq f_u(1-\theta/2) + 3\theta e_u + 1/\omega^2.\end{aligned}$$

Proof of (Q5). Since

$$d'_H(u) = \frac{1}{2}\sum_{v\in N_H(u)}\sum_{w\in N_H(u,v)} \mathbf{I}[v,w\in U'] \leq \frac{1}{2}\sum_{v\in N_H(u)} d_H(u,v)\,\mathbf{I}[v\in U'],$$

Claim 33 implies

$$\mathbf{E}[d'_H(u)] \leq \frac{1-3\theta/4}{2}\sum_{v\in N_H(u)} d_H(u,v) = (1-3\theta/4)d_H(u).$$

We prove concentration in the same way as in the proof of (Q3). For $c \in C(u)$, let

$$T_c = \{\gamma_v(c) : v \in N(N(u))\} \cup \{\eta_v(c) : v \in N(N(u))\}.$$

The random variable $d'_H(u)$ is determined by the set of random variables $\{T_c : c \in C(u)\}$. Observe that

$$d'_H(u) \leq \frac{1}{2} \sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} d^0_H(u,v)\, \mathbf{I}[v \in U'].$$

Fix $c \in C(u)$. Let $x, x' \in \mathcal{A}$ (from Claim 34) such that x and x' differ only in coordinate c. Fix l. As in the proof of (Q3) (see the two paragraphs after (5.19)),

$$\mathbf{I}[v \in U'](x) \neq \mathbf{I}[v \in U'](x')$$

for at most $2\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u,l)$. Further, if $l \leq m-2$, then $d^0_H(u,v) \leq \Delta^{(l+1)/2m}$, and if $l = m-1$, then $d^0_H(u,v) \leq \delta$. Therefore

$$\begin{aligned} |d'_H(u)(x) - d'_H(u)(x')| &\leq \sum_{l=0}^{m-2} \Delta^{1-l/2m}\hat{p}\Delta^{(l+1)/2m} + \Delta^{1-(m-1)/2m}\hat{p}\delta \\ &< m\Delta^{1+1/2m}\hat{p} + \Delta^{1/2+1/2m}\hat{p}\delta. \end{aligned}$$

By (R7) and (R5),

$$C(m\Delta^{1+1/2m}\hat{p} + \Delta^{1/2+1/2m}\hat{p}\delta)^2 \leq \Delta^{1/2}(2\Delta^{559/840})^2 = 4\Delta^{769/420}.$$

By Corollary 11 and Claim 34,

$$\begin{aligned}
\Pr[d'_H(u) > (1-\theta/2)d_H(u) + \Delta^{19/20}] &\le \Pr[d'_H(u) > (1-3\theta/4)d_H(u)/\Pr[\mathcal{A}] + \Delta^{19/20}] \\
&\overset{\text{C.11}}{\le} e^{-2\Delta^{38/20}/C(m\Delta^{1+1/2m}\hat{p}+\Delta^{1/2+1/2m}\hat{p}\delta)^2} + \Pr[\bar{\mathcal{A}}] \\
&\le e^{-(1/2)\Delta^{38/20-769/420}} + \Pr[\bar{\mathcal{A}}] \\
&\le e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}] \\
&\overset{\text{C.34}}{\le} e^{-7\log\Delta} + e^{-10\log\Delta} \\
&\le 2e^{-7\log\Delta}.
\end{aligned}$$

5.4.5 Final step

After the iterative portion of the algorithm, some vertices will still be uncolored. Assuming (R1)-(R7) and Lemmas 29, 30, and 32 hold, we color them using the Asymmetric Local Lemma as follows. Suppose u has not been colored. By (P1), Lemma 32, and (R4),

$$\begin{aligned}
\sum_{c\in C(u)-B^T(u)} p_u^T(c) = \sum_{c\in C(u)} p_u^T(c) - \sum_{c\in B^T(u)} p_u^T(c) &\overset{\text{(P1)}}{\ge} 1 - 1/\log C - |B^T(u)|\hat{p} \\
&\overset{\text{L.32}}{\ge} 1 - o(1) - \epsilon \\
&\overset{\text{(R4)}}{\ge} 1/2.
\end{aligned}$$

For each $c \notin B^T(u)$, define

$$p_u^*(c) := \frac{p_u^T(c)}{\sum_{c \in C(u) - B^T(u)} p_u^T(c)} \leq 2p_u^T(c).$$

For each uncolored vertex u, randomly assign u one color from the distribution given by p_u^*. For an edge $e = uvw \in H^T$, let A_{uvw} denote the event that u, v, and w receive the same color. By definition of T, $T/\omega^2 = o(\omega)$. So by (Q2),

$$e_{uvw}^T \leq e_{uvw}^0 + T/\Delta\omega^2 = 1/C^2 + o(\omega/\Delta) = \omega/\Delta + o(\omega/\Delta).$$

Therefore

$$\Pr[A_{uvw}] = \sum_{c \in C(u)} p_u^*(c) p_v^*(c) p_w^*(c) \leq 8 \sum_{c \in C(u)} p_u^T(c) p_v^T(c) p_w^T(c) = 8e_{uvw}^T \leq 9\omega/\Delta.$$

For each $c \in C(u) \cap C(v)$ and each pair $uv \in G_c^T$, let $B_{uv,c}$ denote the event that u and v both receive color c. By (P3), for each u,

$$\sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] \leq 4 \sum_{c \in C(u)} \sum_{ux \in G_c^T} p_u^T(c) p_x^T(c) = 4f_u^T \leq 64(1 - \theta/4)^T \omega.$$

The event A_{uvw} depends on any event A_e or $B_{f,d}$, where u, v, or w is in the edge e or the edge f. Using (P5),

$$\begin{aligned}
&\sum_{e \in H^T : u \in e} \Pr[A_e] + \sum_{e \in H^T : v \in e} \Pr[A_e] + \sum_{e \in H^T : w \in e} \Pr[A_e] \\
&+ \sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] + \sum_{c \in C(v)} \sum_{vx \in G_c^T} \Pr[B_{vx,c}] \sum_{c \in C(w)} \sum_{wx \in G_c^T} \Pr[B_{wx,c}] \\
&\leq 3(9\omega/\Delta)(1-\theta/3)^T \Delta + 3(64)(1-\theta/4)^T \omega \\
&\leq 219(1-\theta/4)^T \omega \\
&\leq 219 e^{-\theta T/4} \omega \\
&= 219 e^{-5 \log \omega / 4} \omega \\
&= 219 (\frac{1}{\omega})^{5/4} \omega \\
&< 1/4.
\end{aligned}$$

The event $B_{uv,c}$ depends on any event A_e or $B_{f,d}$, where u or v is in e or f. Since

$$\begin{aligned}
&\sum_{e \in H^T : u \in e} \Pr[A_e] + \sum_{e \in H^T : v \in e} \Pr[A_e] + \sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] + \sum_{c \in C(v)} \sum_{vx \in G_c^T} \Pr[B_{vx,c}] \\
&\leq 18(1-\theta/3)^T \omega + 128(1-\theta/4)^T \omega \\
&\leq 1/4,
\end{aligned}$$

the Asymmetric Local Lemma implies that there exists a coloring where none of the events A_{uvw} or $B_{uv,c}$ occur. Since no color in $B^T(u)$ and no color with $p_c^T(u) = 0$ was assigned to u, this coloring, combined with the partial coloring from the algorithm, is a proper list coloring of $H \cup G$.

5.5 Triangle-free hypergraphs

We will derive Theorem 27 as a corollary of the following theorem:

Theorem 35. *Set* $c_0 = 1/259,200$*. Suppose* H *is a rank 3, triangle-free hypergraph with maximum* 3*-degree at most* Δ*, maximum 2-degree at most* $(c_0\Delta \log \Delta)^{1/2}$*, and maximum codegree at most* $\Delta^{6/10}$*. Then*

$$\chi_l(H) \leq (\frac{\Delta}{c_0 \log \Delta})^{1/2}.$$

To prove this using Theorem 28, we need to find values for the parameters ω, ϵ, ω_0, and $\hat{p}$ which satisfy (R1)-(R7), (5.6), (5.7), and (5.8), and $\omega = c_0 \log \Delta$. We will show that the following values satisfy these criteria:

$$\epsilon = 1/72 \qquad \omega = (1/24)(\epsilon/150)\log\Delta \qquad \hat{p} = \Delta^{-11/24} \qquad \omega_0 = 1/19\theta\hat{p}.$$

For (R1),

$$\theta \log(\hat{p}C) = \frac{\epsilon}{\omega}\log(\frac{\Delta^{1/24}}{\sqrt{\omega}}) = \frac{\epsilon}{\omega}(\frac{1}{24}\log\Delta - \frac{1}{2}\log\omega) = 86 - o(1) > 85.$$

The parameters clearly satisfy (R2)-(R7), so all that remains is to show inequalities (5.6), (5.7), and (5.8) hold. Fix a color c. In Claim 36, we first show that that hypergraph $H \cup G_c$ remains triangle-free throughout the algorithm. The next three claims then show that if the hypergraph remains triangle-free, we will have enough independence to derive (5.6), (5.7), and (5.8). Throughout the rest of this section, we will be taking intersections and unions over edges; when we do this, we use the notation e in place of $e \in E(H) \cup E(G_c)$.

Claim 36. *For iteration i, if $H^i \cup G_c^i$ is triangle-free, then $H^{i+1} \cup G_c^{i+1}$ is triangle-free.*

Proof. It suffices to show that when the algorithm creates G_c^{i+1} from G_c^i by adding an edge uv to G_c^i, no triangle is created. Toward a contradiction, suppose that a triangle is created with distinct edges $uv, e, f \in H^{i+1} \cup G_c^{i+1}$ and distinct vertices u, v, w such that $u \in e$, $v \in f$, $w \in e \cap f$, and $u \notin f$, $v \notin e$. Note that $u, v, w \in V(H^i \cup G_c^i)$ and $e, f \in H^i \cup G_c^i$. Since $w \in V(H^i \cup G_c^i)$, w has not been colored. Thus there exists a vertex $x \in V(H^i) - w$ and an edge $uvx \in H^i$ which gave rise to the edge uv. The edges uvx, e, and f form a triangle with vertices u, v, and w in $H^i + G_c^i$, a contradiction. □

In the rest of this section, we define

$$d(u, v) = |\{e \in H \cup G_c : u, v \in e\}|.$$

In addition, we drop the superscript from H^i and G_c^i.

Claim 37. *Suppose $uvw \in H$, $d(u, v) \geq 2$, and $d(w, v) \geq 2$. Then $d(u, w) = 1$.*

Proof. Since $d(u, v) \geq 2$ and $d(w, v) \geq 2$, there exist distinct edges $e, f \neq uvw$ such that $u, v \in e$ and $w, v \in f$. If there exists $x \neq v$ such that $uwx \in H$, then e, f, and uxw form a triangle with corresponding vertices u, v, and w. If $uw \in G_c$, then e, f, and uw form a triangle with vertices u, v, and w. □

Claim 38. *If uvw is an edge and $d(u,w)=1$, then*

$$(\bigcup_{e:u\in e;v\notin e} e-u)\cap(\bigcup_{e:w\in e;v\notin e} e-w)=\emptyset, \tag{5.23}$$

$$(\bigcup_{e:u\in e;v\notin e} e-u)\cap(\bigcup_{e:v\in e;u\notin e} e-v)=\emptyset, \tag{5.24}$$

and

$$(\bigcup_{e:w\in e;v\notin e} e-u)\cap(\bigcup_{e:v\in e;w\notin e} e-v)=\emptyset. \tag{5.25}$$

Proof. Let $x\in U$, and let e be an edge such that $u\in e$, $v\notin e$, and $x\in e-u$. Then $e\neq uvw$, and since $d(u,w)=1$, $x\notin\{u,v,w\}$.

Suppose f is an edge such that $w\in f$, $v\notin f$, and $x\in f-w$. Then, since $x\in f$, $f\neq uvw$. Using $d(u,w)=1$, $u\in e$, $w\in f$ and $e,f\neq uvw$, we get $e\neq f$, $u\notin f$, and $w\notin e$. Since $x\notin uvw$, we obtain a triangle with edges e, f, and uvw and vertices u, w, and x.

