A Study of Well Posedness for Systems of Coupled Non-linear Dispersive Wave Equations

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2014

Chicago, Illinois

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To TATA

ACKNOWLEDGMENTS

I would like to first thank my thesis advisor Jerry L. Bona, who over the course of my graduate studies was always so generous with his knowledge, time and spirit. He has been a constant source of strength and encouragement for me. Always kind and patient, Jerry has been much more than just an advisor, he has been a teacher, mentor, friend and family.

I am grateful to the other members of my thesis committee - Hongqiu Chen, Alexey Cheskidov, Irina Nenciu and Roman Shvydkoy, for their time and guidance throughout the years I have known and learned from them. I especially appreciate Hongqiu's assistance, comments and input while working on this thesis.

Special gratitude goes to my dear parents who have always been an unconditional source of love and support. It is because of them that I discovered very early in my childhood my very first love - Math - and that I wanted to become a mathematician. They contributed enormously to who I am and it would definitely be an understatement to say I have been fortunate to be their daughter. Many thanks also go to Buni, for being a wonderful Grandmother, for raising me and teaching me so many important life lessons.

Special consideration goes to my unique "sorella", Oana, for being loving and supportive and whose ambition, dedication and strong work ethics have always inspired me; and to my brother-in-law, Jesse for teaching me what "enough" means, what really matters and how to relax and enjoy the moment.

ACKNOWLEDGMENTS (Continued)

I would also want to warmly thank my middle and high school Math teachers - Petre Darabut and Oratiu Hoara - for their major contribution in my development as a mathematician. I especially cherish the University years in Timisoara and the lectures of Petre Preda.

My graduate school years in the UIC Math Department have nurtured me in many ways. Steve Hurder and Kari Dueball always made me feel welcome. I would also like to mention Peter Shalen and Irina Nenciu for their inspiring lectures. The friendship of my graduate student colleagues - Natalie McGathey, Kelly Darke, Sarah Cordell and Sanja Pantic - meant a lot to me.

I want to thank all of my friends who have encouraged me throughout the years. Thank you to my close friends - Melania Damian, Cristina Sabliov, Dorin Boldor, Mihaela Topuzu, Mihaela Marcusanu, and Marion and Adrian Pollard - for their support and love even though most of them are many miles away.

Finally, I am infinitely grateful to the three people I can not live without - Andreea, Danel and Gusti. They are my everything. The magnitude of the thanks they deserve blows up in small time, where time t = NOW.

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LIST OF ABBREVIATIONS

BBM	Benjamin Bona Mahony
IVP	Initial Value Problem
RLW	Regularized Long-Wave
gBBM	Generalized Benjamin Bona Mahony
KdV	Korteweg-de Vries
gKdV	Generalized Korteweg-de Vries
IC	Initial Condition
PDE	Partial Differential Equations
RHS	Right Hand Side

SUMMARY

Consideration is given to coupled systems

$$\begin{cases} u_t + u_x - u_{xxt} + P(u,v)_x = 0, \\ v_t + v_x - v_{xxt} + Q(u,v)_x = 0, \end{cases}$$

of two evolution equations of generalized BBM-type, posed for $x \in \mathbb{R}$ and $t \geq 0$ and with u and v real-valued functions of (x,t); the subscripts connote partial differentiation and P and Q are homogeneous polynomials in the variables u and v.

A study of the initial-value problems in which $u(x,t)$ and $v(x,t)$ are specified at $t = 0$, *viz.*

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R},$$

is undertake to analyze and determine conditions on the case of quadratic homogeneous polynomials P and Q under which these initial-value problems are globally well-posed in the L_2-based Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$. The case of cubic homogeneous polynomials P and Q is also investigated.

CHAPTER 1

INTRODUCTION

The focus of this study is **systems of non-linear dispersive wave equations** and the fundamental properties of their solutions. These systems arise in water wave theory, climate modeling and other situations where wave propagation is important.

The particular system of partial differential equations we are investigating is a coupled system of two evolution equations of generalized BBM-type. It has the form

$$\begin{cases} u_t + u_x - u_{xxt} + P(u,v)_x = 0, \\ v_t + v_x - v_{xxt} + Q(u,v)_x = 0, \end{cases} \tag{1.1}$$

posed for $x \in \mathbb{R}$ and $t \geq 0$ and with u and v real-valued functions of (x,t) where subscripts connote partial differentiation. We consider the initial-value problem (IVP henceforth) where the wave profile and the velocity are specified at a starting time taken to be 0, *viz*.

$$u_0(x) = u(x,0) = \phi(x), \quad v_0(x) = v(x,0) = \psi(x), \quad x \in \mathbb{R}. \tag{1.2}$$

For the time being, P and Q are arbitrary homogeneous quadratic polynomials,

$$P(u,v) = Au^2 + Buv + Cv^2,$$

$$Q(u, v) = Du^2 + Euv + Fv^2,$$

with coeficients $A, B, C, D, E, F \in \mathbb{R}$. Later, cubic polynomials will also be considered.

The IVP (1.1) - (1.2) is a natural generalization of the IVP for the BBM-equation or regularized long-wave (RLW) equation,

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0, \quad \eta(x, 0) = f(x), \tag{1.3}$$

The latter was originally proposed as an alternative to the Korteweg-deVries (KdV) equation

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \tag{1.4}$$

by Peregrine (1964, 1967) and Benjamin *et al.* (1972). It models unidirectional propagation of small amplitude, long wavelength water waves. This equation is written in non-dimensional, laboratory coordinates, with x denoting the direction of propagation and t being proportional to elapsed time. As written, it serves as a model for waves propagating in the direction of increasing values of x. The initial condition (IC) represents a snapshot of a disturbance already generated, without necessarily inquiring as to how the motion was initiated. The solution η of either (1.3) or (1.4) provides a prediction of the wave profile at future time. The model subsists on the small amplitude, long wavelength assumptions,

$$\alpha = \frac{a}{h_0} << 1 \quad \text{and} \quad \beta = \frac{h_0}{\lambda} << 1,$$

where a is a typical amplitude, α a typical wavelength and h_0 is the undisturbed depth of the fluid, assumed to be constant. Consequently, the initial data f can be taken in the form $f(x) = \alpha F(\beta x)$, where F is independent of α and β, thereby enforcing the small amplitude, long wavelength presumption.

The IVP (1.3) has a distinguished history on both analytical and experimental sides. The global well-posedness in $H^s(\mathbb{R})$ for any $s \geq 0$ for the BBM IVP was shown in [1]. Comparisons with laboratory experiments may be found in [2].

The generalized BBM equation has the form

$$u_t + u_x + g(u)_x + Lu_t = 0 \tag{1.5}$$

where L is a Fourier-multiplier operator related to the linearized dispersion relation and g is a smooth, real-valued function of a real variable. Local and global well-posedness results for generalized BBM-type equations were presented in an initial form in [3]. The results of local and global well-posedness for the gBBM-equations in weaker L_p-based spaces for appropriate values of p were put forward in [4].

To model two-way propagation of waves in physical systems where nonlinear and dispersive effects are equally important, systems of partial differential equations (PDEs) have been used. The theories developed are local as well as global and use various techniques.

In the case of well-posedness for a system of equations with coupled nonlinearities, Bona, Cohen and Wang [5] improved the global existence results obtained previously by Ash, Cohen and Wang [6] for a system of two KdV equations coupled with quadratic nonlinearities. They provided conditions on the coeficients of the quadratic nonlinear terms so that the IVP is globally well-posed in the L_2-based Sobolev classes $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s > -3/4$.

The purpose of the present work is to analyze and determine conditions under which the initial-value problem (1.1) - (1.2) is globally well-posed in the L_2-based Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$. This is accomplished by first converting the IVP into a system of integral equations. This step does not depend upon the degree of the polynomials P and Q. This conversion is carried out in the first part of Chapter 2, right after Notation is fixed. In the rest of Chapter 2 and through Chapter 5, the focus is on the case when P and Q are homogeneous quadratic polynomials. Conservation Laws are determined in the second part of Chapter 2. These will be used later to derive the *a priori* bounds that lead to global well-posedness. Following the idea in [3], local well-posedness can be assured by applying the contraction-mapping principle in the spaces $H^k(\mathbb{R}) \times H^k(\mathbb{R})$ for $k \geq 1$. Chapter 3 is devoted to local well-posedness, initially deduced for the $L_\infty \times L_\infty$ case and then for $H^k(\mathbb{R}) \times H^k(\mathbb{R})$ for $k \geq 1$. Chapter 4 is concerned with the $L_2 \times L_2$ case, using an approach inspired from the work of Bona & Tzvetkov [1]. The result obtained thus far is extended to $H^s \times H^s$ for any $s \geq 0$ by applying nonlinear interpolation theory in Chapter 5. Attention is then turned to the case when the two polynomials that comprise the nonlinearity of the model are cubic. Similar to the quadratic case, the system is locally well posed in L_2-based spaces. Conditions on the coefficients of the polynomials P and Q are also determined so that the invariants to be used for global well-posedness exist.

CHAPTER 2

PRELIMINARIES

2.1 Notation

The notational conventions and function-space designations used in this paper are set out here. For $1 \leq p < \infty$, $L_p = L_p(\mathbb{R})$ connotes the p^{th}-power Lebesque-integrable functions with the usual modification for the case $p = \infty$. The norm of a function $f \in L_r$ with $1 \leq r \leq \infty$ is written $\|f\|_r$ while the $L_r \times L_r$-norm of a pair (f, g) of such functions is written $\|(f,g)\|_{L_r \times L_r} = \|f\|_r + \|g\|_r$. In general, if X and Y are Banach spaces, then their Cartesian product $X \times Y$ is a Banach space with a product norm defined by $\|(f,g)\|_{X \times Y} = \|f\|_X + \|g\|_Y$. The Sobolev class $H^s = H^s(\mathbb{R})$ for $s \geq 0$ is the class of measurable functions f whose Fourier transform $\widehat{f(\xi)}$ is a measurable function, square integrable with respect to the measure $(1 + |\xi|)^{2s} d\xi$, where

$$\widehat{f(\xi)} = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx.$$

We will usually use simply H^s rather than $H^s(\mathbb{R})$ unless emphasis on the domain of definition of the functions is needed. The norm in $H^s \times H^s$ is denoted $\|\cdot\|_{H^s \times H^s}$. Similarly, for $s \geq 0$ an integer and $p \geq 1$, the space

$$W_p^s(\mathbb{R}) = \left\{ f, f', ..., f^{(s)} \in L_p : \int_{-\infty}^{\infty} \left(|f(x)|^p + |f^{(s)}(x)|^p\right) dx < \infty \right\}$$

is a Banach space endowed with the norm

$$\|f\|_{W_p^s} = \left[\int_{-\infty}^{\infty} \big(|f(x)|^p + |f^{(s)}(x)|^p \big) dx \right]^{\frac{1}{p}}.$$

If X is any Banach space and $T > 0$ given, $C(0,T;X)$ is the class of continuous maps from $[0,T]$ into X with its usual norm

$$\|u\|_{C(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X \, .$$

The subspace $C^1(0,T;X)$ of the elements of $C(0,T;X)$ for which the limit

$$u'(t) = \lim_{h\to 0} \frac{u(t+h) - u(t)}{h}$$

exists in $C(0,T;X)$, is also a Banach space with the obvious norm. For $k \in \mathbb{N}$ the spaces $C^k(0,T;X)$ are defined inductively and by analogy. For convenience and when there couldn't be any confusion created,

$$\int_{-\infty}^{\infty} f(x) \quad \text{is replaced by} \quad \int f.$$

2.2 Integral Equation

The analysis begins with local well-posedness in a reasonably broad set of functional classes. To accomplish this, the given system is converted to an equivalent system of integral equations. This is independent of the form and degree of the polynomials P and Q.