Now suppose that $v,x\in f$ and $u\notin f$. Again, $f\neq uvw$. Because $u\in e$ and $u\notin f$, $e\neq f$. Since $u\notin f$, $v\notin e$, and $x\notin\{u,v,w\}$, e, f, and uvw form a triangle with vertices u, v, and x. By symmetry, this also gives (5.25). □

Claim 39. *If $uv\in G_c$, then*

$$(\bigcup_{e:u\in e;v\notin e} e-u)\cap(\bigcup_{e:v\in e;u\notin e} e-v)=\emptyset. \tag{5.26}$$

Proof. If there exist edges e and f and a vertex x such that $u\in e$, $v\notin e$, $v\in f$, $u\notin f$, and $x\in e-u\cap f-v$, then e, f, and uv form a triangle with vertices u, v, and x in $H\cup G_c$. □

For a set of vertices S, let $\gamma_S(c) = 1$ denote the event that $\gamma_v(c) = 1$ for all $v \in S$, and let $\gamma_S(c) \neq 1$ denote the event that $\gamma_v(c) = 0$ for some $v \in S$.

Claim 40. *For any three vertices x, y, and z,*

$$\Pr[\bigcap_{e:x\in e; y\notin e} \gamma_{e-x}(c) \neq 1] \leq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1] \leq q_x(c)(1+3\theta\hat{p}).$$

Proof. Note first that

$$\Pr[\bigcap_{e:x\in e; y\in e} \gamma_{e-x}(c) \neq 1] \geq \Pr[\gamma_y(c) = 0] \geq 1 - \theta\hat{p}.$$

Similarly,

$$\Pr[\bigcap_{e:x\in e; z\in e} \gamma_{e-x}(c)] \geq 1 - \theta\hat{p}.$$

Since the functions $\mathbf{I}[\bigcap_{x\in e; y\notin e} \gamma_{e-x}(c) \neq 1]$ and $\mathbf{I}[\bigcap_{x\in e; y\in e} \gamma_{e-x}(c) \neq 1]$ are monotone decreasing, the FKG inequality and then the previous two inequalities yield

$$\begin{aligned}
q_x(c) &= \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1 \bigcap_{e:x,y\in e} \gamma_{e-x}(c) \neq 1 \bigcap_{e:x,z\in e} \gamma_{e-x}(c) \neq 1] \\
&\geq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1] \Pr[\bigcap_{e:x\in e, y\in e} \gamma_{e-x}(c) \neq 1] \Pr[\bigcap_{e:x\in e, z\in e} \gamma_{e-x}(c) \neq 1] \\
&\geq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1](1-\theta\hat{p})^2 \\
&\geq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1](1-2\theta\hat{p}).
\end{aligned}$$

Thus

$$\Pr[\bigcap_{e:x\in e;y,z\notin e} \gamma_{e-x}(c) \neq 1] \leq q_x(c)/(1-2\theta\hat{p}) \leq q_x(c)(1+3\theta\hat{p}).$$

□

We can now prove (5.6), (5.7), and (5.8). Suppose uvw is an edge. By Claim 37, we may assume $d(u,w) = 1$. The events $\bigcap_{u\in e;v\notin e} \gamma_{e-u}(c) \neq 1$, $\bigcap_{w\in e;v\notin e} \gamma_{e-w}(c) \neq 1$, and $\bigcap_{v\in e;u,w\notin e} \gamma_{e-v}(c) \neq 1$ depend only on the sets of random variables

$$\{\gamma_x(c) : x \in \bigcup_{e:u\in e;v\notin e} e-u\},$$

$$\{\gamma_x(c) : x \in \bigcup_{e:w\in e;v\notin e} e-w\},$$

and

$$\{\gamma_x(c) : x \in \bigcup_{e:v\in e;u,w\notin e} e-v\},$$

respectively. By (5.23), (5.24), and (5.25), these sets are pairwise disjoint, so the three events are independent of each other. Therefore, applying Claim 40,

$$
\begin{aligned}
&\Pr[c \notin L(u) \cup L(v) \cup L(w)] \\
&= \Pr[\bigcap_{e:u\in e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v\in e} \gamma_{e-v}(c) \neq 1 \bigcap_{e:w\in e} \gamma_{e-w}(c) \neq 1] \\
&\leq \Pr[\bigcap_{e:u\in e;v\notin e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v\in e;u,w\notin e} \gamma_{e-v}(c) \neq 1 \bigcap_{e:w\in e;v\notin e} \gamma_{e-w}(c) \neq 1] \\
&= \Pr[\bigcap_{e:u\in e;v\notin e} \gamma_{e-u}(c) \neq 1] \Pr[\bigcap_{e:v\in e;u,w\notin e} \gamma_{e-v}(c) \neq 1] \Pr[\bigcap_{e:w\in e;v\notin e} \gamma_{e-w}(c) \neq 1] \\
&\overset{\text{C.40}}{\leq} q_u(c)q_v(c)q_w(c)(1+3\theta\hat{p})^3 \\
&< q_u(c)q_v(c)q_w(c)(1+19\theta\hat{p}) \\
&= q_u(c)q_v(c)q_w(c)(1+1/\omega_0).
\end{aligned}
$$

This proves (5.6). The proof of (5.7) is the same, except we start with any two vertices in uvw instead of all three.

Suppose now that $uv \in G_c$ for some color c. By (5.26) and Claim 40,

$$\begin{aligned}
\Pr[c \notin L(u) \cup L(v)] &= \Pr[\bigcap_{e:u \in e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v \in e} \gamma_{e-v}(c) \neq 1] \\
&\leq \Pr[\bigcap_{e:u \in e; v \notin e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v \in e; u \notin e} \gamma_{e-v}(c) \neq 1] \\
&\overset{(5.26)}{=} \Pr[\bigcap_{e:u \in e; v \notin e} \gamma_{e-u}(c) \neq 1] \Pr[\bigcap_{e:v \in e; u \notin e} \gamma_{e-v}(c) \neq 1] \\
&\overset{\text{C.40}}{\leq} q_u(c) q_v(c)(1 + 3\theta\hat{p})^2 \\
&< q_u(c) q_v(c)(1 + 7\theta\hat{p}) \\
&< q_u(c) q_v(c)(1 + 1/\omega_0),
\end{aligned}$$

completing the proof of (5.8) and Theorem 35.

Proof of Theorem 27: Recall that $c_0 = 1/259,200$. Let H be a rank 3, triangle-free hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. The original hypergraph H may have some pairs of vertices with codegree too large to apply Theorem 35, so we will work on a modified hypergraph instead. Let

$$K(u) = \{v \in N(u) : d(u,v) \geq \Delta^{6/10}\}.$$

Define a new hypergraph H' with $V(H') = V(H)$ and

$$E(H') = E(H) - (\bigcup_{u \in V(H)} \bigcup_{v \in K(u)} \{e : u, v \in e\}) + (\bigcup_{u \in V(H)} \bigcup_{v \in K(u)} \{u, v\})$$

Let Δ', Δ'_2, and δ' denote the maximum 3-degree, maximum 2-degree, and maximum codegree of H', respectively. Note that H' is still triangle-free, $\chi_l(H) \leq \chi_l(H')$, $\delta' \leq \Delta^{6/10}$, and $\Delta' \leq \Delta$.

Suppose $\Delta'_2 \leq \sqrt{\Delta}\sqrt{c_0 \log \Delta}$. Since $\Delta' \leq \Delta$ and $\delta' \leq \Delta^{6/10}$, Theorem 35 implies

$$\chi_l(H) \leq \chi_l(H') \leq (\frac{\Delta}{c_0 \log \Delta})^{1/2}.$$

On the other hand, suppose $\Delta'_2 > \sqrt{\Delta}\sqrt{c_0 \log \Delta}$. Then, since

$$\Delta \geq d_H(u) \geq \frac{1}{2} \sum_{v \in N_H(u)} d_H(u,v) \geq \frac{1}{2} \sum_{\substack{v \in N_H(u) \\ d_H(u,v) \geq \Delta^{6/10}}} d_H(u,v) \geq |K(u)|\Delta^{6/10}/2,$$

we have

$$\Delta'_2 \leq \Delta_2 + 2\Delta^{4/10} < \Delta_2 + \Delta'_2/2.$$

Choose Δ'' so that $\Delta'_2 = \sqrt{\Delta''}\sqrt{c_0 \log \Delta''}$. Since $\Delta'_2 > \sqrt{\Delta}\sqrt{c_0 \log \Delta}$, $\Delta'' > \Delta$. Then the maximum 3-degree of H' is at most $\Delta < \Delta''$, the maximum 2-degree of H' is at most $\Delta'_2 \leq \sqrt{\Delta''}\sqrt{c_0 \log \Delta''}$, and the maximum codegree of H' is at most $\Delta^{6/10} < \Delta''^{6/10}$, so Theorem 35 implies

$$\chi_l(H) \leq \chi_l(H') \leq (\frac{\Delta''}{c_0 \log \Delta''})^{1/2} = \frac{\Delta'_2}{c_0 \log \Delta''} < \frac{\Delta'_2}{c_0 \log \Delta'_2} < \frac{2\Delta_2}{c_0 \log 2\Delta_2}.$$

CHAPTER 6

COLORING SPARSE HYPERGRAPHS

Recall that Johansson [2] showed that if G is a triangle-free graph with maximum degree Δ, then

$$\chi(G) = O(\frac{\Delta}{\log \Delta}). \tag{6.1}$$

This improved results of Catlin [20], Lawrence [21], Borodin and Kostochka [22], and Kim [28]. Random graphs show that the $\log \Delta$ factor in (6.1) is optimal. Recently, Frieze and Mubayi [4] generalized (6.1) to linear k-uniform hypergraphs, and in Chapter 5, we proved slightly stronger results for $k = 3$.

Theorem 41 ([4])**.** *Fix $k \geq 3$. If G is a k-uniform, linear hypergraph with maximum degree Δ, then*

$$\chi(G) = O(\frac{\Delta}{\log \Delta})^{\frac{1}{k-1}}.$$

Theorem 42. *If G is a 3-uniform, triangle-free hypergraph with maximum degree Δ, then*

$$\chi(G) = O((\frac{\Delta}{\log \Delta})^{1/2}).$$

Alon, Krivelevich, and Sudakov [29] extended (6.1) by showing that if every vertex $u \in V(G)$ is in at most Δ^2/f triangles, then $\chi(G) = O(\frac{\Delta}{\log f})$, where $\Delta \to \infty$. They used this to show that if G contains no copy of H, where H is a fixed graph such that $H - u$ is biparite for some

$u \in V(H)$, then $\chi(G) = O(\Delta/\log\Delta)$. In this chapter, we give similar improvements to the results of [4] and Chapter 5.

Given two hypergraphs H and A, a map $\phi : V(H) \to V(A)$ is an isomorphism if for all $E \subset V(H)$, $\phi(E) \in A$ if and only if $E \in H$. If there exists an isomorphism $\phi : V(H) \to V(A)$, we say H is isomorphic to A and denote this by $H \cong_\phi A$. Given a hypergraph H and $v \in V(H)$, let

$$\Delta_{H,v}(G) = \max_{u \in V(G)} |\{A \subset G : A \cong_\phi H \text{ and } \phi(u) = v\}|$$

and

$$\Delta_H(G) = \min_{v \in V(H)} \Delta_{H,v}(G).$$

For example, suppose G is a d-regular graph, and H is the path with edges xy and yz. Then $\Delta_{H,x}(G) = \Delta_{H,z}(G) = d(d-1)$, while $\Delta_{H,y}(G) = \binom{d}{2}$. Thus $\Delta_H(G) = \binom{d}{2}$. Another example appears after the statement of Theorem 45.

We improve the result of [54] with the following theorem.

Theorem 43. *Let G be a 3-uniform hypergraph with maximum degree Δ. Let $\mathcal{T}$ denote the set of 3-uniform triangles. If*

$$\Delta_H(G) \le \Delta^{(n(H)-1)/2}/f$$

for all $H \in \mathcal{T}$, then

$$\chi(G) = O((\frac{\Delta}{\log f})^{1/2}).$$

Notice that the hypotheses of Theorem 43 are satisfied when G is linear and $f = \Delta^{1/2}$, so Theorem 43 implies Theorem 41 for 3-uniform hypergraphs.

Given a rank k hypergraph G and $A \subset V(G)$, let

$$d_j(A) = |\{B \in G : |B| = j, A \subset B\}|$$

for each $1 \leq j \leq k$. When the hypergraph is k-uniform, we write $d(A)$ instead of $d_k(A)$. For $1 \leq l \leq j \leq k$, define the maximum (j,l)-degree of G, denoted $\Delta_{j,l}(G)$, to be $\max_{A \subset V(G):|A|=l} d_j(A)$ and the maximum j-degree of G to be $\Delta_{j,1}(G)$.

Consider the random greedy algorithm for forming an independent set I in a hypergraph G: at each step of the algorithm, a vertex v is chosen at random from the set of vertices $V(G) - I$ such that $I \cup \{v\}$ contains no edge of G. The algorithm terminates when no such v exists. Notice that when the vertex set of G consists of the edges of a complete graph, and the edges of G correspond to triangles in this graph, the random greedy independent set algorithm reduces to the triangle-free process (see [55], [48], [49]). Bennett and Bohman [56] have recently shown that this algorithm terminates with a large independent set I with high probability, unifying many of the previous results on H-free processes. Define the *b-codegree* of a pair of distinct vertices v, v' to be the number of edges $e, e' \in G$ such that $v \in e$, $v' \in e'$, and $|e \cap e'| = b$. Let $\Gamma_b(G)$ be the maximum b-codegree of G.

Theorem 44 (Bennett, Bohman [56]). *Let $k \geq 2$ and $\epsilon > 0$ be fixed. Let G be a k-uniform, D-regular hypergraph on N vertices such that $D > N^{\epsilon}$. If*

$$\Delta_{k,l}(G) < D^{\frac{k-l}{k-1}-\epsilon} \text{ for } l = 2, \ldots, k-1$$

and $\Gamma_{k-1}(G) < D^{1-\epsilon}$ then the random greedy independent set algorithm produces an independent set I in G with

$$|I| = \Omega(N(\frac{\log N}{D})^{\frac{1}{k-1}})$$

with probability $1 - \exp\{-N^{\Omega(1)}\}$.

Our next theorem improves and extends Bennett and Bohman's result on the independence number to chromatic number when $k \geq 3$. We also weaken the hypothesis by not requiring any condition on $\Gamma_{k-1}(G)$. Note that an important aspect of [56] is that the random greedy procedure results in a large independent set. We do not make any such claims in our result below.

Theorem 45. *Fix $k \geq 3$. Let G be a k-uniform hypergraph with maximum degree Δ. If*

$$\Delta_{k,l}(G) \leq \Delta^{\frac{k-l}{k-1}}/f \text{ for } l = 2, \ldots, k-1,$$

then

$$\chi(G) = O((\frac{\Delta}{\log f})^{\frac{1}{k-1}}).$$

A construction similar to the one in [29] shows that this bound is tight (up to a constant factor) for all f. The following example shows that Theorem 43 implies Theorem 45 and also motivates our definition of Δ_H: Suppose G is a 3-uniform hypergraph with maximum 3-degree Δ and maximum $(3,2)$-degree at most $\Delta^{1/2}/f$. Recall that $F_5 = \{abc, abd, ced\}$. Notice $\Delta_{F_5,e} \leq \Delta^2$, while $\Delta_{F_5,a} \leq \Delta^2/f^2$; thus $\Delta_{F_5} \leq \Delta^2/f^2$. We also have $\Delta_{C_3} \leq \Delta^{5/2}/f$ and $\Delta_{K_4^-} \leq 4\Delta^{3/2}/f$. Theorem 43 therefore implies $\chi(G) = O((\Delta/\log f)^{1/2})$.

Theorem 43 and Theorem 45 both follow from a general partitioning lemma, which is a generalization of the main result of [29] to hypergraphs.

Definition. Let G be a rank k hypergraph. G is $(\Delta, \omega_2, \ldots, \omega_k)$-*sparse* if G has maximum k-degree at most Δ, and for all $1 \leq l < j \leq k$, G has maximum (j,l)-degree at most $\Delta^{\frac{j-l}{k-1}}\omega_j$.

Recall that a hypergraph H is *connected* if for all $u, v \in V(H)$, there exists a sequence of edges $e_1, \ldots, e_n \in H$ such that $u \in e_1$, $v \in e_n$, and $e_i \cap e_{i+1} \neq \emptyset$ for $1 \leq i < n$.

Lemma 46. *Fix $k \geq 2$. Let G be a rank k hypergraph, and let $\mathcal{H}$ be a finite family of fixed, connected hypergraphs. Let $f = \Delta^{O(1)}$, where f is sufficiently large. Suppose that*

- *G is $(\Delta, \omega_2, \ldots, \omega_k)$-sparse, where $\omega_j = \omega_j(\Delta) = f^{o(1)}$ for all $2 \leq j \leq k$.*
- *For all $H \in \mathcal{H}$, $\Delta_H(G) \leq \Delta^{\frac{n(H)-1}{k-1}}/f^{n(H)}$.*

Then $V(G)$ can be partitioned into $O(\Delta^{\frac{1}{k-1}}/f)$ parts such that the hypergraph induced by each part is $\mathcal{H}$-free and has maximum j-degree at most $2^{2k}f^{j-1}\omega_j$, for each $1 \leq j \leq k$.

Our main new contribution is a technical lemma (Lemma 50 in Section 6.2) that allows us to prove a large deviation inequality for positive random variables with mean less than 1. This may be of independent interest.

6.0.1 Organization

In Section 6.1, we use Lemma 46 to prove Theorems 43 and 45. The remaining sections are devoted to Lemma 46. In Section 6.2, we prove our main probabilistic lemma, which is used in Section 6.3. Section 6.3 contains the proof of Lemma 46.

6.0.2 Notation

Given a hypergraph G and $A \subset V(G)$, we define the $(j - |A|)$-uniform hypergraph

$$L_{G,j}(A) = \{B - A : B \in G \text{ with } A \subset B \text{ and } |B| = j\}.$$

Let $N_{G,j}(A) = V(L_{G,j}(A))$. When G is k-uniform, we drop the subscript j. When G is clear from the context, we drop the subscript G.

6.1 Proof of Theorem 43 and 45

We first prove a simple bound on the chromatic number of general rank k hypergraphs. The proof is based on a proof of Erdős and Lovász [17] for k-uniform hypergraphs. This bound will allow us to assume that f is sufficiently large in the proofs of Theorems 43 and 45.

6.1.1 Chromatic number of rank k hypergraphs

Lemma 47. *Fix $k \geq 2$. Let G be a rank k hypergraph with maximum j-degree Δ_j, $j = 2, \ldots, k$. Then*

$$\chi(G) \leq \max_{j=2}^{k}(4k(k-1)\Delta_j)^{1/(j-1)}.$$

Proof. Set

$$r = \max_{j=2}^{k}(4k(k-1)\Delta_j)^{1/(j-1)}.$$

Then $\Delta_j \leq \frac{r^{j-1}}{4k(k-1)}$ for each $j = 2, \ldots, k$. Assign each vertex $u \in V(G)$ a color chosen uniformy at random from $[r]$. For each edge $e \in G$, let B_e denote the event that each vertex in e received the same color. Then $\Pr[B_e] = r^{1-|e|}$. The event B_e depends only on events B_f such that $B_e \cap B_f \neq \emptyset$. Since

$$\sum_{f \in H: e \cap f \neq \emptyset} \Pr[B_f] = \sum_{j=2}^{k} \sum_{\substack{f \in H: e \cap f \neq \emptyset, \\ |f| = j}} \Pr[B_f] \leq \sum_{j=2}^{k} |e| \Delta_j r^{1-j} \leq 1/4,$$

the Asymmetric Local Lemma implies that there exists a coloring where no event B_e occurs. □

6.1.2 Proof of Theorem 43

We will use the following theorem from Chapter 5.

Theorem 48. *Suppose G is a rank 3, triangle-free hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. Then*

$$\chi(G) = O(\max\{(\frac{\Delta}{\log \Delta})^{1/2}, \frac{\Delta_2}{\log \Delta_2}\}).$$

In fact, we will prove the following stronger result.

Theorem 49. *Let G be a rank 3 hypergraph with maximum 3-degree at most Δ and maximum 2-degree at most Δ_2. Let $\mathcal{H}$ denote the family of rank 3 triangles. If*

$$\Delta_H(G) \leq \max\{\Delta^{1/2}, \Delta_2\}^{n(H)-1}/f$$

for all $H \in \mathcal{H}$, then

$$\chi(G) = O(\max\{\frac{\Delta}{\log f})^{1/2}, \frac{\Delta_2}{\log f}\}).$$

Proof. By Lemma 47, we may assume that f is sufficiently large. We consider two cases.

Case 1: $\Delta_2 \leq (\Delta \log f)^{1/2}$.

Set $g = \frac{f}{\log^{5/2} f}$. Since $\Delta_2 \leq (\Delta \log f)^{1/2}$ and $n(H) \leq 6$ for all $H \in \mathcal{H}$,

$$\Delta_H(G) \leq \max\{\Delta^{1/2}, \Delta_2\}^{n(H)-1}/f \leq (\Delta \log f)^{\frac{n(H)-1}{2}}/f \leq \Delta^{\frac{n(H)-1}{2}}/g \tag{6.2}$$

for each $H \in \mathcal{H}$.

Let

$$K = \{A \subset V(G) : |A| = 2 \text{ and } d_3(A) \geq \Delta^{1/2}\}.$$

Define the rank 3 hypergraph

$$G' = G - \{E \in G : A \subset E \text{ for some } A \in K\} \cup K.$$

In other words, if a pair of vertices has high codegree, then we replace all 3-edges containing the pair with a 2-edge between the pair. Hence any proper coloring of G' is also a proper coloring of G.

We will now apply Lemma 46 to G' with the function g. Set $\omega_2 = \log^{1/2} g$ and $\omega_3 = 1$. We first check that G' is $(2\Delta, \omega_2, \omega_3)$-sparse. G' clearly has maximum 3-degree at most 2Δ. Also, by definition of K, G' has maximum $(3,2)$-degree at most $\Delta^{1/2} < (2\Delta)^{1/2}\omega_3$. Let $u \in V(G)$. Since $|L_{G,3}(u)| \leq \Delta$, u is in at most $2\Delta^{1/2}$ sets in K. Thus, G' has maximum $(2,1)$-degree at most

$$(\Delta \log f)^{1/2} + 2\Delta^{1/2} < (2\Delta \log g)^{1/2} = (2\Delta)^{1/2}\omega_2,$$

which shows that G' is $(2\Delta, \omega_2, \omega_3)$-sparse. Note also that $\omega_j = f^{o(1)}$ for $j = 2, 3$.

Let $H' \in \mathcal{H}$. Suppose $T' \subset G'$ is a copy of H' which is not in G. Since each $e \in T' - G$ corresponds to at least $\Delta^{1/2}$ edges of size 3 in G, T' corresponds to at least $(\Delta^{1/2} - 5)^{|T'-G|}$ copies of H in G, where H is the triangle obtained by replacing each $e \in T' - G$ with a distinct size 3 edge containing e and some vertex outside of T'. Thus

$$\Delta_{H'}(G')(\Delta^{1/2} - 5)^{n(H)-n(H')} \leq \Delta_H(G).$$

Using (6.2), this implies

$$\Delta_{H'}(G') \leq 2\Delta_H(G)\Delta^{(n(H')-n(H))/2} \leq 2\Delta^{(n(H')-1)/2}/g \leq 2\Delta^{(n(H')-1)/2}/(g^{1/6})^{n(H')}.$$

Setting $g' = g^{1/6}$, Lemma 46 therefore implies that there exists a partition of $V(G') = V(G)$ into $O(\Delta^{1/2}/g')$ parts such that each part is $\mathcal{H}$-free and has maximum j-degree at most $O(g'^{j-1}\omega_j)$. By Theorem 42, we may properly color each part with

$$O(\max\{\frac{g'}{\log^{1/2} g'}, \frac{g'\omega_2}{\log(g'\omega_2)}\}) = O(\frac{g'}{\log^{1/2} g'})$$

different colors, resulting in a total of $O((\frac{\Delta}{\log g})^{1/2}) = O((\frac{\Delta}{\log f})^{1/2})$ colors.