Let U denote the vector

$$U = \begin{pmatrix} u \\ \\ v \end{pmatrix}$$

of dependent variables, U_0 the vector

$$U_0 = \begin{pmatrix} u_0 \\ \\ v_0 \end{pmatrix}$$

of initial data and M the vector of non-linearities

$$M(U) = \begin{pmatrix} P(u,v) \\ \\ Q(u,v) \end{pmatrix}.$$

Then the system can be written as

$$U_t + U_x - U_{xxt} = -M_x,$$

with

$$U(x,0) = U_0(x).$$

Rearranging this equation leads to

$$U_t - U_{xxt} = -U_x - M_x,$$

or

$$(I - \partial_x^2)U_t = -\partial_x(U + M).$$

Inverting the operator $(I - \partial_x^2)$ subject to boundness at $\pm\infty$ leads to

$$U_t(x,t) = -(I - \partial_x^2)^{-1}\partial_x\big[U(x,t) + M(U(x,t))\big]. \tag{2.1}$$

For now, assume that U is bounded or at least is not growing exponentially fast as $x \to \pm\infty$. Under that assumption, after integrating by parts, we obtain the system

$$U_t(x,t) = \int_{-\infty}^{\infty} K(x-y)\big[U(y,t) + M(U(y,t))\big]\, dy, \tag{2.2}$$

where the kernel K is applied to the vector U componentwise and

$$K(x) = \frac{1}{2}\mathrm{sgn}(x)e^{-|x|}.$$

Integrating the relation above with respect to t leads to the integral equation

$$U(x,t) = U_0(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y)\Big[U(y,s) + M\big(U(y,s)\big)\Big]\, dyds. \tag{2.3}$$

If we look at this integral formula as

$$U = \mathcal{A}U \tag{2.4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U(x,t) := U_0 + \int_0^t \int_{-\infty}^{\infty} K(x-y)\big[U(y,s) + M(U(y,s))\big]\, dyds, \tag{2.5}$$

the problem of finding a solution to the integral equation becomes a fixed-point problem for $\mathcal{A}$. A solution can be inferred to exist by showing that $\mathcal{A}$ is a contraction and then applying the Contraction Mapping Theorem. This is the strategy for establishing local well posedness pursued in Chapter 3.

2.3 Hamiltonian Structure and Conservation Laws

As we will see, local well-posedness theory doesn't depend on assumptions about the coefficients of P and Q. To move to global in time well-posedness, we need *a priori* estimates for the solution. To prepare for that, observe that the system has a Hamiltonian structure which sometimes turns out to be helpful in estimating the growth of the spacial norms of the solution as functions of time. Assume that we have in hand a solution (u, v) of the system which is not growing exponentially for $x \to \pm\infty$. That such solutions exist, at least locally in time, is a consequence of the theory in Chapter 3.

2.3.1 Energy invariant

Consider the following quadratic functional:

$$\Omega(u,v) := \int \Big(au^2 + buv + cv^2 + du_x^2 + eu_xv_x + fv_x^2\Big)dx,$$

where a, b, c, d, e, f are real numbers to be determined. A formal calculation where integration by parts is used with the assumption that (u, v) is the solution of the system and u and v and their derivatives do not make any contribution at $x = \pm\infty$, leads to the formulas

$$\begin{aligned}
\frac{d}{dt}\Omega =& \int \Big(2auu_t + bu_tv + buv_t + 2cvv_t + 2du_{xt}u_x + eu_{xt}v_x + eu_xv_{xt} + 2fv_xv_{xt}\Big)dx \\
=& \int \Big(2auu_t + bu_tv + buv_t + 2cvv_t - 2du_{xxt}u - eu_{xxt}v - euv_{xxt} - 2fvv_{xxt}\Big)dx \\
=&2\int u\Big(au_t - du_{xxt}\Big)dx + \int v\Big(bu_t - eu_{xxt}\Big)dx+ \\
&+ \int u\Big(bv_t - euv_{xxt}\Big)dx + 2\int v\Big(cv_t - fv_{xxt}\Big)dx \\
=&2\int \Big(a-d\Big)uu_tdx - 2d\int uP_xdx + \int\Big(b-e\Big)vu_tdx - e\int vP_xdx+ \\
&+ \int\Big(b-e\Big)uv_tdx - e\int uQ_xdx + 2\int\Big(c-f\Big)vv_tdx - 2f\int vQ_xdx.
\end{aligned}$$

To simplify things a bit assume $a = d$, $b = e$ and $c = f$. By expanding P_x and Q_x, the terms in the sum above may be written in the form

$$
\begin{aligned}
\frac{d}{dt}\Omega =& - 2a \int \Big(2Au^2u_x + Buvu_x + Bu^2v_x + 2Cuvv_x\Big)dx \\
& - b \int \Big(2Auvu_x + Buvv_x + Bv^2u_x + 2Cv^2v_x\Big)dx \\
& - b \int \Big(2Du^2u_x + Eu^2v_x + Euvu_x + 2Fuvv_x\Big)dx \\
& - 2c \int \Big(2Duvu_x + Ev^2u_x + Euvv_x + 2Fv^2v_x\Big)dx \\
=& \int \Big(- 4Aa - 2Db\Big)u^2u_xdx + \int \Big(2Cb - 4Fc\Big)v^2v_xdx \\
& - \int \Big[\Big(2Ba + Eb + 2Ab + 4Dc\Big)uvu_x + \Big(2Ba + Eb\Big)u^2v_x\Big]dx \\
& - \int \Big[\Big(Bb + 2Ec + 4Ca + 2Fb\Big)uvv_x + \Big(Bb + 2Ec\Big)v^2u_x\Big]dx
\end{aligned}
$$

The first two terms of the sum above are integrals which have integrands that are x-derivatives. Thus, for smooth solutions going to 0 at $\pm\infty$, they vanish without further assumptions. The last two integrals in the sum would vanish without futher assumptions on (u, v) if and only if

$$
\begin{aligned}
2Ba + Eb &= 2Ab + 4Dc, \\
Bb + 2Ec &= 4Ca + 2Fb. \qquad (2.6)
\end{aligned}
$$

Thus a, b and c must solve the two equations

$$\begin{aligned} 2Ba + \Big(E - 2A\Big)b - 4Dc &= 0, \\ 4Ca + \Big(2F - B\Big)b - 2Ec &= 0, \end{aligned} \tag{2.7}$$

for it to be the case that the time derivative above is zero. This pair of equations always has a non-trivial solution.

For such values of a, b and c, Ω is an invariant of the temporal evolution of smooth solutions of the given system. Furthermore one can easily see by calculation that

$$\begin{aligned} \frac{\partial}{\partial u}\Big(bP + 2cQ\Big) &= 2\Big(Ab + 2Dc\Big)u + \Big(Bb + 2Ec\Big)v, \\ \frac{\partial}{\partial v}\Big(2aP + bQ\Big) &= \Big(2Ba + Eb\Big)u + 2\Big(2Ca + Fb\Big)v. \end{aligned}$$

Therefore, according to (2.7),

$$\frac{\partial}{\partial u}\Big(bP + 2cQ\Big)(u, v) = \frac{\partial}{\partial v}\Big(2aP + bQ\Big)(u, v).$$

It follows that there is a cubic polynomial $R(u, v)$ such that

$$\frac{\partial R}{\partial u} = 2aP + bQ \quad \text{and} \quad \frac{\partial R}{\partial v} = bP + 2cQ. \tag{2.8}$$

2.3.2 Second Conservation Law

For the same a, b and c, we search for a second Conservation Law. To calculate the time derivative of the functional

$$\int_{-\infty}^{\infty} \left(au^2 + buv + cv^2\right) dx,$$

integration by parts and the assumption that (u, v) is a solution of the system which, along with their first few derivatives, do not have any contribution at $x = \pm\infty$. Using the relations

$$\int u_t(u - u_{xt} + P) = 0,$$

$$\int u_t(v - v_{xt} + Q) + v_t(u - u_x + P) = 0,$$

$$\int v_t(v - v_{xt} + Q) = 0,$$

it follows that

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Big(au^2+buv+cv^2\Big)dx &= \int\Big(2auu_t+bu_tv+buv_t+2cvv_t\Big)dx = \\
=&2a\int uu_t dx + b\int\big(u_tv+uv_t\big)dx+2c\int vv_t dx\\
=&2a\int\big[u_t(u-u_{xt}+P)\big]dx+2a\int u_tu_{xt}dx-2a\int u_tPdx\\
&+b\int u_t\big(v-v_{xt}+Q\big)dx+b\int u_tv_{xt}dx-b\int u_tQdx\\
&+b\int v_t\big(u-u_{xt}+P\big)dx+b\int v_tu_{xt}dx-b\int v_tPdx\\
&+2c\int\Big[v_t\big(v-v_{xt}+Q\big)\Big]dx+2c\int v_tv_{xt}dx-2c\int v_tQdx\\
=&-\int\Big[\big(2aP+bQ\big)u_t+\big(bP+2cQ\big)v_t\Big]dx+b\int\frac{d}{dx}(u_tv_t)dx\\
=&-\int\left(\frac{\partial R}{\partial u}u_t+\frac{\partial R}{\partial v}v_t\right)dx\\
=&-\frac{d}{dt}\int R\,dx.
\end{aligned}$$

Consequently, we find that

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big(au^2+buv+cv^2+R(u,v)\Big)dx=0.$$

Therefore if the functional $\Phi(u,v)$ is defined by

$$\Phi(u,v):=\int_{-\infty}^{\infty}\Big(au^2+buv+cv^2+R(u,v)\Big)dx, \tag{2.9}$$

then $\Phi(u, v)$ is also an invariant of the temporal evolution of smooth solutions of our system and it serves as a Hamiltonian for the given system as long as $4ac - b^2$ is not 0.

Both these conserved quantities Ω and Φ are going to be used further for obtaining *a priori* bounds on solutions of the system when the quadratic form q defined by

$$q(u, v) = au^2 + buv + cv^2, \tag{2.10}$$

vanishes only at origin which is the case exactly when its discriminant is negative. Of course, neither a nor c can be 0 if the discriminant is negative. Without loss of generality, assume both a and c are positive.