Case 2: $\Delta_2 > (\Delta \log f)^{1/2}$.

Set $\Delta' = \frac{\Delta_2^2}{\log f}$. Then $\Delta < \Delta'$. Thus G has maximum 3-degree at most Δ', and

$$\Delta_H(G) \leq \max\{\Delta^{1/2}, \Delta_2\}^{n(H)-1}/f < \max\{\Delta'^{1/2}, \Delta_2\}^{n(H)-1}/f.$$

for all $H \in \mathcal{H}$. We may therefore apply case 1 with Δ' in the role of Δ to obtain a coloring with at most $O((\frac{\Delta'}{\log f})^{1/2}) = O(\frac{\Delta_2}{\log f})$ colors. □

6.1.3 Proof of Theorem 45

Let G be a k-uniform hypergraph with maximum degree Δ such that $\Delta_{k,l}(G) \leq \Delta^{\frac{k-l}{k-1}}/f$ for $l = 2, \ldots, k-1$. As in the proof of Theorem 48, we may assume that f is sufficiently large. Set $\omega_j = 1$ for $2 \leq j \leq k$. Observe that G is $(\Delta, \omega_2, \ldots, \omega_k)$-sparse. For each $l = 2, \ldots, k-1$, let H_l be the k-uniform hypergraph consisting of two edges which share exactly l vertices. Set $\mathcal{H} = \{H_2, \ldots, H_{k-1}\}$ and $f' = f^{1/(2k-2)}$. Then for $a \in V(H_l)$ with $d_{H_l}(a) = 2$,

$$\Delta_{H_l} \leq \Delta_{H_l,a} \leq \Delta \binom{k-1}{l-1} \Delta_{k,l} \leq \binom{k-1}{l-1} \Delta^{\frac{2k-1-l}{k-1}}/f \leq \binom{k-1}{l-1} \Delta^{\frac{2k-1-l}{k-1}}/f'^{n(H_l)},$$

so Lemma 46 implies that there exists a partition of $V(G)$ into $O(\Delta^{\frac{1}{k-1}}/f')$ parts such that each part is $\mathcal{H}$-free and has maximum degree at most $O(f'^{k-1})$. Since each part is $\mathcal{H}$-free, each part is linear and can be colored with $O(\frac{f'}{\log^{1/(k-1)} f'})$ colors by Theorem 41. Using a different set of colors for each part gives a proper coloring of G with $O((\frac{\Delta}{\log f})^{\frac{1}{k-1}})$ colors.

6.2 Deviation inequality

To motivate our next lemma, which is the main novel ingredient in this work, we outline the proof of Lemma 46, which appears in Section 6.3. We are given a hypergraph G and a fixed hypergraph H, and we know that each $u \in V(G)$ is in very few copies of H. Our goal is to produce a coloring of G such that the subhypergraph induced by each color is H-free. We randomly color the vertices of G, hoping to remove all copies of H in the induced subgraph of each color. Consider some $u \in V(G)$. Since u is in few copies of H, the expected number of these which remain in u's color is much less than 1. We would like to conclude that with very low probability, u is in more than c copies of H in the subgraph induced by u's color, for some constant c. We could then use the Local Lemma to find a coloring such that u is in at most c copies of H in the subgraph induced by u's color, which would bring us close to our goal of removing all copies of H from the subgraph. To draw this conclusion, we could try to let F_u be the set of copies of H which contain u, and let F'_u be the set of copies of H which contain u and whose vertices all receive u's color. Then we could apply Theorem 12 to bound F'_u and make the above conclusion. However, in our case, "very low probability" requires us to set $\lambda = \Omega(\log |V(F_u)|)$. Since M_0 and M_1 are always at least 1 (the empty product is taken to be 1), this only allows us to conclude that u is in $\Omega(\log |V(F_u)|)$ copies of H. Since $|V(F_u)|$ will be tending to ∞, this is not small enough.

Frieze and Mubayi [4] overcame this by using a two-step random process. However, their proof used the linearity of G and the assumption that H is a triangle, and we were not able to easily duplicate their method. Our approach, which can be viewed as a generalization of the method in [29], is to instead bound the *transversal number* of the hypergraph F'_u. A *transversal* of F_u is a subset $K \subset V(F_u)$ such that $A \cap K \neq \emptyset$ for all $A \in F_u$. In other words, the subhypergraph of F_u induced by $V(F_u) - K$ has no edges. The *transversal number* of F_u, denoted $\tau(F_u)$, is the minimum size of a transversal of F_u.

Returning to our outline, we will use Lemma 50 below to show that the probability that $\tau(F'_u)$ is large is very low.

Lemma 50. *Suppose F is an s-uniform hypergraph, and z_i, $i \in V(F)$ are independent random indicator variables with* $\Pr[z_i = 1] = p$*, for all $i \in V(F)$. Let*

$$F' = \{A \in F : \forall i \in A, z_i = 1\}.$$

Suppose there exists $\alpha > 0$ *such that* $|F|p^{(1-\alpha)s} < 1$*. Then for any* $c \geq e2^s s\alpha$*,*

$$\Pr[\tau(F') > s^2(c/\alpha)^{s+1}] \leq s^2|V(F)|^{s-1}p^c.$$

We then repeat this for all $v \in V(G)$ in the same color class as u. Let K_v be the set of vertices in a transversal of size at most $\tau(F'_v)$. Following [29], we create a 2-graph on these vertices, with an edge from v to each vertex in K_v. Since $\tau(F'_v)$ is bounded by a constant, this graph has constant out-degree and can be properly colored with a constant number of colors. Since the neighborhood of v in this graph is a transversal and all of the vertices in the neighborhood of v received a different color than v, none of the edges of F_v (which correspond to copies of H containing v) could survive in v's new color class. Thus the subgraph induced by each new color contains no copies of H.

6.2.1 Proof of Lemma 50

We will need the following simple proposition to prove Lemma 50. It is a straightforward generalization of the well-known fact that a graph with many edges has either a large matching or a large star. Let F be a k-uniform hypergraph. Recall that $M \subset F$ is a *matching* if $A \cap B = \emptyset$ for any $A, B \in M$ with $A \neq B$. When F is k-uniform, recall that for all $A \subset V(F)$,

$$L_F(A) = \{B - A : B \in F \text{ with } A \subset B\}.$$

Proposition 51. *If* F *is a* k*-uniform hypergraph, then there exists* $A \subset V(F)$ *such that* $L_F(A)$ *contains a matching of size at least* $\frac{|F|^{1/k}}{k-|A|}$.

Proof. We induct on k. If $k = 1$, the claim holds with $A = \emptyset$. Assume the result for k, and let F be a $k+1$-uniform hypergraph with maximum degree Δ. By the greedy coloring algorithm, F can be partitioned into $k\Delta$ matchings, so F contains a matching with $\frac{|F|}{k\Delta}$ edges. Thus $\Delta > |F|^{k/(k+1)}$ or F contains a matching with at least $\frac{|F|}{k|F|^{k/(k+1)}} = \frac{|F|^{1/(k+1)}}{k}$ edges. In the second case, we are done (with $A = \emptyset$), so assume there exists $u \in V(F)$ with $d(u) > |F|^{k/(k+1)}$. Set $A = \{u\}$ and consider the k-uniform hypergraph $L_F(A)$. By induction, there exists $B \subset V(L_F(A))$ such that $L_{L(A)}(B)$ contains a matching of size at least

$$\frac{|L_F(A)|^{1/k}}{(k - |B|)} = \frac{|L_F(A)|^{1/k}}{k + 1 - |A \cup B|} > \frac{(|F|^{k/(k+1)})^{1/k}}{k + 1 - |A \cup B|} = \frac{|F|^{1/(k+1)}}{k + 1 - |A \cup B|}.$$

Since $B \subset V(L_F(A)) \subset V(F)$ and $L_{L(A)}(B) = L_F(A \cup B)$, this completes the proof. □

Proof of Lemma 50. For $k = 0, 1, \ldots, s$, set $\tau_k = |F|p^{(1-\alpha)k}$. For $k = 1, \ldots, s$, let

$$H_k = \{A \subset V(F) : |A| = k, d(A) > \tau_k, \text{ and } \forall B \subsetneq A, d(B) \le \tau_{|B|}\}.$$

Fix k, where $1 \le k \le s$. Let $B \in \binom{V(F)}{b}$, where $b < k$. Suppose there exists $A \in H_k$ with $B \subsetneq A$. By definition of H_k, $d(B) \le \tau_b$. Since each such A corresponds to at least τ_k sets in F, and each of these τ_k sets is counted at most $\binom{s}{k}$ times under this correspondence, this implies

$$\tau_k |\{A \in H_k : B \in \binom{A}{b}\}| \le \binom{s}{k} d(B) < 2^s \tau_b. \tag{6.3}$$

Consider the k-uniform hypergraph $Z_k = \{A \in H_k : z_i = 1 \ \forall i \in A\}$. Suppose $|Z_k| \geq (c/\alpha)^k$. Then by Proposition 51, there exists $B \in \binom{V(Z_k)}{b}$, where $b < k$, such that the $(k-b)$-uniform hypergraph $L_{Z_k}(B)$ contains a matching of size at least $x_b := \frac{c/\alpha}{(k-b)}$. Each edge in this matching corresponds to an edge $A \in H_k$ with $B \subset A$. By (6.3), there are at most $2^s\tau_b/\tau_k$ edges. Thus the probability that such a B exists is at most

$$\begin{aligned}
\sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} \binom{2^s\tau_b/\tau_k}{x_b} p^{(k-b)x_b} &\leq \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} (\frac{e2^s\tau_b}{\tau_k x_b})^{x_b} p^{(k-b)x_b} \\
&< \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} (\frac{\tau_b p^{k-b}}{\tau_k})^{x_b} \\
&= \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} p^{((1-\alpha)b+k-b-(1-\alpha)k)x_b} \\
&= \sum_{b=1}^{k-1} \sum_{B \in \binom{V(H_k)}{b}} p^{\alpha(c/\alpha)} \\
&< k|V(F)|^{k-1}p^c.
\end{aligned}$$

Therefore, for each $k = 1, \ldots, s$,

$$\Pr[|Z_k| > (c/\alpha)^k] < k|V(F)|^{k-1}p^c.$$

Hence

$$\begin{aligned}\Pr[|\bigcup_{k=1}^{s} Z_k| \geq s(c/\alpha)^s]] &< \Pr[|\bigcup_{k=1}^{s} Z_k| \geq \sum_{k=1}^{s}(c/\alpha)^k] \\ &< \sum_{k=1}^{s} k|V(F)|^{k-1}p^c \\ &< s^2|V(F)|^{s-1}p^c.\end{aligned}$$

Since each edge in $\bigcup_{k=1}^{s} Z_k$ contains at most s vertices, this implies

$$\Pr[|\bigcup_{k=1}^{s} V(Z_k)| \geq s^2(c/\alpha)^s] < s^2|V(F)|^{s-1}p^c.$$

We now claim that $\bigcup_{k=1}^{s} V(Z_k)$ is a transversal of F'. Suppose $A \in F$ and $z_i = 1$ for all $i \in A$. Since $d(A) = 1 > |F|p^{(1-\alpha)s} = \tau_s$, $A \in H_s$ or $d(B) > \tau_b$ for some $B \in \binom{A}{b}$. If $A \in H_s$, then $A \in Z_s$ and $A \subset V(Z_s)$, so assume the second case. Choose a minimal set $B \in \binom{A}{b}$ with $d(B) > \tau_b$. Then $B \in H_b$, so $B \in Z_b$ and hence $A \cap V(Z_b) \neq \emptyset$. □

6.3 Proof of Lemma 46

We break the proof of Lemma 46 into two steps. In step 1, we prove Lemma 52, which is a slight variant of Lemma 46 when $f \geq \Delta^\epsilon$. In Section 6.3.2, we will use the same argument as in [29] to show that the proof of Lemma 46 can be reduced to Lemma 52.