2.4 Positive Definite Condition on the Coefficients

Referring back to the system (2.6), the coefficient matrix is

$$N = \begin{pmatrix} 2B & E - 2A & -4D \\ 4C & 2F - B & -2E \end{pmatrix}.$$

When $rank(N) = 2$, then a, b, c are given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} E(E - 2A) + 2D(B - 2F) \\ -2BE + 8CD \\ B(B - 2F) + 2C(E - 2A) \end{pmatrix},$$

up to a non-zero scalar multiple.

By calculation, it easily follows that for these values, if

$$\begin{aligned} 4ac =& BE(E-2A)(B-2F) + 2CE(E-2A)^2 + 2BD(B-2F)^2 \\ &+ 4CD(E-2A)(B-2F) \\ =& (BE+4CD)(E-2A)(B-2F) + 2CE(E-2A)^2 + 2BD(B-2F)^2 \\ >& (BE-4CD)^2 = b^2; \end{aligned} \tag{2.11}$$

then the quadratic form q defined in (2.10) is positive definite.

The case $rank(N) = 1$ has three subcases, the complete analysis dependents on the quantity of $(E-2A)^2 + 8BD$ or $(2F-B)^2 + 8CE$. If the bigger one is strictly positive, the system (1.1) can be decoupled, the positive definite condition holds true. If the bigger quantity is equal to 0 then both are equal to zero and the method used is not applicable. A different method would show that the global well-posedness holds true. If smallest one of the quantities is negative, the result is not true anymore. The system can be written as a single complex BBM-equation instead of two equations. Numerical methods show that the system's solution blows up in finite time. The details will be pursued later as a companion result of this thesis and omitted here. However, as a remark, a few particular examples of the coefficient choices are presented.

In the case that $B = C = D = E = 0$ and $A, F \neq 0$ then both quantities $(E-2A)^2+8BD$ and $(2F - B)^2 + 8CE$ are strictly positive, the initial system is decoupled and by choosing $a = c = 1$ and $b = 0$, one obtains a time invariant Ω under the flow generated by the (1.1).

In the case that $A = \frac{1}{2}$, $B = C = D = F = 0$, and $E = 1$ then both quantities $(E - 2A)^2 + 8BD$ and $(2F - B)^2 + 8CE$ are equal to zero. The system looks as follows

$$\begin{cases} u_t + u_x - u_{xxt} + uu_x = 0, \\ v_t + v_x - v_{xxt} + (uv)_x = 0, \end{cases} \tag{2.12}$$

The local and global well-posedness results hold true.

In the case that $A = \frac{1}{2}$, $B = D = F = 0$, $C = -\frac{1}{2}$ and $E = 1$ then $(E - 2A)^2 + 8BD$ and $(2F - B)^2 + 8CE$ are equal to zero. The system looks as follows

$$\begin{cases} u_t + u_x - u_{xxt} + uu_x - vv_x = 0, \\ v_t + v_x - v_{xxt} + (uv)_x = 0, \end{cases} \tag{2.13}$$

or equivalently, as a complex-BBM equation

$$(u + iv)_t + (u + iv)_x - (u + iv)_{xxt} + (u + iv)(u + iv)_x = 0. \tag{2.14}$$

This equation is not globally well-posed, based on numerical calculations.

With a, b and c determined, the next step is to find the coefficients of the cubic polynomial

$$R(u, v) := \alpha u^3 + \beta u^2 v + \gamma uv^2 + \delta v^3,$$

appearing in the second conservation law. A calculation yields

$$\alpha = \frac{1}{3}(2aA + bD), \quad \beta = bA + 2cD,$$
$$\gamma = 2aC + bF, \quad \delta = \frac{1}{3}(bC + 2cF). \tag{2.15}$$

Because of (2.6), β and γ may also be written as

$$\beta = aB + \frac{1}{2}bE, \quad \gamma = \frac{1}{2}bB + cE.$$

CHAPTER 3

WELL-POSEDNESS IN INTEGER ORDER SOBOLEV SPACES

3.1 Local Well Posedness

Local well-posedness is proved via the Contraction Mapping Principle.

3.1.1 Local Well-Posedness in $L_\infty \times L_\infty$

Consider the space

$$X_T := L_\infty\big(\mathbb{R} \times [0,T]\big) \times L_\infty\big(\mathbb{R} \times [0,T]\big) \tag{3.1}$$

with its usual norm,

$$\left\|U - \tilde{U}\right\|_{X_T} := \sup_{x\in\mathbb{R}, t\in[0,T]} |u(x,t) - \tilde{u}(x,t)| + \sup_{x\in\mathbb{R}, t\in[0,T]} |v(x,t) - \tilde{v}(x,t)|, \tag{3.2}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{and} \qquad \tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

$(X_T, \|.\|_{X_T})$ is a Banach space. To establish local existence via the Contraction Mapping Principle, the domain for the operator $\mathcal{A}$ will be restricted to the closed subset

$$B_R(0) := \{U \in X_T : \|U\|_{X_T} \leq R\},$$

where $R > 0$ and $T > 0$ will be determined presently.

Consider $U, \tilde{U} \in B_R(0)$, and look for an estimate of $\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T}$. The aim is to choose specific values of R and T so that $\mathcal{A}$ is a contraction on $B_R(0)$. Using the definition of the operator $\mathcal{A}$, one calculates thusly for $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v})$

$$\begin{aligned}
\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T} =& \sup_{x\in\mathbb{R}} \left| \int_0^T \int_{-\infty}^{\infty} K(x-y)\big[u(y,t) + P(u,v) - \tilde{u}(y,t) - P(\tilde{u}.\tilde{u})\big]\, dyds \right| \\
&+ \sup_{x\in\mathbb{R}} \left| \int_0^T \int_{-\infty}^{\infty} K(x-y)\big[v(y,t) + Q(u,v) - \tilde{v}(y,t) - Q(\tilde{u}.\tilde{u})\big]\, dyds \right| \\
\leq& \sup_{x\in\mathbb{R}} \int_0^T \int_{-\infty}^{\infty} \Big|K(x-y)\Big|\Big|(u-\tilde{u})\big(1 + A(u+\tilde{u}) + Bv\big) \\
&+ (v-\tilde{v})\big(C(v+\tilde{v}) + B\tilde{u}\big)\Big| dyds \\
&+ \sup_{x\in\mathbb{R}} \int_0^T \int_{-\infty}^{\infty} \Big|K(x-y)\Big|\Big|(u-\tilde{u})\big(D(u+\tilde{u}) + E\tilde{v}\big) \\
&+ (v-\tilde{v})\big(1 + F(v+\tilde{v}) + Eu\big)\Big| dyds \\
\leq& \sup_{x\in\mathbb{R}} \int_0^T \int_{-\infty}^{\infty} |K(x-y)| \Big[|u-\tilde{u}|\big(1 + |A|(|u| + |\tilde{u}|) + |B||v|\big) \\
&+ |v-\tilde{v}|\big(|C|(|v| + |\tilde{v}|) + |B||\tilde{u}|\big)\Big]\, dyds \\
&+ \sup_{x\in\mathbb{R}} \int_0^T \int_{-\infty}^{\infty} |K(x-y)| \Big[|u-\tilde{u}|\big(|D|(|u| + |\tilde{u}|) + |E||\tilde{v}|\big) \\
&+ |v-\tilde{v}|\big(1 + |F|(|v| + |\tilde{v}|) + |E||u|\big)\Big]\, dyds. \qquad (3.3)
\end{aligned}$$

Furthermore, as $U, \tilde{U} \in B_R(0)$, it follows that the inequality can be continued, *viz.*

$$\begin{aligned} \left\| \mathcal{A}U - \mathcal{A}\tilde{U} \right\|_{X_T} \leq \ & c_R^1 \sup_{x \in \mathbb{R}} \int_0^T \int_{-\infty}^{\infty} \left| K(x-y) \right| \left[|u - \tilde{u}| + |v - \tilde{v}| \right] dyds \\ & + c_R^2 \sup_{x \in \mathbb{R}} \int_0^T \int_{-\infty}^{\infty} \left| K(x-y) \right| \left[|u - \tilde{u}| + |v - \tilde{v}| \right] dyds \\ \leq \ & \left\| U - \tilde{U} \right\|_{X_T} C_R \ \sup_{x,t} \int_0^t \int_{-\infty}^{\infty} |K| dyds \\ \leq \ & \left\| U - \tilde{U} \right\|_{X_T} C_R \ T, \end{aligned} \tag{3.4}$$

where c_R^1, c_R^2 and C_R are constantsonly dependant on R. The choices of R and T are to be made such that $\mathcal{A}$ maps $B_R(0)$ into itself and so that $\mathcal{A}$ is a contraction on $B_R(0)$. In particular, one would need to have

$$C_R \ T < 1.$$

Choose T and R so that

$$C_R \ T = \frac{1}{2},$$

for example. For any $U \in B_R(0)$, we need $\|\mathcal{A}U\|_{X_T} \leq R$. Since

$$\begin{aligned} \|\mathcal{A}U\|_{X_T} &= \|\mathcal{A}U - 0\|_{X_T} \leq \|\mathcal{A}U - \mathcal{A}0\|_{X_T} + \|\mathcal{A}0 - 0\|_{X_T} \\ &\leq \frac{1}{2} \|U - 0\|_{X_T} + \|U_0\|_{\infty} \leq \frac{1}{2} R + \|U_0\|_{\infty} \leq R, \end{aligned} \tag{3.5}$$

which is satisfied by choosing $R = 2 \ \|U_0\|_\infty$. Then, the choice

$$T = \frac{1}{2 \ C_R},$$

allows us to apply the Contraction Mapping Theorem to the integral equation (2.3). The result is existence of a unique $U \in X_T$ that is solution of the given system.

3.1.2 Local Existence in Other Spaces

Lemma 1. *Suppose that Z is a space embedded in L_∞ such that, for any $u, v \in Z$, there exist global constants C_1 and C_2 satisfying*

$$(i) \quad \|uv\|_Z \le C_1 \|u\|_Z \|v\|_Z \qquad \textit{and} \qquad (ii) \quad \left\| \int_{-\infty}^{\infty} K(x-y)u(y)dy \right\|_Z \le C_2 \|u\|_Z . \tag{3.6}$$

Then there is a unique local solution for the initial-value problem for (1.1) in the space $C\big(0, T; Z \times Z\big)$.

Proof. The space $C\big(0, T; Z \times Z\big)$ carries the norm

$$\left\| U - \tilde{U} \right\|_{C(0,T;Z\times Z)} := \sup_{t\in[0,T]} \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_Z + \sup_{t\in[0,T]} \|v(\cdot,t) - \tilde{v}(\cdot,t)\|_Z , \tag{3.7}$$

where $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v})$.

Let $Z_T := C(0, T; Z)$. Then for the easiness of notation, the $C(0, T; Z \times Z)$-norm and the $Z \times Z$-norm of a vector U will respectively be denoted by $\|U\|_{Z_T}$ and $\|U\|_Z$ henceforth.

The collection $C\big(0,T;Z\times Z\big)$ is a Banach space of course. Restrict the operator $\mathcal{A}$ defined in (2.5) to

$$B_{R,Z}(0) := \{U \in C(0,T;Z\times Z) : \|U\|_{Z_T} \le R\}.$$

Consider $U,\tilde{U} \in B_{R,Z}(0)$, and look for an estimate of $\left\|U-\tilde{U}\right\|_{Z_T}$. The aim is to choose specific values of R and T so that $\mathcal{A}$ is a contraction on $B_{R,Z}(0)$. Notice that

$$\begin{aligned}
\left\|\mathcal{A}U-\mathcal{A}\tilde{U}\right\|_{Z_T} &= \sup_{t\in[0,T]}\left\|\int_0^t\int_{-\infty}^{\infty}K(x-y)\Big[u(y,s)+P(u,v)-\tilde{u}(y,s)-P(\tilde{u}.\tilde{u})\Big]\,dyds\right\|_Z \\
&\quad+\sup_{t\in[0,T]}\left\|\int_0^t\int_{-\infty}^{\infty}K(x-y)\Big[v(y,s)+Q(u,v)-\tilde{v}(y,s)-Q(\tilde{u}.\tilde{u})\Big]\,dyds\right\|_Z \\
&\le \sup_{t\in[0,T]}\int_0^t\left\|K*\Big[(u-\tilde{u})(1+A(u+\tilde{u})+Bv)+(v-\tilde{v})(C(v+\tilde{v})+B\tilde{u})\Big]\right\|_Z ds \\
&\quad+\sup_{t\in[0,T]}\int_0^t\left\|K*\Big[(u-\tilde{u})(D(u+\tilde{u})+E\tilde{v})+(v-\tilde{v})(1+F(v+\tilde{v})+Eu)\Big]\right\|_Z ds \\
&\le T\left\|\int_{-\infty}^{\infty}K(x-y)\big[(u-\tilde{u})(1+A(u+\tilde{u})+Bv)+(v-\tilde{v})(C(v+\tilde{v})+B\tilde{u})\big]\,dy\right\|_{Z_T} \\
&\quad+T\left\|\int_{-\infty}^{\infty}K(x-y)||\big[(u-\tilde{u})(D(u+\tilde{u})+E\tilde{v})+(v-\tilde{v})(1+F(v+\tilde{v})+Eu)\big]dy\right\|_{Z_T}.
\end{aligned}$$

Applying (ii) from (3.6), the triangle inequality and (3.6) part (i) yields

$$\begin{aligned}
\left\|\mathcal{A}U-\mathcal{A}\tilde{U}\right\|_{Z_T} \leq\ & TC_2\Big[\left\|(u-\tilde{u})\Big(1+A(u+\tilde{u})+Bv\Big)+(v-\tilde{v})\Big(C(v+\tilde{v})+B\tilde{u}\Big)\right\|_{Z_T} \\
&+\left\|(u-\tilde{u})\Big(D(u+\tilde{u})+E\tilde{v}\Big)+(v-\tilde{v})\Big(1+F(v+\tilde{v})+Eu\Big)\right\|_{Z_T}\Big] \\
\leq\ & TC_2\Big[\left\|u-\tilde{u}\right\|_{Z_T}+\left\|(u-\tilde{u})\Big(A(u+\tilde{u})+Bv\Big)\right\|_{Z_T}+\left\|(v-\tilde{v})\Big(C(v+\tilde{v})+B\tilde{u}\Big)\right\|_{Z_T} \\
&+\left\|(u-\tilde{u})\Big(D(u+\tilde{u})+E\tilde{v}\Big)\right\|_{Z_T}+\left\|v-\tilde{v}\right\|_{Z_T}+\left\|(v-\tilde{v})\Big(F(v+\tilde{v})+Eu\Big)\right\|_{Z_T}\Big] \\
\leq\ & TC_2\Big[\left\|u-\tilde{u}\right\|_{Z_T}+C_1\left\|u-\tilde{u}\right\|_{Z_T}\left\|A(u+\tilde{u})+Bv\right\|_{Z_T}+C_1\left\|v-\tilde{v}\right\|_{Z_T}\left\|C(v+\tilde{v})+B\tilde{u})\right\|_{Z_T} \\
&+C_1\left\|u-\tilde{u}\right\|_{Z_T}\left\|D(u+\tilde{u})+E\tilde{v}\right\|_{Z_T}+\left\|v-\tilde{v}\right\|_{Z_T}+C_1\left\|v-\tilde{v}\right\|_{Z_T}\left\|F(v+\tilde{v})+Eu)\right\|_{Z_T}\Big] \\
\leq\ & TC_2\Big[\left\|u-\tilde{u}\right\|_{Z_T}+C_1\left\|u-\tilde{u}\right\|_{Z_T}\Big(|A|(\left\|u\right\|_{Z_T}+|A|\left\|\tilde{u}\right\|_{Z_T}+|B|\left\|v\right\|_{Z_T}\Big) \\
&+C_1\left\|v-\tilde{v}\right\|_{Z_T}\Big(|C|(\left\|v\right\|_{Z_T}+|C|\left\|\tilde{v}\right\|_{Z_T}+|B|\left\|\tilde{u}\right\|_{Z_T}\Big) \\
&+C_1\left\|u-\tilde{u}\right\|_{Z_T}\Big(|D|(\left\|u\right\|_{Z_T}+|D|\left\|\tilde{u}\right\|_{Z_T}+|E|\left\|\tilde{v}\right\|_{Z_T}\Big)+\left\|v-\tilde{v}\right\|_{Z_T} \\
&+C_1\left\|v-\tilde{v}\right\|_{Z_T}\Big(|F|\left\|v\right\|_{Z_T}+|F|\left\|\tilde{v}\right\|_{Z_T}+|E|\left\|u\right\|_{Z_T}\Big)\Big].
\end{aligned} \tag{3.8}$$

As $U,\tilde{U}\in B_{R,Z}(0)$, this can be further bounded, *viz.*

$$\begin{aligned}
\left\|\mathcal{A}U-\mathcal{A}\tilde{U}\right\|_{Z_T} \leq\ & TC_2\Big[\left\|u-\tilde{u}\right\|_{Z_T}\Big(1+C_1R(2|A|+|B|+2|D|+|E|)\Big) \\
&+\left\|v-\tilde{v}\right\|_{Z_T}\Big(1+C_1R(2|C|+|B|+2|F|+|E|)\Big)\Big] \\
\leq\ & TC_2\big[1+C_R\big]\Big(\left\|u-\tilde{u}\right\|_{Z_T}+\left\|v-\tilde{v}\right\|_{Z_T}\Big) \\
=\ & TC_2\big[1+C_R\big]\left\|U-\tilde{U}\right\|_{Z_T}.
\end{aligned}$$

Let $R = 2\left\|U_0\right\|_Z$ and demand that

$$TC_2\big(1 + C_R\big) = \frac{1}{2}.$$

Then $\mathcal{A}$ is contractive on $B_{R,Z}(0)$ and for any $U \in B_R(0)$,

$$\|\mathcal{A}U\|_{Z_T} \leq R.$$

based on the same argument as in (3.5). Therefore, the Contraction Mapping Theorem can be applied to the integral equation (2.3) and the result is a unique U that is solution of the given system. $\square$

This Lemma provides the local well-posedness on H^1 , H^2 and in general in H^s, for $s > \frac{1}{2}$ as they are Banach algebras which satisfy both (i) and (ii) in (3.6).

3.2 Smoothness and Regularity

Consider $U_0, V_0 \in L_\infty(\mathbb{R})$ as initial data, fix $T > 0$, X_T defined as in (3.1) and let $U, V \in X_T$ be the corresponding solutions of the initial-value problem (1.1) - (1.2). To check the continuous dependence of solutions on the initial data, start locally. The overall conclusion follows from iteration of the argument below:

$$U = \mathcal{A}_{U_0}U = U_0 + \int_0^t \int_{-\infty}^{\infty} K(x-y)\Big(U(y,s) + M\Big)dyds,$$

and

$$V = \mathcal{A}_{V_0}V = V_0 + \int_0^t \int_{-\infty}^{\infty} K(x-y)\Big(V(y,s) + M\Big)dyds.$$

Using the fact that $\mathcal{A}$ is a contraction in X_T, at least for small T,

$$\begin{aligned} \|U - V\|_{X_T} &= \|\mathcal{A}_{U_0}U - \mathcal{A}_{V_0}V\|_{X_T} \\ &\leq \|\mathcal{A}_{U_0}U - \mathcal{A}_{U_0}V\|_{X_T} + \|\mathcal{A}_{U_0}V - \mathcal{A}_{V_0}V\|_{X_T} \\ &\leq \theta \|U - V\|_{X_T} + \|U_0 - V_0\|_{\infty}, \end{aligned}$$

where $0 < \theta < 1$. Therefore,

$$\|U - V\|_{X_T} \leq \frac{1}{1-\theta} \|U_0 - V_0\|_{\infty}$$

and the mapping of the initial data U_0 to the solution U is seen to be locally Lipschitz continuous. Moreover, as the initial-value problem (1.1) - (1.2) was solved locally thus far by converting it into a system of integral equations and applying a Picard iteration type of approach, there is automatically strong regularity results for the flow map. The argument for smoothness of the flow map has been presented in several contexts before (see, *e.g.* Bona, Sun and Zhang [7] or Bona and Tzvetkov [1]).

Notice that if $U_0 \in C_b(\mathbb{R}) \times C_b(\mathbb{R})$, the solution determined via the last proposition lies in $C([0,T] \times \mathbb{R}) \times C([0,T] \times \mathbb{R})$.

Let U be a continuous bounded solution of the initial-value problem (1.1) - (1.2). Since U solves the system of integral equations it follows immediately that U is differentiable with respect to t and that

$$U_t(x,t) = \int_{-\infty}^{\infty} K(x-y)\Big[U(y,t) + M\big(U(y,t)\big)\Big]\, dy,$$

or, what is the same,

$$U_t(x,t) = -(I-\partial_x^2)^{-1}\partial_x\Big[U(x,t) + M\big(U(x,t)\big)\Big].$$

Clearly U_t is a C^1-function in both x and t and therefore

$$U_{tt}(x,t) = \int_{-\infty}^{\infty} K(x-y)\Big[U_t(y,t) + M_t\Big]\, dy.$$

The latter is also a C^1-function in x and t. By induction, one can conclude that U is a C^∞-function in t and $\partial_t^k U$ is a C^1-function in x for any $k \geq 1$.