6.3.1 Step 1: f large

Lemma 52. *Fix $k \geq 2$ and $\epsilon \in (0, \frac{1}{k-1})$. Let G be a rank k hypergraph, and let $\mathcal{H}$ be a finite family of fixed hypergraphs. Suppose that*

- *G is $(\Delta, \omega_2, \ldots, \omega_k)$-sparse, where $\omega_j = \omega_j(\Delta) = \Delta^{o(1)}$ for all $2 \leq j \leq k$.*
- *$\Delta_H(G) \leq \Delta^{\frac{n(H)-1}{k-1} - n(H)\epsilon}$ for all $H \in \mathcal{H}$.*

Then $V(G)$ can be partitioned into $O(\Delta^{\frac{1}{k-1}-\epsilon})$ parts such that the hypergraph induced by each part is $\mathcal{H}$-free and has maximum j-degree at most $2\Delta^{(j-1)\epsilon}\omega_j$, for each $j \in [k]$.

Proof. Let $N = \max_{H \in \mathcal{H}} n(H)$. Color the vertices of G uniformly at random with $r = \Delta^{\frac{1}{k-1}-\epsilon}$ colors. Fix u, and for each $v \in V(G)$, let z_v be a random indicator variable which is 1 if v receives the same color as u and 0 otherwise. Note that $\Pr[z_v = 1] = 1/r$. For each $H \in \mathcal{H}$, choose $v_H \in V(H)$ such that $\Delta_{H,v_H}(G) = \Delta_H(G)$. Define a $(n(H) - 1)$-uniform hypergraph

$$T_H(u) = \{V(A) - u : A \subset G \text{ with } A \cong_\phi H \text{ and } \phi(u) = v_H\}.$$

Since to each $A \subset G$ with $A \cong_\phi H$ and $\phi(u) = v_H$ we may associate the set $V(A) - u \in T_H(u)$,

$$v(T_H(u)) \leq (N-1)|T_H(u)| \leq (N-1)\Delta_H(G).$$

Also, let

$$T'_H(u) = \{A \in T_H(u) : \forall v \in A, z_v = 1\}.$$

We define $k + |\mathcal{H}|$ bad events for each u.

- $A_{u,j}$: For $1 \leq j \leq k$, $A_{u,j}$ denotes the event

$$Z_{u,j} := \sum_{A \in L_{H,j}(u)} \prod_{i \in A} z_i \geq 2\Delta^{(j-1)\epsilon}\omega_j.$$

- $B_{u,H}$: For each $H \in \mathcal{H}$, $B_{u,H}$ denotes the event $\tau(T'_H(u)) > (n(H)-1)^2(c/\alpha_H)^{n(H)}$, where

$$\alpha_H = 1 - \frac{n(H)\epsilon - \frac{n(H)-1}{k-1}}{(\epsilon - \frac{1}{k-1})(n(H)-1)} = \frac{\epsilon}{(\frac{1}{k-1} - \epsilon)(n(H)-1)} > 0$$

and

$$c = \frac{(N^2+3N)2^N}{\frac{1}{k-1} - \epsilon} > 3N2^N.$$

To bound the probability of $A_{u,j}$, we apply Theorem 12 to the $(j-1)$-uniform hypergraph $L_{H,j}(u)$. Let $A \subset V(L_{H,j}(u))$ with $|A| \leq j-1$. Then

$$\begin{aligned} M_A = d_j(A \cup \{u\})/r^{j-|A|-1} &\leq \Delta_{j,|A|+1}(G)/r^{j-|A|-1} \\ &\leq \Delta^{\frac{j-|A|-1}{k-1}}\omega_j/r^{j-|A|-1} \\ &= \Delta^{(j-|A|-1)\epsilon}\omega_j. \end{aligned}$$

Therefore $\mathbf{E}[Z_u] = M_0 \leq \Delta^{(j-1)\epsilon}\omega_j$ and $M_1 \leq \Delta^{(j-2)\epsilon}\omega_j$. Setting $\lambda = (k+3N+2)\log\Delta$, we obtain constants a and b such that

$$\begin{aligned}\Pr[A_{u,j}] = \Pr[Z_{u,j} \geq 2\Delta^{(j-1)\epsilon}\omega_j] &< \Pr[Z_{u,j} \geq \Delta^{(j-1)\epsilon}\omega_j + a\Delta^{\epsilon(j-1)-\epsilon/2}\lambda^{j-1}] \\ &\leq b(j\Delta)^{j-2}\Delta^{-(j+3N+2)} \\ &< \Delta^{-3N}.\end{aligned}$$

To bound $B_{u,H}$, we apply Lemma 50 with $F = T_H(u)$, $F' = T'_H(u)$, and $p = 1/r$. Since

$$|T_H(u)|p^{(1-\alpha_H)(n(H)-1)} \leq \Delta^{\frac{n(H)-1}{k-1} - n(H)\epsilon} p^{(1-\alpha_H)(n(H)-1)} = 1,$$

$\alpha_H \in (0,1)$, and $c > 3N2^N > e2^N N\alpha_H > e2^{n(H)-1}(n(H)-1)\alpha_H$, Lemma 50 implies

$$\begin{aligned}\Pr[B_{u,H}] &\leq (n(H)-1)^2 v(T_H(u))^{n(H)-2} p^c \\ &= (n(H)-1)^2 v(T_H(u))^{n(H)-2} \Delta^{(-N^2-3N)2^N} \\ &< (n(H)-1)^2 v(T_H(u))^{n(H)-2} \Delta^{-N^2-3N} \\ &< N^2 (N\Delta^{\frac{N-1}{k-1}})^{N-1} \Delta^{-N^2-3N} \\ &< \Delta^{-3N}.\end{aligned}$$

Each event $A_{u,j}$ is determined by the colors assigned to the vertices in $N_j(u) \cup \{u\}$. Each event $B_{u,H}$ is determined by the colors assigned to the vertices in $T_H(u) \cup \{u\}$. Thus $A_{u,j}$ depends on:

- Events of the form $A_{v,i}$ where $N_j(u) \cap N_i(v) \neq \emptyset$. There are at most

$$(j\Delta^{\frac{j-1}{k-1}}\omega_j)(\sum_{l=1}^{k} l\Delta^{\frac{l-1}{k-1}}\omega_l) < (j\Delta)(k^2\Delta) < \Delta^N$$

  such events.

- Events of the form $B_{v,H}$, where $N_j(u) \cap V(T_H(v)) \neq \emptyset$. Since $|N_j(u)| \leq j\Delta$ and $|V(T(v,H))| \leq N\Delta^{\frac{N-1}{k-1}}$, there are at most $(j\Delta)|\mathcal{H}|\Delta^{\frac{N-1}{k-1}} \leq \Delta^N$ such events.

Also, $B_{u,H}$ depends on:

- Events of the form $A_{v,j}$ where $V(T_H(u)) \cap N_j(v) \neq \emptyset$. There at most $(k^2\Delta)N\Delta^{\frac{N-1}{k-1}} \leq \Delta^N$ such events.

- Events of the form $B_{v,H'}$, where $V(T_H(u)) \cap V(T_{H'}(v)) \neq \emptyset$. There are at most $|\mathcal{H}|N^2\Delta^{2\frac{N-1}{k-1}} \leq \Delta^{2N}$ such events.

Since the probability of each event is at most Δ^{-3N}, the Local Lemma implies that there exists a coloring of $V(G)$ with r colors so that none of the events $A_{u,j}$ or $B_{u,H}$ occur.

Fix a color, and consider the subhypergraph G' induced by the vertices which received that color. For each $u \in V(G')$ and each $H \in \mathcal{H}$, let $K(u,H)$ be a minimum sized transversal of $T'(u,H)$. Create a simple graph W on $V(G')$ with edge set $\{\{u,v\} : u \in V(G'), v \in K(u,H) \text{ for some } H \in \mathcal{H}\}$. Consider any subgraph of W on n' vertices. Since no event $B_{u,H}$ occurs, the number of edges in this subgraph is at most $n'|\mathcal{H}|(N-1)^2(c/\alpha)^N$; it therefore contains a vertex with degree at most $2|\mathcal{H}|(N-1)^2(c/\alpha)^N$. W is therefore $2|\mathcal{H}|(N-1)^2(c/\alpha)^N$-degenerate and can thus be properly colored with $2|\mathcal{H}|(N-1)^2(c/\alpha)^N + 1$ new colors. Since each of the $K(u,H)$ is a transversal, the subhypergraph induced by each of these new colors is $\mathcal{H}$-free. Repeating this for each of the original r colors results in a partition of F into at most

$$r(2|\mathcal{H}|(N-1)^2(c/\alpha)^N+1) = O(\Delta^{\frac{1}{k-1}-\epsilon})$$

parts, where each part is $\mathcal{H}$-free and has maximum j-degree at most $2\Delta^{(j-1)\epsilon}\omega_j$. □

6.3.2 Step 2: f small

When $f < \Delta^\epsilon$, we use the same random halving argument as in [29]. We recursively divide the hypergraph into two parts until we obtain a set of hypergraphs, each with maximum degree small enough to apply Lemma 52. The halving step is accomplished with Lemma 54, while Proposition 55 is used to analyze the recursion.

Proposition 53. *Let G be a rank k, $(\Delta, 2\omega, \ldots, 2\omega)$-sparse hypergraph. Then for any $A \subset V(G)$ and $h \geq |A|$, G contains at most $2^{(h+1)T}\omega^T\Delta^{\frac{h-|A|}{k-1}}$ connected hypergraphs on h vertices which contain A, where $T = 2^h$.*

Proof. Starting with any edge that intersects A, we try to greedily grow a connected subgraph which contains A and $h-|A|$ other vertices. At step i, we add an edge of size j_i which contains $l_i \geq 1$ vertices already in the subgraph and $a_i \geq 0$ vertices in A which are not already in the subgraph. There are at most $2^h\Delta_{j_i,l_i+a_i}$ choices for this edge. Since one edge is added at every step, this process terminates after at most $T = 2^h$ steps, resulting in a total of $2^{hT}\prod_{i=1}^{T}\Delta_{j_i,l_i+a_i}$

subgraphs containing A. Since at each step we add $j_i - (l_i + a_i)$ vertices outside of A to the subgraph, $\sum_{i=1}^{T} j_i - (l_i + a_i) = h - |A|$. Thus

$$2^{hT} \prod_{i=1}^{T} \Delta_{j_i, l_i + a_i} \leq 2^{(h+1)T} \omega^T \prod_{i=1}^{T} \Delta^{\frac{j_i - (l_i + a_i)}{k-1}} = 2^{(h+1)T} \omega^T \Delta^{\frac{h-|A|}{k-1}}.$$

□

Lemma 54. *Let G be a rank k hypergraph, and let $\mathcal{H}$ be a finite family of fixed, connected hypergraphs. Suppose that G is $(\Delta, 2\omega_2, \ldots, 2\omega_k)$-sparse, where $\omega_j = \omega_j(\Delta) = \Delta^{o(1)}$ for all $2 \leq j \leq k$. Then for Δ sufficiently large, there exists a partition of $V(G)$ into two subhypergraphs G_1 and G_2 such that*

- *For each $i = 1, 2$ and $1 \leq l < j \leq k$, we have $\Delta_{j,l}(G_i) \leq \Delta_{j,l}(G)/2^{j-l} + \Delta^{\frac{j-l}{k-1} - \frac{1}{2k}}$*
- *For each $i = 1, 2$ and $H \in \mathcal{H}$, we have $\Delta_H(G_i) \leq \Delta_H(G)/2^{n(H)-1} + \Delta^{\frac{n(H)-1}{k-1} - \frac{1}{2k}}$.*

Proof. Let $N = \max_{H \in \mathcal{H}} n(H)$. Color the vertices of G uniformly at random with the colors 1 and 2. For each $A \subset V(G)$, let $d'_j(A)$ denote the j-degree of A in the subhypergraph induced by the minimum color of a vertex in A.