Considering the spatial regularity, start with the fixed point $U = \mathcal{A}U$ of the system (2.3) written as

$$U(x,t) = U_0 + \int_0^t K * \Big[U(y,s) + M\Big] ds \tag{3.9}$$

and initial data U_0 that lies in $C^1_b(\mathbb{R}) \times C^1_b(\mathbb{R})$. This is an important assumption which will influence the conclusion drawn at the end of this paragraph. Then U is also a C^1 function and direct calculation yields

$$\partial_x \Big[K * (U+M)\Big] = \big(U+M\big) - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big[U(y,t)+M\Big]dy. \tag{3.10}$$

Moreover, as $U+M$ is C^1, (3.10) can be differentiated with respect to x to obtain

$$\begin{aligned}\partial_x^2 \Big[K * \big(U+M\big)\Big] &= \big(U+M\big)_x + \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sgn}(x-y)e^{-|x-y|}\Big[U(y,t)+M\Big]dy \\ &= \big(U+M\big)_x + \Big[K * (U+M)\Big].\end{aligned}$$

Therefore,

$$(1-\partial_x^2)\big[K * (U+M)\big] = -(U+M)_x,$$

so U will always have one derivative and the convolution on the RHS makes it even smoother. Consequently, by substituting the result above in (3.9), we find that

$$(1-\partial_x^2)\,U_t = \;\; (U+M)_x.$$

Thus U is a solution of the initial system (1.1) and it has reasonable regularity.

However, it is easily shown that U can not have more spacial regularity than the initial data U_0. Assume that for some $t>0$, $U(\cdot,t)$ is a C^k-function in x while $U_0(\cdot,t)$ is a $C^{k-1}\setminus C^k$-

function in x. Then $(U+M)$ is a C^k-function in x and the result after convolution with K, would be a C^{k+1}-function in x. Integrating this function with respect to t, would still preserve the C^{k+1} property. Therefore from (3.9) it follows that $U_0(\cdot, t)$ is a C^k-function in x, contrary to the assumption made. $\square$

Furthermore, the smoothing associated with the temporal derivative for $U \in H^k \times H^k$ is approached just as in the L_∞ case, to conclude that U_t is spatially smoother and it lies in $C(0, \infty; H^{k+1} \times H^{k+1})$. As for the spatial regularity, the system solution has at most the same regularity as the initial data.

3.3 *A* priori Estimates and Global Well Posedness for Smooth Solutions

As seen in the previous sections, local well-posedness theory doesn't depend on assumptions about the coefficients of P and Q. To move to global in time well-posedness, *a priori* estimates for the solution are needed. These will be derived below assuming that u and v are smooth solutions of the system which are not growing exponentially for $x \to \pm\infty$ and assuming the condition for the quadratic form q to be positive definite is satisfied by its coefficients (see section 2.4).

In this case, the integrand of the Ω is positive definite too. Consequently, there is a $\lambda > 0$ such that

$$\int_{-\infty}^{\infty} \Big[u^2(x,t) + v^2(x,t) + u_x^2(x,t) + v_x^2(x,t)\Big] dx \leq \lambda\, \Omega(u, v) = \lambda\, \Omega(u_0, v_0) = M_1. \tag{3.11}$$

3.3.1 H^1- Bounds

Assume now that the initial condition (u_0, v_0) is in $H^1 \times H^1$ and let (u, v) be the solution pair of our system corresponding to this data. By definition,

$$\|(u,v)\|^2_{H^1\times H^1} = \int_{-\infty}^{\infty} \left(u^2 + v^2 + u_x^2 + v_x^2\right) dx.$$

Hence, the relation (3.11) can also be written as

$$\|(u,v)\|^2_{H^1\times H^1} \leq M_1, \tag{3.12}$$

so the $H^1 \times H^1$- norm of (u, v) is also uniformly bounded independently of t.

Therefore the local well-posedness result obtained in section 3.1.2 can be extended to conclude the existence of globally defined solution that lies in $C(0, T; H^1 \times H^1)$ for any $T > 0$. The solution is unique and depends continuously on the initial data in $H^1 \times H^1$.

3.3.2 Differential Form of the Gronwall Lemma

This well-known result will be used going forward ([8], page 708).

Lemma 2. *(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$ which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\,\eta(t) + \psi(t), \tag{3.13}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \le e^{\int_0^t \phi(s)\,ds}\left(\eta(0) + \int_0^t \psi(s)\,ds\right) \tag{3.14}$$

for all $0 \le t \le T$.

3.3.3 H^2 - Bounds

Consider again the solution (u,v) of the system (1.1), whose first few partial derivatives lie in $L_2 \times L_2$. To estimate the growth of the $H^2 \times H^2$-norm of a solution $U = (u,v)$, consider the following calculation:

$$\begin{aligned}
\frac{d}{dt}\int \left(u_x^2 + v_x^2 + u_{xx}^2 + v_{xx}^2\right)dx = \; & 2\int (u_x u_{xt} + v_x v_{xt} + u_{xx}u_{xxt} + v_{xx}v_{xxt})dx \\
= \; & 2\int \Big(- u_t u_{xx} + u_{xx}u_{xxt} - v_t v_{xx} + v_{xx}v_{xxt}\Big)dx \\
= \; & 2\int \Big(u_{xx}(u_{xxt} - u_t) + v_{xx}(v_{xxt} - v_t)\Big)dx \\
= \; & \int \Big(u_{xx}(u_x + P_x) + v_{xx}(v_x + Q_x)\Big)dx \\
= \; & 2\int \Big(u_{xx}P_x + v_{xx}Q_x\Big)dx \\
= \; & \int \Big(2Auu_xu_{xx} + Bvu_xu_{xx} + Buv_xu_{xx} + 2Cvv_xu_{xx} + \\
& + 2Duu_xv_{xx} + Euv_xv_{xx} + Evu_xv_{xx} + 2Fvv_xv_{xx}\Big)dx.
\end{aligned} \tag{3.15}$$

The integrand on the RHS is a polynomial whose general monomial term is of the form rs_xw_{xx} where r, s, w are either u or v. To obtain a bound on the $H^2 \times H^2$ norm of the solution, let M_1

be the previously derived time-independent bound in $H^1 \times H^1$. Then, elementary estimates imply that

$$\left| \int (r s_x w_{xx}) dx \right| \leq \|r\|_\infty \|s_x\|_2 \|w_{xx}\|_2 \leq \|r\|_2^{1/2} \|r_x\|_2^{1/2} \|s_x\|_2 \|w_{xx}\|_2 \quad \leq M_1^2 \|w_{xx}\|_2 . \quad (3.16)$$

Let

$$N(t) = \left[\int \Big(u_x^2(x,t) + v_x^2(x,t) + u_{xx}^2(x,t) + v_{xx}^2(x,t) \Big) dx \right]^{\frac{1}{2}}$$

Applying the estimate (3.16) systematically to (3.15) yields

$$\begin{aligned} \frac{d}{dt} N^2(t) &\leq C_1 (\|u_{xx}(\cdot,t)\|_2 + \|v_{xx}(\cdot,t)\|_2) \\ &\leq C_1 \, N(t), \end{aligned}$$

where C_1 is a constant that depends on M_1 and on the absolute values of the polynomial coefficients, but is time-independent. Hence,

$$2N(t) \, \frac{d}{dt} N(t) \leq C_1 \, N(t)$$

implies that

$$\frac{d}{dt} N(t) \leq \frac{C_1}{2}.$$

It follows that

$$N(t) \leq N(0) + \frac{1}{2} C_1 \, t, \quad (3.17)$$

for any $t > 0$.

The relation (3.17) together with (3.12) lead to

$$\|(u(\cdot,t),v(\cdot,t))\|^2_{H^1\times H^1} + N^2(t) \leq M_1 + \Big(N(0) + \frac{1}{2}C_1\,t\Big)^2$$

thus

$$\|(u(\cdot,t),v(\cdot,t))\|^2_{H^2\times H^2} \leq M_1 + \Big(N(0) + \frac{1}{2}C_1\,t\Big)^2.$$

Hence

$$\|(u(\cdot,t),v(\cdot,t))\|_{H^2\times H^2} \leq C_2 + C_3\,t,$$

where $C_2, C_3 > 0$ are t-independent constants that depend on M_1 and $N(0)$, so on $H^2 \times H^2$-norm of $(u(\cdot,0),v(\cdot,0))$.

Therefore the local well-posedness result in time obtained in section 3.1.2 can be extended to any time interval $[0,T]$. It concludes the existence of globally defined solution in $C(0,\infty;H^2 \times H^2)$. The solution is unique and depends continuously on the initial data in $H^2 \times H^2$.

3.3.4 H^k - Bounds

The *a priori* $H^k \times H^k$ bounds are determined inductively. The claim is that for any positive integer k, the $H^k \times H^k$ norm of the solution is bounded on bounded time intervals.

The cases $k = 1$ and $k = 2$ are in hand. Assuming now that for every j with $2 \leq j \leq k$, there is a positive constant

$$c_{j-1} = c_{j-1}\Big(\big\|\big(u(\cdot,0), v(\cdot,0)\big)\big\|_{H^{j-1}\times H^{j-1}}\,; T\Big)$$

such that for any solution (u, v) of the system (1.1), the inequality

$$\|(u(\cdot,t), v(\cdot,t)\|_{H^{j-1}\times H^{j-1}} \leq c_{j-1} \tag{3.18}$$

holds. Then there is a positive constant

$$c_k = c_k\big(\|(u(\cdot,0), v(\cdot,0))\|_{H^k\times H^k}\,; T\big)$$

such that for $0 \leq t \leq T$

$$\|(u(\cdot,t), v(\cdot,t))\|_{H^k\times H^k} \leq c_k. \tag{3.19}$$

Proof. To begin with, integration by parts and (1.1) implies

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\left[\left(\partial_x^{(k-1)}u\right)^2+\left(\partial_x^{(k)}u\right)^2\right]dx &= \int\left(\partial_x^{(k-1)}u\ \ \partial_x^{(k-1)}u_t+\partial_x^{(k)}u\ \ \partial_x^{(k)}u_t\right)dx\\
&= \int\left(\partial_x^{(k-1)}u\ \ \partial_x^{(k-1)}u_t-\partial_x^{(k-1)}u\ \ \partial_x^{(k+1)}u_t\right)dx\\
&= \int\partial_x^{(k-1)}u\ \ \left(\partial_x^{(k-1)}u_t-\partial_x^{(k-1)}u_{xxt}\right)dx\\
&= -\int\partial_x^{(k-1)}u\ \ \partial_x^{(k-1)}\left(u_x-P_x\right)dx\\
&= -\int\left(\partial_x^{(k-1)}u\right)\left(\partial_x^{(k)}P\right)dx\\
&= \int\left(\partial_x^{(k)}u\right)\left(\partial_x^{(k-1)}P\right)dx,\\
\frac{1}{2}\frac{d}{dt}\int\left[\left(\partial_x^{(k-1)}v\right)^2+\left(\partial_x^{(k)}v\right)^2\right]dx &= \int\left(\partial_x^{(k)}v\right)\left(\partial_x^{(k-1)}Q\right)dx
\end{aligned}
$$

The results above will be used in the following estimate

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int&\left[\left(\partial_x^{(k-1)}u\right)^2+\left(\partial_x^{(k-1)}v\right)^2+\left(\partial_x^{(k)}u\right)^2+\left(\partial_x^{(k)}v\right)^2\right]dx =\\
&\le\int\left(\partial_x^{(k)}u\right)\left(\partial_x^{(k-1)}P\right)+\left(\partial_x^{(k)}v\right)\left(\partial_x^{(k-1)}Q\right)dx. \qquad (3.20)
\end{aligned}
$$

The next step is to estimate the monomials that the polynomial above is made of. These have the form

$$\begin{aligned}
\int \left| \left(\partial_x^{(k)} w\right) \left(\partial_x^{(j)} r\right) \left(\partial_x^{(k-1-j)} s\right) \right| dx &\le \left\| \partial_x^{(k)} w \right\|_2 \left\| \partial_x^{(j)} r \right\|_\infty \left\| \partial_x^{(k-1-j)} s \right\|_2 \\
&\le \left\| \partial_x^{(k)} w \right\|_2 \left\| \partial_x^{(j)} r \right\|_{H^1} \|s\|_{H^{k-1-j}} \\
&\le \left\| \partial_x^{(k)} w \right\|_2 \|r\|_{H^j} \|s\|_{H^{k-1-j}} \\
&\le \left\| \partial_x^{(k)} w \right\|_2 \|r\|_{H^j} \|s\|_{H^{k-1-j}} \\
&\le c_j c_{k-1-j} \left\| \partial_x^{(k)} w \right\|_2 , \qquad (3.21)
\end{aligned}$$

where r, s and w are either u or v. Thus,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int \left[\left(\partial_x^{(k-1)} u\right)^2 + \left(\partial_x^{(k-1)} v\right)^2 + \left(\partial_x^{(k)} u\right)^2 + \left(\partial_x^{(k)} v\right)^2 \right] dx \\
&\quad \le c \left(\left\| \partial_x^{(k)} u \right\|_2 + \left\| \partial_x^{(k)} v \right\|_2 \right) \\
&\quad \le c \left(\int \left[\left(\partial_x^{(k-1)} u\right)^2 + \left(\partial_x^{(k-1)} v\right)^2 + \left(\partial_x^{(k)} u\right)^2 + \left(\partial_x^{(k)} v\right)^2 \right] dx \right)^{\frac{1}{2}} , \qquad (3.22)
\end{aligned}$$

where c is a constant. Solving the ordinary differential inequality yields

$$\left\| \partial_x^{(k-1)} u \right\|_2 + \left\| \partial_x^{(k-1)} v \right\|_2 + \left\| \partial_x^{(k)} u \right\|_2 + \left\| \partial_x^{(k)} v \right\|_2 \le c\,T.$$

This together with (3.18) indicates the following:

$$\|(u(.,t),v(.,t))\|_{H^k\times H^k} \leq \|(u(.,0),v(.,0))\|_{H^k\times H^k} + cT. \tag{3.23}$$

Therefore, there is a positive constant

$$c_k = c_k\big(\,\|(u(.,0),v(.,0))\|_{H^k\times H^k}\,;T\big)$$

such that

$$\|(u(.,t),v(.,t))\|_{H^k\times H^k} \leq c_k. \tag{3.24}$$

This completes the induction. $\square$

CHAPTER 4

WELL-POSEDNESS IN $L_2 \times L_2$

4.1 Local Existence

Consider the Banach space $C\big(0,T;L_2(\mathbb{R})\times L_2(\mathbb{R})\big)$ with its usual norm

$$\left\|U-\tilde{U}\right\|_{C(0,T;L_2(\mathbb{R})\times L_2(\mathbb{R}))} := \sup_{t\in[0,T]} \|u(\cdot,t)-\tilde{u}(\cdot,t)\|_2 + \sup_{t\in[0,T]} \|v(\cdot,t)-\tilde{v}(\cdot,t)\|_2\,,$$

where

$$U = \begin{pmatrix} u \\ \\ v \end{pmatrix} \qquad \text{and} \qquad \tilde{U} = \begin{pmatrix} \tilde{u} \\ \\ \tilde{v} \end{pmatrix}.$$

Let $X_T := C\big(0,T;L_2(\mathbb{R})\big)$. For simplicity, $C(0,T;L_2(\mathbb{R})\times L_2(\mathbb{R}))$-norm of a vector U will be denoted by $\|U\|_{X_T}$.

To establish local existence via the Contraction Mapping Principle, the domain for the operator $\mathcal{A}$ will be restricted to the closed subset

$$B_R(0) := \{U \in C\big(0,T;L_2(\mathbb{R})\times L_2(\mathbb{R})\big) : \|U\|_{X_T} \le R\},$$

where $R>0$ and $T>0$ will be determined. Consider $U,\tilde{U}\in B_R(0)$, and look for an estimate of $\left\|\mathcal{A}U-\mathcal{A}\tilde{U}\right\|_{X_T}$. The aim is to choose specific values of R and T so that $\mathcal{A}$ is a contraction on $B_R(0)$. Using the definition of the operator $\mathcal{A}$, applying the triangle inequality and Holder's inequality one calculates that for $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v})$,

$$\begin{aligned}
\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T} &= \sup_{t\in[0,T]} \left\| \int_0^t K * \big[u(x,t) + P(u,v) - \tilde{u}(x,t) - P(\tilde{u},\tilde{v})\big] ds \right\|_2 \\
&\quad + \sup_{t\in[0,T]} \left\| \int_0^t K * \big[v(x,t) + Q(u,v) - \tilde{v}(x,t) - Q(\tilde{u},\tilde{v})\big] \, ds \right\|_2 \\
&\le T \left\|K * (u-\tilde{u})\right\|_{X_T} + T \left\|K * (v-\tilde{v})\right\|_{X_T} \\
&\quad + T \left\| K * \Big[(u-\tilde{u})\big(A(u+\tilde{u}) + Bv\big) + (v-\tilde{v})\big(C(v+\tilde{v}) + B\tilde{u}\big) \Big] \right\|_{X_T} \\
&\quad + T \left\| K * \Big[(u-\tilde{u})\big(D(u+\tilde{u}) + E\tilde{v}\big) + (v-\tilde{v})\big(F(v+\tilde{v}) + Eu\big) \Big] \right\|_{X_T} \\
&\le 2T \Bigg[\left\|u-\tilde{u}\right\|_{X_T} \left(\frac{1}{2} + (|A|+|D|)(\|u\|_{X_T} + \|\tilde{u}\|_{X_T}) + B\,\|v\|_{X_T} + |E|\,\|\tilde{v}\|_{X_T} \right) \\
&\quad + \left\|v-\tilde{v}\right\|_{X_T} \left(\frac{1}{2} + (|C|+|F|)(\|v\|_{X_T} + \|\tilde{v}\|_{X_T}) + |B|\,\|\tilde{u}\|_{X_T} + |E|\,\|u\|_{X_T} \right) \Bigg].
\end{aligned} \tag{4.1}$$

Using the fact that $U, \tilde{U} \in B_R(0)$, the inequality becomes

$$\begin{aligned}
\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T} &\le 2T \Bigg[\left\|u-\tilde{u}\right\|_{X_T} \left(\frac{1}{2} + 2R(|A|+|D|) + |B|R + |E|R \right) \\
&\quad + \left\|v-\tilde{v}\right\|_{X_T} \left(\frac{1}{2} + 2R(|C|+|F|) + |B|R + |E|R \right) \Bigg] \\
&\le 2T \left(\frac{1}{2} + R \; C_R \right) \left\|U - V\right\|_{X_T}.
\end{aligned} \tag{4.2}$$

Let $R = 2\,\|U_0\|_{X_T}$ and demand that

$$T\left(\frac{1}{2} + R\ C_R\right) = \frac{1}{4}.$$

Then $\mathcal{A}$ is contractive on $B_R(0)$ and for any $U \in B_R(0)$,

$$\|\mathcal{A}U\|_{X_T} \leq R, \tag{4.3}$$

as in (3.5). Therefore, the Contraction Mapping Theorem can be applied to the integral equation (2.3) and the result is a unique U that is solution of the given system. Moreover, using the same argument as in the previous chapter, the solution found above is seen to be continuously dependent on the initial data.

4.2 Long Time Solutions for Small Initial Data

Now that the $L_2 \times L_2$ local existence has been established, the next step is to look for long time solutions when one starts off with L_2-small initial data. One of the forms the system (1.1) could be written in is

$$U_t(x,t) = \int_{-\infty}^{\infty} K * \big[U + M\big]\, dx. \tag{4.4}$$

If R denotes the operator

$$R = -(I - \partial_x^2)^{-1}\partial_x, \tag{4.5}$$

then since R is a skew-adjoint operator, $\int UR(U) = 0$, Young's inequality therefore implies that

$$\int_{-\infty}^{\infty} UU_t dx = \int U\Big[R(U+M)\Big]dx = \int U\Big(K * M\Big)dx =$$
$$\le \|U\|_2 \|K * M\|_2 \le \|U\|_2 \|K\|_2 \|M\|_1 \tag{4.6}$$

To continue the last estimate, the norm $\|M\|_1$ needs to be calculated and estimated in more detail. Using the triangle and the Cauchy-Schwartz inequalities, it follows that

$$\begin{aligned}
\|M\|_1 &= \left\|Au^2 + Buv + Cv^2\right\|_1 + \left\|Du^2 + Euv + Fv^2\right\|_1 \\
&\le \Big(|A| + |D|\Big) \left\|u^2\right\|_1 + \Big(|B| + |E|\Big) \|uv\|_1 + \Big(|C| + |F|\Big) \left\|v^2\right\|_1 \\
&\le \Big(|A| + |D|\Big) \|u\|_2^2 + \Big(|B| + |E|\Big) \|u\|_2 \|v\|_2 + \Big(|C| + |F|\Big) \|v\|_2^2 \\
&\le c \ \Big(\|u\|_2^2 + \|v\|_2^2\Big) \\
&\le c \|U\|_2^2 .
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} UU_t dx \le \|U\|_2 \|K\|_2 \, c \, \|U\|_2^2$$
$$\le \|U\|_2 \, \tilde{c} \, \|U\|_2^2 = \tilde{c} \, \|U\|_2^3 , \tag{4.7}$$

where c and $\tilde{c}$ are constants.

As $U \in C(0,T;L_2)$ and $U_t \in C(0,T;H^1)$ it follows that $UU_t \in C^1(0,T;L_2)$ and

$$\int_{-\infty}^{\infty} UU_t \, dx = \frac{1}{2}\frac{d}{dt}\left\| U \right\|_2^2 \tag{4.8}$$

holds true.

The ordinary differential inequality

$$\frac{1}{2}\frac{d}{dt}\left\| U \right\|_2^2 \leq \tilde{c}\left\| U \right\|_2^3,$$

that follows from (4.7) and (4.8), implies that

$$\frac{d}{dt}\left\| U \right\|_2 \leq \tilde{c}\left\| U \right\|_2^2.$$

Solving the inequality, it follows that

$$-\frac{1}{\left\| U \right\|_2} + \frac{1}{\left\| U_0 \right\|_2} \leq \tilde{c}T \quad \text{or} \quad \left\| U(T) \right\|_2 \leq \frac{\left\| U_0 \right\|_2}{1 - \tilde{c}T\left\| U_0 \right\|_2}.$$

In conclusion, if the L_2-norm is small to start with, then one can infer existence of an L_2-solution over a long time.