For each $H \in \mathcal{H}$, choose $v_H \in V(H)$ such that $\Delta_{H,v_H}(G) = \Delta_H(G)$. For each $u \in V(G)$, define a $(n(H) - 1)$-uniform hypergraph

$$T_H(u) = \{V(A) - u : A \subset G \text{ with } A \cong_\phi H \text{ and } \phi(u) = v_H\}.$$

Also, for each $H \in \mathcal{H}$, let

$$T'_H(u) = \{A \in T_H(u) : \forall v \in A, v \text{ receives the same color as } u\}.$$

Define the following bad events:

- $C_{A,j}$: For each $A \subset V(G)$ with $d_j(A) > 0$, $C_{A,j}$ denotes the event

$$d'_j(A) > \Delta_{j,|A|}(G)/2^{j-|A|} + \Delta^{\frac{j-|A|}{k-1} - \frac{1}{2k}}.$$

- $B_{u,H}$: For each $H \in \mathcal{H}$, $B_{u,H}$ denotes the event

$$|T'_H(u)| > \Delta_H(G)/2^{n(H)-1} + \Delta^{\frac{n(H)-1}{k-1} - \frac{1}{2k}}.$$

We use Theorem 10 to bound the probability of each event. The random variable $d'_j(A)$ is determined by the colors of the vertices in $N_j(A)$. If $v \in N_j(A)$, changing v's color affects $d'_j(A)$ by at most $d_j(A \cup \{v\}) \leq \Delta_{j,|A|+1}$. Also,

$$\begin{aligned}\sum_{v \in N_j(A)} d_j(A \cup \{v\})^2 \leq \Delta_{j,|A|+1}(G) \sum_{v \in N_j(A)} d_j(A \cup \{v\}) &< \Delta_{j,|A|+1}(G) j d_j(A) \\ &\leq j \Delta_{j,|A|+1}(G) \Delta_{j,|A|}(G) \\ &\leq 4j \Delta^{\frac{2j-2|A|-1}{k-1}} \omega_j^2.\end{aligned}$$

Consequently, Theorem 10 and $\omega_j(\Delta) = \Delta^{o(1)}$ imply

$$\begin{aligned}\Pr[C_{A,j}] &< \Pr[d_j'(A) > \Delta_{j,|A|}(G)/2^{j-|A|} + (4j\Delta^{\frac{2j-2|A|-1}{k-1}}\omega_j^2 2N\log\Delta)^{1/2}] \\ &\le 2e^{-4N\log\Delta} \\ &< \Delta^{-3N}.\end{aligned}$$

Let $T_H(u,v)$ denote the set of copies of H in G containing both u and v. The random variable $|T_H'(u)|$ is determined by the colors of the vertices in $V(T_H(u))$. Changing the color of $v \in V(T_H(u))$ affects $|T_H'(u)|$ by at most $|T_H(u,v)|$, which, by Proposition 53 (with $|A| = 2$), is at most $\omega\Delta^{\frac{n(H)-2}{k-1}}$, where $\omega = 2^{(n(H)+1)2^{n(H)}} \max_{j=1}^{k} \omega_j^{2^{n(H)}} = \Delta^{o(1)}$. Note also that Proposition 53 (with $|A| = 1$) implies $|T_H(u)| < \omega\Delta^{\frac{n(H)-1}{k-1}}$. Since

$$\begin{aligned}\sum_{v\in V(T(u,H))} |T_H(u,v)|^2 &\le \omega\Delta^{\frac{n(H)-2}{k-1}} \sum_{v\in V(T(u,H))} |T_H(u,v)| \\ &\le \omega\Delta^{\frac{n(H)-2}{k-1}} n(H)|T_H(u)| \\ &\le n(H)\Delta^{\frac{n(H)-2}{k-1}+\frac{n(H)-1}{k-1}}\omega^2 \\ &= n(H)\Delta^{\frac{2n(H)-3}{k-1}}\omega^2,\end{aligned}$$

$$\begin{aligned}\Pr[B_{u,H}] &< \Pr[T_H'(u) > \Delta_H/2^{n(H)-1} + (n(H)\Delta^{\frac{2n(H)-3}{k-1}}\omega^2 2N\log\Delta)^{1/2}] \\ &\le 2e^{-4N\log\Delta} \\ &< \Delta^{-3N}.\end{aligned}$$

The event $C_{A,j}$ is determined by the colors of the vertices in $N_j(A) \cup A$. The event $B_{u,H}$ is determined by the colors of the vertices in $T_H(u) \cup \{u\}$. Thus $C_{A,j}$ depends on:

- Events of the form $C_{B,i}$, where $N_j(A) \cap N_i(B) \neq \emptyset$. A vertex $u \in N_j(A)$ is in at most $i\Delta 2^i$ sets $N_i(B)$, so there are at most $(j\Delta)k^2\Delta 2^k < \Delta^{2N}$ such events.
- Events of the form $B_{u,H}$, where $N_j(A) \cap V(T_H(u)) \neq \emptyset$. Since $|V(T_H(u))| \leq N\Delta^{\frac{N-1}{k-1}}$, there are at most $(j\Delta)|\mathcal{H}|N\Delta^{\frac{N-1}{k-1}} < \Delta^{2N}$ such events.

Also, $B_{u,H}$ depends on:

- Events of the form $B_{v,H'}$, were $V(T_H(u)) \cap V(T_{H'}(v)) \neq \emptyset$. There are at most $|\mathcal{H}|N^2\Delta^{2\frac{N-1}{k-1}} < \Delta^{2N}$ such events.
- Events of the form $C_{A,j}$, were $N_j(A) \cap V(T_H(u)) \neq \emptyset$. There are most $(N\Delta^{\frac{N-1}{k-1}})(k^2\Delta 2^k) < \Delta^{2N}$ such events.

Since the probability of each event is at most Δ^{-3N}, the Local Lemma implies that there exists a 2-coloring of $V(G)$ such that no event $C_{A,j}$ or $B_{u,H}$ holds. □

Proposition 55. *Let $a, b, m \geq 1$ be fixed and $s_0 = d_0^{a/b} g$ for $g > 0$. Suppose that the sequences d_t and s_t have initial values d_0 and s_0 and satisfy*

$$d_{t+1} = \frac{d_t}{2^b} + d_t^{1-1/m} \quad \text{and} \quad s_{t+1} = \frac{s_t}{2^a} + d_t^{a/b-1/m}.$$

Then there exists $D > 0$ such that

$$d_t \leq 2d_0 2^{-bt} \quad \text{and} \quad s_t \leq d_t^{a/b} g + d_t^{a/b-1/(2m)}$$

for all $d_t \geq D$.

Proof. Note that for any constant c, as $d_0 \to \infty$,

$$(c + d_0^{1/m} 2^{-bt/m})^m = d_0 2^{-bt} + \sum_{i=0}^{m-1} \binom{m}{i} c^{m-i} d_0^{i/m} 2^{-\frac{bti}{m}} = d_0 2^{-bt} + o(d_0) \leq 2 d_0 2^{-bt}.$$

To prove the first inequality, we will thus prove by induction the tighter bound (for D sufficiently large)

$$d_t \leq (\frac{2^b}{m(2^{b/m} - 1)} + d_0^{1/m} 2^{-bt/m})^m.$$

This is clear for d_0, so assume the bound for d_t. Then

$$\begin{aligned} d_{t+1} = \frac{d_t}{2^b} + d_t^{1-1/m} \leq \frac{1}{2^b}(\frac{2^b}{m} + d_t^{1/m})^m \leq \frac{1}{2^b}(\frac{2^b}{m} + \frac{2^b}{m(2^{b/m} - 1)} + d_0^{1/m} 2^{-bt/m})^m \\ = (\frac{2^b}{m(2^{b/m} - 1)} + d_0^{1/m} 2^{-b(t+1)/m})^m. \end{aligned}$$

For the second inequality, we first prove by induction $s_{t+1} = \frac{s_0}{x^{t+1}} + \sum_{k=0}^{t} \frac{d_k^y}{x^{t-k}}$, where $x = 2^a$ and $y = a/b - 1/m$. This is clear for $t = 0$, so assume the bound for s_t. Then

$$s_{t+1} = \frac{s_t}{x} + d_t^y = \frac{s_0}{x^{t+1}} + \frac{1}{x} \sum_{k=0}^{t-1} \frac{d_k^y}{x^{t-1-k}} + d_t^y = \frac{s_0}{x^{t+1}} + \sum_{k=0}^{t} \frac{d_k^y}{x^{t-k}}, \tag{6.4}$$

completing the induction. Using $d_k \leq 2d_0 2^{-bk}$ and $a - by = b/m \geq 0$,

$$\sum_{k=0}^{t} \frac{d_k^y}{x^{t-k}} \leq \frac{(2d_0)^y}{x^t} \sum_{k=0}^{t} (\frac{x}{2^{by}})^k = \frac{(2d_0)^y}{x^t} \sum_{k=0}^{t} 2^{(a-by)k} < \frac{(2d_0)^y}{x^t} \frac{2^{(t+1)(a-by)}}{2^{a-by} - 1}$$
$$= \frac{(2d_0)^y 2^{(t+1)(a-by)-at}}{2^{b/m} - 1} \tag{6.5}$$

By definition of d_t, $d_t \geq d_0 2^{-bt}$. Using this with (6.4) and with (6.5),

$$\begin{aligned} s_t < s_0 2^{-at} + \frac{(2d_0)^y 2^{a-by-tby}}{2^{b/m} - 1} &\leq s_0 (\frac{d_t}{d_0})^{a/b} + \frac{(2d_t 2^{bt})^y 2^{a-by-tby}}{2^{b/m} - 1} \\ &\leq d_0^{a/b} g (\frac{d_t}{d_0})^{a/b} + \frac{d_t^y 2^{y+a-by}}{2^{b/m} - 1} \\ &= d_t^{a/b} g + \frac{d_t^{a/b-1/m} 2^{y+a-by}}{2^{b/m} - 1} \\ &< d_t^{a/b} g + d_t^{a/b-1/(2m)}, \end{aligned}$$

where the last inequality assumes

$$d_t \geq D > (\frac{2^{y+a-by}}{2^{b/m} - 1})^{2m}.$$

□

Proof of Lemma 46. Define sequences

$$d_{t+1} = \frac{d_t}{2^{k-1}} + d_t^{1-\frac{1}{2k}}$$
$$r_{j,l,t+1} = \frac{r_{j,l,t}}{2^{j-l}} + d_t^{\frac{j-l}{k-1}-\frac{1}{2k}}, 1 \leq l < j \leq k$$
$$s_{H,t+1} = \frac{s_{H,t}}{2^{n(H)}-1} + d_t^{\frac{n(H)-1}{k-1}-\frac{1}{2k}}, H \in \mathcal{H},$$

where $d_0 = \Delta$, $r_{j,l,0} = \Delta^{\frac{j-l}{k-1}}\omega_j(\Delta)$, and $s_{H,0} = \Delta^{\frac{n(H)-1}{k-1}}/f^{n(H)}$. Since $r_{j,l,0} \leq 2\Delta^{\frac{j-l}{k-1}}\omega_j(\Delta)$, we may apply Lemma 54 to G to obtain hypergraphs $G_{1,1}$ and $G_{1,2}$, each with maximum degree at most $d_1 = r_{k,1,1}$, maximum (j,l)-degree at most $r_{j,l,1}$, and maximum H-degree at most $s_{H,1}$. We apply this halving step a total of T times, where $T = \lceil \frac{1}{k-1}\log_2 \frac{2d_0}{f^{4Nk}} \rceil$ and $N = \max_{H\in\mathcal{H}} n(H)$. Specifically, at each step, we apply Lemma 54 (with parameters $d_t = r_{k,1,t}$, $r_{j,l,t}$, $s_{H,t}$) to each of the hypergraphs $G_{t,1},\ldots,G_{t,2^t}$. Since $\omega_j(\Delta) = f^{o(1)}$, and for $t \leq T$,

$$d_t \geq d_0 2^{-(k-1)T} \geq d_0 2^{-(k-1)(1+\frac{1}{k-1}\log_2 \frac{2d_0}{f^{4Nk}})} = 2^{-k} f^{4Nk}, \tag{6.6}$$