4.3 Global Well-Posedness in $L_2 \times L_2$

Based on the results obtained in the two preceding sections, local well-posedness will be extended to global well-posedness using a low-frequency/high-frequency decomposition method.

Fix $T > 0$. If the initial data (u_0, v_0) is small enough in the $L_2 \times L_2$-norm, then the system (1.1) is known to have a unique solution in $C(0, T; L_2 \times L_2)$. Consider initial data of arbitrary size. Split the initial data $(u_0, v_0) \in L_2 \times L_2$ into two pieces, one "small" and the other "smooth". This is possible in the following manner:

$$
\begin{aligned}
u_0 &= u_{01} + u_{02} = (u_0 - \Phi_\epsilon * u_0) + \Phi_\epsilon * u_0, \\
v_0 &= v_{01} + v_{02} = (v_0 - \Phi_\epsilon * v_0) + \Phi_\epsilon * v_0,
\end{aligned}
$$

where Φ_ϵ is a standard mollifier. The important properties that are to be used here are

$$
\begin{aligned}
&(i)\ \ \Phi_\epsilon * w_0 \text{ and } \Phi_\epsilon * v_o \text{ are smooth in } H^k, \\
&(ii)\ \ \Phi_\epsilon * w_0 \to w_0 \text{ in } L_2 \text{ when } \epsilon \to 0.
\end{aligned} \tag{4.9}
$$

Hence, by choosing $\epsilon > 0$ small enough, we can insure that

$$
\begin{aligned}
&u_{01} = (u_0 - \Phi_\epsilon * u_0) \text{ is small in } L_2\text{-norm and } u_{02} = \Phi_\epsilon * u_0 \text{ is smooth,} \\
&v_{01} = (v_0 - \Phi_\epsilon * v_0) \text{ is small in } L_2\text{-norm and } v_{02} = \Phi_\epsilon * v_0 \text{ is smooth.}
\end{aligned} \tag{4.10}
$$

Applying the previous results to the system (1.1) with initial data (u_{01}, v_{01}), it is seen that there is a unique solution (u, v) that lies in $C([0, T]; L_2 \times L_2)$.

Consider now the initial-value problem

$$\begin{cases} w_t + w_x - w_{xxt} + P(w,z)_x + \Big(2Auw + Buz + Bvw + 2Cvz\Big)_x = 0 \\ z_t + z_x - z_{xxt} + Q(w,z)_x + \Big(2Duw + Euz + Evw + 2Fvz\Big)_x = 0 \\ w_0 = u_{02}, \;\; z_0 = v_{02} \end{cases} \tag{4.11}$$

with the initial data $(u_{02}, v_{02}) \in H^k \times H^k$. If this system has a solution (u', v') in $C\big([0,T]; H^k \times H^k\big)$ then $(u + u', v + v')$ will solve the initial system (1.1).

An *a priori* bound for (w, z) on $H^1 \times H^1$-norm is necessary to extend the local existence for (4.11) to the interval $[0, T]$. To obtain an estimation of the $H^1 \times H^1$-norm, the following formal calculation comes to our rescue. The computation is easily justified by use of the continuous dependence result and the time derivative makes sense as all the components are in $C(0,T;H^1)$.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \Big(w^2+w_x^2+z^2+z_x^2\Big)dx = \int \Big(ww_t+w_xw_{xt}+zz_t+z_xz_{xt}\Big)dx\\
&= \int \Big[w(w_t-w_{xxt})+z(z_t-z_{xxt})\Big]dx\\
&= -\int \Big[w\Big(w_x+P_x+(2Auw+Buz+Bvw+2Cvz)_x\Big)\\
&\quad + z\Big(z_x+Q_x+(2Duw+Euz+Evw+2Fvz)_x\Big)\Big]dx\\
&= -\int \Big[\frac{B}{2}(w^2)_xz+(C-\frac{E}{2})w(z^2)_x+Au_x(w^2)+B(uwz)_x\\
&\quad + (2D-B)uz_xw+2Du_xwz+\frac{B}{2}v_xw^2+2C(vwz)_x+(E-2C)vzw_x\\
&\quad + D(w^2)_xz+\frac{E}{2}u_xz^2+Ev_xzw+Fv_xz^2\Big]dx\\
&= -\int \Big[\Big(Au_x+\frac{B}{2}v_x\Big)w^2+\Big(\frac{E}{2}u_x+Fv_x\Big)z^2+\Big(\frac{B}{2}+C-\frac{E}{2}\Big)w(z^2)_x\\
&\quad + D(w^2)_xz+\Big((2D-B)u+(E-2C)v\Big)zw_x+\Big(2Du_x+Ev_x\Big)wz\Big]dx. \qquad (4.12)
\end{aligned}
$$

The integrand on the RHS is a polynomial whose monomial terms are of the form u_xrs, v_xrs, $r(s^2)_x$, urs_x or vrs_x where r,s are either w or z. To obtain a bound in the $H^1\times H^1$ norm of the solution (w,z), the following estimates are used:

$$
\left|\int ur_xsdx\right| \le \int |ur_xs|dx \le \|u\|_2\,\|s\|_\infty\,\|r_x\|_2 \le \|u\|_2\,c\,\|s\|_{H^1}\,\|r\|_{H^1}\,, \qquad (4.13)
$$

$$\left|\int (u_x r s)dx\right| = \left|\int (u r_x s + u r s_x)dx\right| \leq \int |u r_x s|dx + \int |u r s_x|dx$$
$$\leq \|u\|_2 \left(c\,\|s\|_{H^1}\,\|r\|_{H^1} + \tilde{c}\,\|r\|_{H^1}\,\|s\|_{H^1}\right)$$
$$\leq c\,\|u\|_2\,\|s\|_{H^1}\,\|r\|_{H^1} \tag{4.14}$$

and

$$\left|\int r(s^2)_x dx\right| \leq 2\,\|r\|_2\,\|s\|_\infty\,\|s_x\|_2$$
$$\leq 2\,\|r\|_{H^1}\,c\,\|s\|_{H^1}\,\|s\|_{H^1}$$
$$\leq c\,\|r\|_{H^1}\,(\|s\|_{H^1})^2. \tag{4.15}$$

Applying the just derived estimates to (4.12) yields

$$\frac{d}{dt}\,\|(w,z)\|^2_{H^1\times H^1} \leq c\,\|(u,v)\|_2\left(\|(w,z)\|_{H^1\times H^1}\right)^2 + \tilde{c}\left(\|(w,z)\|_{H^1\times H^1}\right)^3 \tag{4.16}$$

where the constants c and $\tilde{c}$ are independent of $t \in [0,T]$. Consequently,

$$\frac{d}{dt}\,\|(w,z)\|_{H^1\times H^1} \leq c_1\,\|(w,z)\|_{H^1\times H^1} + \tilde{c}\left(\|(w,z)\|_{H^1\times H^1}\right)^2, \tag{4.17}$$

from which it follows via a Gronwall-type argument that

$$\|(w,z)\|_{H^1\times H^1} \leq \frac{\|(u_{02},v_{02})\|_{H^1\times H^1}}{1 - c_2\,e^{c_1 T}\,\|(u_{02},v_{02})\|_{H^1\times H^1}}, \tag{4.18}$$

where the constants c_1 and c_2 are independent of $t \in [0, T]$. Hence, local well-posedness on $[0, T]$ is insured. $\square$

CHAPTER 5

INTERPOLATION AND WELL-POSEDNESS IN $H^S \times H^S$

To extend the results obtained for $L_2 \times L_2$ and the $H^k \times H^k$ spaces, $k = 1, 2, \cdots$, nonlinear interpolation theory will be used. The outcome will be global well-posedness in the L_2-based Hilbert spaces $H^s \times H^s$ for all $s \geq 0$.

Nonlinear interpolation of Banach spaces and their operators goes back to the work of Peetre [9] and Peetre-Lions [10]. The following version of the theory will suffice for our purposes (see [11]). More general results of this nature are available.

Theorem 1. *Let r and s, with $r > s$ be two non-negative real numbers. Suppose that for some $T > 0$, the operator A is defined on both $H^r \times H^r$ and $H^s \times H^s$ and maps these spaces continuously into $C([0,T]; H^r \times H^r)$ and $C([0,T]; H^s \times H^s)$, respectively. Suppose in addition that A respects the inequalities*

(i) $\|Af - Ag\|_{C([0,T];H^s \times H^s)} \leq c_0(\|f\|_{H^s \times H^s} + \|g\|_{H^s \times H^s})\|f - g\|_{H^s \times H^s}$

and

(ii) $\|Ah\|_{H^r \times H^r} \leq c_1(\|h\|_{H^s \times H^s})\|h\|_{H^r \times H^r}$,

for some continuous functions c_0 and c_1.

Then, for any $b \in [s, r]$, A maps $H^b \times H^b$ continuously into $C([0,T]; H^b \times H^b)$ and

$$\|Af\|_{C([0,T];H^b \times H^b)} \leq c_b(\|f\|_{H^s \times H^s})\|f\|_{H^b \times H^b}$$

where, for $\gamma > 0, c_b(\gamma)$ may be taken in the form

$$c_b(\gamma) = 4c_0(4\gamma)^{1-\theta}c_1(3\gamma)^\theta$$

with c_0 and c_1 as in (i) and (ii) and $\theta = \frac{b-s}{r-s}$.

Starting with (u_0, v_0) in $H^k \times H^k$ with k integer, as initial data for the system (1.1), denote by A the mapping that associates to (u_0, v_0) the pair (u, v) in $C(0, T; H^k \times H^k)$ that solves the system (1.1). So

$$A : H^k \times H^k \quad \to \quad C\big([0, T]; H^k \times H^k\big),$$

for any k non-negative integer and for any $T > 0$. As seen in the previous chapter, the Contraction Mapping Theorem, which was used to prove the local existence, also assures that the mapping A is locally Lipschitz in $H^k \times H^k$. The latter result can be iterated to conclude that for any $T > 0$, there is continuous function c_k^T such that

$$\begin{aligned} &\|A(u_0, v_0) - A(\widetilde{u}_0, \widetilde{v}_0)\|_{C([0,T];H^k\times H^k)} \\ &\quad \le c_k^T\Big(\|u_0\|_{H^k} + \|v_0\|_{H^k} + \|\widetilde{u}_0\|_{H^k} + \|\widetilde{v}_0\|_{H^k}\Big)\|(u_0 - \widetilde{u}_0, v_0 - \widetilde{v}_0)\|_{H^k\times H^k}. \end{aligned} \tag{5.1}$$

Of course, it is probably the case that c_k^T grows with T, but T is fixed in the discussion.