$\omega_j(\Delta) = d_t^{o(1)}$. Also, recall that f is sufficiently large, so by (6.6), we may assume that if $t \leq T$, then d_t is sufficiently large to apply Proposition 55. Thus Proposition 55 (with each $r_{j,l,t}$ in the role of s_t, $g = \omega_j(\Delta), b = k-1, a = j-l$, and $m = 2k$) implies

$$r_{j,l,t} \leq d_t^{\frac{j-l}{k-1}}\omega_j(\Delta) + d_t^{\frac{j-l}{k-1}-\frac{1}{4k}} \leq 2d_t^{\frac{j-l}{k-1}}\omega_j(\Delta).$$

Thus each $G_{t,i}$ is $(d_t, 2\omega_2, \ldots, 2\omega_k)$-sparse, so we may apply Lemma 54 to obtain 2^{t+1} new hypergraphs $G_{t+1,1}, \ldots, G_{t+1,2^{t+1}}$ such that each $G_{t+1,i}$ has maximum degree $d_{t+1} = r_{k,1,t+1}$, maximum (j,l)-degree $r_{j,l,t+1}$, and maximum H-degree $s_{H,t+1}$. In the final step, we obtain 2^T hypergraphs $G_{T,1}, \ldots, G_{T,2^T}$, each with maximum degree d_T, maximum (j,l)-degree $r_{j,l,T}$ and maximum H-degree $s_{H,T}$. By Proposition 55,

$$d_T \leq 2d_0 2^{-(k-1)T} \leq f^{4Nk} < 2^k f^{4Nk},$$

and

$$r_{j,l,T} \leq 2d_T^{\frac{j-l}{k-1}} \omega_j(\Delta) \leq (2^k f^{4Nk})^{\frac{j-l}{k-1}} \omega_j(\Delta).$$

Proposition 55 (with each $s_{H,T}$ in the role of s_T, $g = 1/f, b = k-1, a = n(H)-1$, and $m = 2k$) also yields

$$\begin{aligned} s_{H,T} &\leq d_T^{\frac{n(H)-1}{k-1}} / f^{n(H)} + d_T^{\frac{n(H)-1}{k-1} - \frac{1}{4k}} \\ &\leq (f^{4Nk})^{\frac{n(H)-1}{k-1}} / f^{n(H)} + (f^{4Nk})^{\frac{n(H)-1}{k-1} - \frac{1}{4k}} \\ &\leq 2(f^{4Nk})^{\frac{n(H)-1}{k-1}} / f^{n(H)} \\ &= (f^{4Nk})^{\frac{n(H)-1}{k-1} - \frac{n(H)}{4Nk}} \\ &\leq (2^k f^{4Nk})^{\frac{n(H)-1}{k-1} - \frac{n(H)}{4Nk}}. \end{aligned}$$

We may therefore apply Lemma 52 (with $\Delta = 2^k f^{4Nk}, \omega_j(\Delta)$, and $\epsilon = \frac{1}{4Nk}$) to partition each of the hypergraphs $G_{T,1}, \ldots, G_{T,2^T}$ into $O(f^{4Nk(\frac{1}{k-1} - \frac{1}{4Nk})})$ parts such that the hypergraph induced by each part is $\mathcal{H}$-free and has maximum j-degree at most $2(2^k f^{4Nk})^{\frac{j-1}{4Nk}}\omega_j < 2^{2k} f^{j-1}\omega_j$ for each $1 \leq j \leq k$. Summing over each of the 2^T hypergraphs, this results in a total of

$$2^T O(f^{\frac{4Nk}{k-1}-1}) \leq 2^{1+\frac{1}{k-1}\log_2 \frac{2d_0}{f^{4Nk}}} O(f^{\frac{4Nk}{k-1}-1}) = O(\Delta^{\frac{1}{k-1}}/f)$$

$\mathcal{H}$-free hypergraphs, each with maximum j-degree at most $2^{2k} f^{j-1}\omega_j(\Delta)$.

CHAPTER 7

HYPERGRAPHS WITH LOW INDEPENDENCE NUMBER

Recall from Chapter 3 that Turán [5] showed that $\alpha(G) \geq \frac{n}{d+1}$ for any graph G with n vertices and average degree d. Spencer [57] extended Turán's bound to hypergraphs, showing that for all $k \geq 1$ there is a c_k so that every $(k+1)$-uniform hypergraph H with average degree d satisfies $\alpha(H) \geq c_k \frac{n}{d^{1/k}}$.

When G is a graph, Turán's bound can be substantially improved if G is forbidden from containing a fixed subgraph. Ajtai, Komlós, and Szemerédi [1] showed that if G is triangle-free, then

$$\alpha(G) \geq \frac{1}{100} \frac{n}{d} \log d. \tag{7.1}$$

Shearer [39] later improved the constant to $1 + o(1)$ (with $d \to \infty$). Ajtai, Erdős, Komlós, and Szemerédi [58] showed that if $t \geq 4$ and G is K_t free, then $\alpha(G) \geq c_t \frac{n}{d} \log \log d$. Subsequently, Shearer [59] improved their bound to $\alpha(G) \geq c'_t \frac{n}{d} \frac{\log d}{\log \log d}$.

When the edges of the hypergraph have size at least 3, much less is known. A hypergraph is *linear* (or contains no 2-cycles) if every pair of vertices is contained in at most one edge. Let H be a $(k+1)$-uniform hypergraph with n vertices and average degree d. Ajtai, Komlós, Pintz, Spencer, and Szemerédi [13] showed that there exists a positive constant c_k such that if H contains no 2, 3, or 4 cycles, then

$$\alpha(H) \geq c_k \frac{n}{d^{1/k}} \log^{1/k} d. \tag{7.2}$$

Duke, Lefmann, and Rödl [14] answered a question of Spencer and showed that the same bound holds for linear hypergraphs. The bounds (7.1) and (7.2) have found applications in number theory [1], discrete geometry [60], coding theory [61], and Ramsey theory [62].

Let $K_4^{(3)}$ denote the complete 3-uniform hypergraph on 4 vertices, and let $K_4^{-(3)}$ denote $K_4^{(3)}$ minus an edge. Motivated by applications of (7.2), Ajtai, Erdős, Komlós and Szemerédi posed the following question in 1981.

Question 56 (Ajtai-Erdős-Komlós-Szemerédi [58])**.** *Is there a function $\omega(d) \to \infty$ such that if a 3-uniform hypergraph H contains no $K_4^{(3)}$ (or even $K_4^{-(3)}$), then $\alpha(H) \geq \frac{n}{d^{1/2}}\omega(d)$?*

Define the 3-uniform hypergraphs $F_5 = \{abc, abd, cde\}$ and $C_3 = \{abc, cde, efa\}$. In Chapter 5, we answered Question 56 positively if, in addition to $K_4^{-(3)}$, F_5 and C_3 are also forbidden.

A hypergraph is *c-sparse* if every vertex subset S spans at most $c|S|^2$ edges. By Spencer's extension of Turán's bound, every c-sparse hypergraph H with n vertices satisfies $\alpha(H) \geq c'_k\sqrt{n}$. For linear 3-uniform hypergraphs, (7.2) implies $\alpha(H) \geq c'_k\sqrt{n \log n}$. In 1986, de Caen (see [63]) conjectured that a similar improvement holds even for c-sparse hypergraphs (observe that linear implies $\frac{1}{2}$-sparse).

Conjecture 57 (De Caen [63])**.** *For every positive c, there is a function $\omega(n) \to \infty$ such that every c-sparse 3-uniform hypergraph H with n vertices satisfies $\alpha(H) \geq \omega(n)\sqrt{n}$.*

Recently, Kostochka, Mubayi, and Verstraëte [64] proved bounds of the form $\alpha(H) > c(\frac{n}{d} \log \frac{n}{d})^{1/2}$ as long as every pair of vertices lies in at most $d < n/\log^{12} n$ edges. Based on their results, they posed a stronger version of de Caen's conjecture: for every positive c, there is a function $\omega(n) \to \infty$ such that every c-sparse 3-uniform hypergraph H with n vertices and average degree d satisfies $\alpha(H) \geq \omega(n)\frac{n}{d^{1/2}}$.

Recall that a proper coloring of a hypergraph H is a partition of $V(H)$ into independent sets. The chromatic number of H, denoted $\chi(H)$, is the minimum parts needed in a proper coloring of H. Erdős and Lovász [17] showed that every $(k+1)$-uniform hypergraph with maximum degree Δ has $\chi(H) \leq c_k \Delta^{1/k}$. Strengthening (7.2), Frieze and Mubayi [4] showed that every $(k+1)$-uniform linear hypergraph with maximum degree Δ satisfies $\chi(H) \leq c_k'(\frac{\Delta}{\log \Delta})^{1/k}$. In [3; 4], the same authors conjectured a stronger positive answer to the question of Ajtai, Erdős, Komlós, and Szemerédi.

Conjecture 58 (Frieze-Mubayi [3; 4])**.** *If F is a nontrivial $(k+1)$-uniform hypergraph and H is an F-free $(k+1)$-uniform hypergraph with maximum degree Δ, then $\chi(H) \leq c_F(\Delta/\log \Delta)^{1/k}$.*

Gyárfás and Mubayi (see Elekes [65]) showed that Conjecture 58, with $k = 2$ and $F = K_9^{(3)}$, could be applied to resolve an old question of Erdős in discrete geometry. Guruswami and Sinop [66] showed that Conjecture 58 (with $k \geq 3$ and F any $(k+1)$-uniform hypergraph with chromatic number at least 3) would imply that certain hardness results in computer science cannot be improved.

Let T_k be the k-uniform hypergraph with $k+1$ edges $e_1, \ldots, e_k, f$ where for all $i > j$ we have $e_i \cap e_j = S$ with $|S| = k-1$ and $f \supset e_i - e_j$ for all $i < j$. In other words, k edges share the same set of $k-1$ points and the last edge contains the remaining vertex from each of the k edges. A k-uniform hypergraph has independent neighborhoods if it contains no copy of T_k. Note that this a generalization of triangle-freeness to hypergraphs. Bohman, Frieze, and Mubayi [18] conjectured a weaker version of Conjecture 58: if H is a 3-uniform hypergraph with maximum degree Δ and independent neighborhoods, then $\chi(H) = o(\Delta^{1/2})$.

We construct hypergraphs which negatively answer Question 56 and provide counterexamples to Conjectures 57 and 58, as well as the related conjectures from [64] and [18]. Each $(k+1)$-uniform hypergraph H that we construct is the line graph of a k-partite k-uniform hypergraph. In particular, we take only those sets of edges which are isomorphic to some fixed $(k+1)$-uniform hypergraph F_k. We then use extremal results for F_k to bound $\alpha(H)$. Kostochka, Pudlák, and Rödl [67] have used related ideas to provide explicit constructions of graphs which give constructive lower bounds of Ramsey numbers.

Our constructions also provide the correct order of magnitude for some new 3-uniform Ramsey numbers. Let F be a 3-uniform hypergraph. Recall that $r(F, t)$ is the smallest n so that every red-blue coloring of the edges of $K_n^{(3)}$ contains a red F or a blue complete $K_t^{(3)}$. Let P_s denote the 3-uniform hypergraph with vertex set $[s+2]$ and edge set $\{\{i, i+1, i+2\} : i \in [s]\}$. P_s is called the 3-uniform tight path. Results of Phelps and Rödl [68] imply that the Ramsey number of P_2 satisfies

$$r(P_2, t) = \Theta(t^2/\log t).$$

For $s \geq 4$, we show

$$r(P_s, t) = \Theta(t^2).$$

This improves on the previously known

$$c_s t^2 / \log t < r(P_s, t) < c'_s t^2,$$

for all $s \geq 3$. The order of magnitude of $r(P_3, t)$ remains open.

7.1 Construction

Fix $k \geq 2$. A *k-simplex* is a collection of $k+1$ sets with empty intersection, every k of which have nonempty intersection. The extremal problem for simplices was introduced by Erdős [69] in 1971 and has a long history [70; 89; 90; 88]. Mubayi introduced the notion of strong simplices [71] (see also [72; 93]). A *strong k-simplex* S_k is the k-uniform hypergraph with vertex set $\{v_1, v'_1, \ldots, v_k, v'_k\}$ and edge set $\{e, e_1, \ldots, e_k\}$ where $e = \{v_1, \ldots, v_k\}$ and $e_i = e \cup \{v'_i\} - v_i$ (e is called the central edge). Given disjoint sets $X_1, \ldots, X_k$ with each $X_i \cong [n]$, a *positive strong k-simplex* S_k^+ is a k-partite strong simplex in $X_1 \times \cdots \times X_k$ with $v'_i > v_i$ for each i. In other words, the vertices of the central edge lie below the other vertex in their corresponding part.

A *k-cluster* is a collection of $k+1$ sets with empty intersection whose union has size at most $2k$. The notion of a k-cluster was introduced by Mubayi [71] to generalize a problem of Katona. Extremal results for clusters are more recent [73; 92; 88]. The family of *special k-clusters* $\mathcal{D}_k$ is the k-uniform hypergraph family that is defined inductively as follows: $\mathcal{D}_2 = \{D_2\}$, where

D_2 is the path with three edges. For $k \geq 3$, $\mathcal{D}_k$ is the family of k-uniform hypergraphs which can be constructed as follows: begin with any $D_{k-1} \in \mathcal{D}_{k-1}$, which is assumed inductively to have $2(k-1)$ vertices and two disjoint edges a and b. Then D_k is a member of $\mathcal{D}_k$ if it can be formed by adding two new vertices x, y to D_{k-1}, enlarging all edges of D_{k-1} by including x, and enlarging a by including y. Thus D_k has $2k$ vertices and $k+1$ edges, two of which are disjoint. We will use D_k to denote an arbitrarily chosen member of $\mathcal{D}_k$.

Let $F_k \in \{S_k^+, D_k\}$. Let $X_1, \ldots, X_k$ be disjoint sets each isomorphic to $[n]$. Define the $(k+1)$-uniform hypergraph $H(F_k)$ with vertex set $X_1 \times \cdots \times X_k$ and edge set

$$H(F_k) = \{A \subset X_1 \times \cdots \times X_k : A \cong F_k\}.$$

In words, the vertices of $H(F_k)$ correspond to edges in the complete k-partite k-uniform hypergraph with parts of size n and the edges correspond to copies of positive strong k-simplices (or special k-clusters). For example, the edges of $H(D_2)$ correspond to 3-edge paths in the complete bipartite graph $[n] \times [n]$, while the edges of $H(S_2^+)$ correspond to 3-edge paths in $[n] \times [n]$ which open upward. The edges of $H(S_2^+)$ may also be viewed as sets of three points in the $n \times n$ grid which form an L shape.

7.1.1 Bounding the independence number

Fix a k-uniform hypergraph F_k. The Zarankiewicz number $z(n, F_k)$ is the maximum number of edges in a k-partite k-uniform hypergraph with parts of size n that contains no copy of F_k. Since copies of F_k correspond to edges of $H(F_k)$,

$$\alpha(H(F_k)) \leq z(n, F_k).$$

We may thus use the two lemmas below to bound $\alpha(H(S_k^+))$ and $\alpha(H(F_k^+))$. For a vertex v in a k-uniform hypergraph H, define its link to be the $(k-1)$-uniform hypergraph $L_v = \{S \subset V(H) : v \notin S, S \cup \{v\} \in H\}$. The degree of a set T, written $d_H(T)$ is the number of edges containing T.

Lemma 59. *Fix $k \geq 2$ and $D_k \in \mathcal{D}_k$. Then $z(n, D_k) \leq kn^{k-1}$.*

Proof. We proceed by induction on k. The base case $k = 2$ follows from the fact that $D_2 = P_3$, the path with 3 edges. For the induction step, suppose we are given k-partite H with $|H| > kn^{k-1}$ with parts $X_1, \ldots, X_k$ each of size n. For each $v \in X_1$, let L_v be the link $(k-1)$-uniform hypergraph of v. Let $A_v \subset L_v$ comprise those $(k-1)$-sets T with $d_H(T) = 1$ and $B_v = L_v - A_v$. Then

$$kn^{k-1} < |H| = \sum_{v \in X_1} |L_v| = \sum_{v \in X_1} |A_v| + \sum_{v \in X_1} |B_v| \leq n^{k-1} + \sum_{v \in X_1} |B_v|. \tag{7.3}$$

Consequently, there exists $x \in X_1$ with $|B_x| > (k-1)n^{k-2}$. Let D_{k-1} be the member of $\mathcal{D}_{k-1}$ that gives rise to D_k in the inductive construction of D_k. Apply induction to B_x to obtain a copy of D_{k-1} in $X_2 \times \cdots \times X_k$. To form D_k, begin by enlarging each edge of D_{k-1} with x. Add another edge by enlarging one of the two disjoint edges a, b of D_{k-1} (say a) by some other vertex $y \in X_1$. Note that y exists since $a \in B_x$. We have thus obtained a copy of D_k, where $a \cup \{y\}$ and $b \cup \{x\}$ are the disjoint edges. □

Lemma 60. *Fix $k \geq 2$. Then $z(n, S_k^+) \leq 2kn^{k-1}$.*

Proof. We proceed by induction on k. If H is bipartite and $|H| > 4n$, then H contains a cycle, and the smallest vertex in either part of this cycle lies in a copy of S_2^+. For the induction step, suppose we are given a k-partite $H \subset X_1 \times \cdots \times X_k$ with $|H| > 2kn^{k-1}$, where each $X_i \cong [n]$. For each $v \in X_1$, define L_v, A_v, and B_v as in the proof of Lemma 59. Let B_v^+ be the set of all $S \in B_v$ such that there exists $v' > v$ with $S \in L_{v'}$. We will find $x \in X_1$ with $|B_x^+| > 2(k-1)n^{k-2}$ and then apply induction. Now

$$\sum_{v \in X_1} |B_v^+| = \sum_{\substack{S \in X_2 \times \cdots \times X_k: \\ d(S) \geq 2}} (d(S) - 1) \geq \sum_{\substack{S \in X_2 \times \cdots \times X_k: \\ d(S) \geq 2}} d(S) - n^{k-1} = \sum_{v \in X_1} |B_v| - n^{k-1}.$$

Thus, as in (7.3) from the proof of Lemma 59,

$$2kn^{k-1} < |H| \leq n^{k-1} + \sum_{v \in X_1} |B_v| \leq 2n^{k-1} + \sum_{v \in X_1} |B_v^+|.$$

Consequently, there exists $x \in X_1$ with $|B_x^+| > 2(k-1)n^{k-2}$. Apply induction to B_x to obtain a copy of S_{k-1}^+ in $X_2 \times \cdots \times X_k$. To form S_k^+, begin by enlarging each edge of S_{k-1}^+ with x. Add another edge by enlarging the central edge e by some other vertex $y \in X_1$ with $y > x$. Note that y exists since $e \in B_x^+$ and $d_H(e) > 1$. We have thus obtained a copy of S_k^+, where $e \cup \{x\}$ is the central edge. □

7.1.2 Properties of $H(S_k^+)$

By Lemma 60, $\alpha(H(S_k^+)) \leq 2kn^{k-1}$. The maximum degree of $H(S_k^+)$, denoted Δ, is at most $(k+1)n^k$. Thus

$$\alpha(H(S_k^+)) \leq 2kn^{k-1} = 2k(k+1)^{1/k}\frac{n^k}{((k+1)n^k)^{1/k}} \leq 2k(k+1)^{1/k}\frac{n^k}{\Delta^{1/k}},$$

and

$$\chi(H(S_k^+)) \geq \frac{\Delta^{1/k}}{2k(k+1)^{1/k}}.$$

7.1.2.1 Independent neighborhoods

Recall that T_{k+1} is the $(k+1)$-uniform hypergraph with $k+2$ edges $e_1, \ldots, e_{k+1}, f$ where for all $i > j$ we have $e_i \cap e_j = S$ with $|S| = k$ and $f \supset e_i - e_j$ for all $i < j$. Suppose $S_{k,1}^+, \ldots, S_{k,k+1}^+$ satisfy $S_{k,i}^+ \cap S_{k,j}^+ = S$, for $i < j$ and $|S| = k$. Since each strong k-simplex is positive, they must share a single central edge. Thus the edges in $(S_{k,1}^+ \cup \cdots \cup S_{k,k+1}^+) - S$ share a single vertex and so do not form a positive strong k-simplex. Therefore $H(S_k^+)$ does not contain any copy of T_{k+1}, disproving Conjecture 58 and the weaker conjecture of [18].

7.1.2.2 $K_4^{(3)}$ and $K_4^{-(3)}$

Recall that $K_4^{(3)}$ is the complete 3-uniform hypergraph on 4 vertices, and $K_4^{-(3)}$ is $K_4^{(3)}$ minus an edge. Observe that every link graph of $H(S_2^+)$ consists of components which are stars and complete bipartite graphs. Since $K_4^{-(3)}$ has a vertex whose link graph is a triangle, $H(S_2^+)$ contains no copy of $K_4^{-(3)}$. Since $\alpha(H(S_2^+)) \leq 4\sqrt{3}n^2/\Delta^{1/2}$, and the average degree of $H(S_2^+)$ is at most Δ, this negatively answer Question 56.

7.1.2.3 Ramsey numbers for 3-uniform tight paths

Recall that the tight path P_s is the 3-uniform hypergraph with vertex set $[s+2]$ and edge set $\{\{i, i+1, i+2\} : i \in [s]\}$. It is easy to prove that for fixed s, we have $\text{ex}(n, P_s) = O(n^2)$ and this immediately implies that

$$r(P_s, t) = O(n^2).$$

Indeed, if we have a P_s-free 3-uniform hypergraph on $c_s n$ vertices (c_s large), then its average degree is at most $c_s' n$, so it has an independent set of size at least $t = c_s'' n^{1/2}$.

Theorem 61. *Fix $s \geq 4$. Then $r(P_s, t) = \Theta(t^2)$.*

Proof. It suffices to prove the lower bound. Observe that $H(S_2^+)$ contains no P_4. This is because every link graph of $H(S_2^+)$ has one component that is a complete bipartite graph and all of its other components are stars; further, the edges of the bipartite component all have codegree one. A P_4 has one link graph that is a 3-edge path and one of the edges in this link graph has codegree 2. □

7.1.3 Properties of $H(D_k)$

Observe that $H(S_2^+)$ is 1-sparse, so $H(S_2^+)$ immediately provides a counterexample to Conjecture 57. However, for $k \geq 3$, $H(S_k^+)$ is not c-sparse for any constant c, so one may ask whether or not for $k \geq 3$ and every positive c there is a function $\omega(n) \to \infty$ such that every c-sparse $(k+1)$-uniform hypergraph H with n vertices satisfies $\alpha(H) \geq \omega(n)n^{1/k}$. We now show that $H(D_k)$ provides a counterexample to this generalization of de Caen's conjecture.

By Lemma 59, $\alpha(H(D_k)) \leq kn^{k-1}$, so it suffices to show that $H(D_k)$ is 2^{2k^2-2k-1}-sparse. Let $S \subset V(H(D_k))$. The vertex set of a copy of D_k is determined by the two disjoint edges in D_k. There are at most $\binom{2k}{k}^{k-1}$ possibilities for the remaining $k-1$ edges. Therefore we may associate every pair of vertices in S to at most $\binom{2k}{k}^{k-1}$ edges in the subgraph induced by S. Since every edge corresponds to at least one pair of vertices, the number of edges in S is at most

$$\binom{2k}{k}^{k-1}\binom{|S|}{2} < 2^{2k^2-2k-1}|S|^2.$$

This disproves Conjecture 57, as well as the stronger conjecture in [64].

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2012 Counting independent sets in triangle-free graphs (with D. Mubayi)
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