Start by considering A as a mapping of $L^2 \times L^2$ into the space $C([0,T]; L^2 \times L^2)$ and of $H^1 \times H^1$ into $C([0,T]; H^1 \times H^1)$. Because of the inequalities (3.11) and (4.2), the conditions (ii) and (i) in Theorem 1 are respectively satisfied and it is thereby concluded by applying the Theorem 1 that for any fixed $T > 0$ and s with $0 < s < 1, s \neq \frac{1}{2}$, A maps $H^s \times H^s$ continuously into $C([0,T]; H^s \times H^s)$ and respects the the inequality

$$\begin{aligned} &\|A(u_0, v_0)\|_{C([0,T];H^s\times H^s)} \\ &\qquad \leq c_s^T\big(\|u_0\|_{L^2} + \|v_0\|_{L^2}\big)\|(u_0, v_0)\|_{H^s\times H^s}, \end{aligned} \tag{5.2}$$

where $c_s^T(y) = 4c_0^T(4y)^{1-\theta}c_1^T(3y)^\theta$ and c_0^T and c_1^T are as in formulas (5.1) respectively for $k = 0$ and $k = 1$.

Re-interpolating between $s = 0$ and $s = \frac{3}{4}$, say, the same conclusion holds for $s = \frac{1}{2}$. Iteratively, one can repeat these arguments for $1 < s < 2$, $2 < s < 3$ and so on. It follows that the system (1.1) is globally well posed in all the L_2-based Hilbert spaces $H^s \times H^s, s \geq 0$, as long as the coefficients of the polynomials P and Q satisfy (2.6) and (2.11).

CHAPTER 6

CUBIC CASE

In this chapter, the polynomials P and Q appearing in the our system

$$\begin{cases} u_t + u_x - u_{xxt} + P(u,v)_x = 0, \\ v_t + v_x - v_{xxt} + Q(u,v)_x = 0. \end{cases}$$

are taken to be homogeneous, cubic polynomials, *viz.*

$$\begin{aligned} \Pi(u,v) &= A_{11}u^3 + A_{12}u^2v + A_{13}uv^2 + A_{14}v^3, \\ \Psi(u,v) &= A_{21}u^3 + A_{22}u^2v + A_{23}uv^2 + A_{24}v^3, \end{aligned} \tag{6.1}$$

with all the coefficients $A_{ij} \in \mathbb{R}$, $i = 1, 2$, $j = 1, 2, 3, 4$. The analysis of the cubic case is a little more involved than the quadratic one.

6.1 Invariants

Consider the quadratic functional

$$\Omega(u,v) := \int \Big(\alpha u^2 + \beta uv + \gamma v^2 + \alpha u_x^2 + \beta u_x v_x + \gamma v_x^2\Big) dx,$$

where α, β, γ are real numbers to be determined. A formal calculation where integration by parts is used freely and it is presumed both u and v vanish at $x = \pm\infty$, leads to the formula

$$\frac{d}{dt}\Omega = -2\alpha \int u\Pi_x dx - \beta \int v\Pi_x dx - \beta \int u\Psi_x dx - 2\gamma \int v\Psi_x dx.$$

Writing Π_x and Ψ_x out in detail yields

$$\begin{aligned}
\frac{d}{dt}\Omega = &\int (-6A_{11}\alpha - 3A_{12}\beta)u^3 u_x dx + \int (-3A_{14}\beta - 6A_{24}\gamma)v^3 v_x dx \\
&- \int [(4A_{12}\alpha + 3A_{11}\beta + 2A_{22}\beta + 6A_{21}\gamma)u^2 v u_x + (2A_{12}\alpha + A_{22}\beta)u^3 v_x] dx \\
&- \int [(6A_{14}\alpha + 2A_{13}\beta + 3A_{24}b + 4A_{23}\gamma)uv^2 v_x + (A_{13}\beta + 2A_{23}\gamma)v^3 u_x] dx \\
&- \int (2A_{13}\alpha + 2A_{12}\beta + A_{23}\beta + 4A_{22}\gamma)uv^2 u_x dx \\
&- \int (4A_{13}\alpha + A_{12}\beta + 2A_{23}\beta + 2A_{22}\gamma)u^2 v v_x dx. \qquad (6.2)
\end{aligned}$$

The first two terms in the sum above are integrals which have integrands that are x-derivatives. Thus, for smooth solutions going to 0 at $\pm\infty$, they vanish without further assumptions. The last three integrals in the sum would vanish without further assumptions on (u, v) if and only if

$$
\begin{aligned}
2A_{12}\alpha + A_{22}\beta &= 3A_{11}\beta + 6A_{21}\gamma, \\
A_{13}\beta + 2A_{23}\gamma &= 6A_{14}\alpha + 3A_{24}\beta, \\
2A_{13}\alpha + A_{23}\beta &= A_{12}\beta + 2A_{22}\gamma. \qquad (6.3)
\end{aligned}
$$

Thus α, β and γ must solve the three equations

$$
\begin{cases}
2A_{12}\alpha + (A_{22} - 3A_{11})\beta - 6A_{21}\gamma = 0, \\
6A_{14}\alpha + (3A_{24} - A_{13})\beta - 2A_{23}\gamma = 0, \\
2A_{13}\alpha + (A_{23} - A_{12})\beta - 2A_{22}\gamma = 0,
\end{cases} \qquad (6.4)
$$

for it to be the case that the time derivative of $\Omega(u, v)$ is zero. This system of equations has a non-trivial solution only if its determinant

$$
\Delta = \begin{vmatrix} 2A_{12} & A_{22} - 3A_{11} & -6A_{21} \\ 6A_{14} & 3A_{24} - A_{13} & -2A_{23} \\ 2A_{13} & A_{23} - A_{12} & -2A_{22} \end{vmatrix} \qquad (6.5)
$$

is zero, which is to say,

$$
\begin{aligned}
-4\Big[(A_{12}A_{22} - A_{13}A_{21})(3A_{24} - A_{13}) + (A_{23} - A_{12})(9A_{21}A_{14} - A_{12}A_{23}) \\
+ (3A_{11} - A_{22})(3A_{22}A_{14} - A_{13}A_{23})\Big] = 0. \qquad (6.6)
\end{aligned}
$$

For example, one possibility to have $\Delta = 0$ is that

$$\begin{cases} A_{12} = 3A_{14}, \\ A_{22} - 3A_{11} = 3A_{24} - A_{13}, \\ 3A_{21} = A_{23}. \end{cases} \tag{6.7}$$

In this case, the cubics that couple the original system's unknowns u and v would have the form

$$\Pi(u,v) = \widetilde{A}u^3 + 3\widetilde{B}u^2v + \widetilde{C}uv^2 + \widetilde{B}v^3,$$

$$\Psi(u,v) = \widetilde{D}u^3 + \left[3(\widetilde{A} + \widetilde{E}) - \widetilde{C}\right]u^2v + 3\widetilde{D}uv^2 + \widetilde{E}v^3,$$

where $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}$ are real numbers. With this choice of polynomial coefficients, the values of α, β and γ from Ω are

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} (\widetilde{C} - 3\widetilde{E})(3\widetilde{A} - \widetilde{C} + 3\widetilde{E}) - 9\widetilde{D}(\widetilde{B} - \widetilde{D}) \\ -6\widetilde{B}(\widetilde{C} - 3\widetilde{E}) + 18\widetilde{A}\widetilde{B} - 6\widetilde{C}\widetilde{D} \\ \widetilde{C}(\widetilde{C} - 3\widetilde{E}) - 9\widetilde{B}(\widetilde{B} - \widetilde{D}) \end{pmatrix},$$

up to a scalar multiple.

If α, β and γ are a non-trivial solution of (6.4), it follows immediately that

$$\frac{\partial}{\partial u}\Big(\beta\Pi + 2\gamma\Psi\Big) = \Big(3A_{11}\beta + 6A_{21}\gamma\Big)u^2 + 2(A_{12}\beta + 2A_{22}\gamma)uv + \Big(A_{13}\beta + 2A_{23}\gamma\Big)v^2,$$
$$\frac{\partial}{\partial v}\Big(2\alpha\Pi + \beta\Psi\Big) = \Big(2A_{12}\alpha + A_{22}\beta\Big)u^2 + 2\Big(2A_{13}\alpha + A_{23}\beta\Big)uv + \Big(6A_{14}\alpha + 3A_{24}\beta\Big)v^2,$$

whence,

$$\frac{\partial}{\partial u}\Big(\beta\Pi + 2\gamma\Psi\Big)(u,v) = \frac{\partial}{\partial v}\Big(2\alpha\Pi + \beta\Psi\Big)(u,v).$$

In consequence, there is a quartic polynomial $\Theta(u,v)$ such that

$$\frac{\partial\Theta}{\partial u} = 2\alpha\Pi + \beta\Psi \quad \text{and} \quad \frac{\partial\Theta}{\partial v} = \beta\Pi + 2\gamma\Psi. \tag{6.8}$$

6.1.1 Second Conservation Law

In case there is a non-trivial solution (α, β, γ) of (6.4), a second Conservation Law is now contemplated. Consider the time derivative of the functional

$$\int_{-\infty}^{\infty}\Big(\alpha u^2 + \beta uv + \gamma v^2\Big)dx,$$

and following the same calculations as in Chapter 2 for the quadratic case to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big(\alpha u^2 + \beta uv + \gamma v^2\Big)dx = -\frac{d}{dt}\int \Theta(u,v)dx. \tag{6.9}$$

If the functional $\Phi(u,v)$ is defined by

$$\Phi(u,v) := \int_{-\infty}^{\infty} \Big(\alpha u^2 + \beta uv + \gamma v^2 + \Theta(u,v)\Big)dx, \tag{6.10}$$

then $\Phi(u,v)$ is also an invariant of the temporal evolution of smooth solutions of our system. It serves as a Hamiltonian for the given system as long as $4\alpha\gamma - \beta^2$ is not 0.

Both these conserved quantities Ω and Φ will be used to obtain *a priori* bounds on solutions of the system when the quadratic form ρ defined by

$$\rho(u,v) = \alpha u^2 + \beta uv + \gamma v^2, \tag{6.11}$$

vanishes only at origin, thus when its discriminant is negative. The details however will be pursued later as a companion result of this thesis and are omitted here.

6.2 Local Well Posedness

The key component to proving local well-posedness is once again the Contraction Mapping Principle.

6.2.1 Local Well-Posedness in L_∞ and C_b

Consider again the Banach space

$$X_T := L_\infty\big(\mathbb{R}\times[0,T]\big) \times L_\infty\big(\mathbb{R}\times[0,T]\big)$$

endowed with its usual norm (see 3.2). To establish local existence via the Contraction Mapping principle, the domain of the operator $\mathcal{A}$ will be restricted to the usual closed ball

$$B_R(0) := \{U \in X_T : \|U\|_{X_T} \leq R\},$$

with $R > 0$ and $T > 0$ to be determined.

Consider $U, \tilde{U} \in B_R(0)$, and look for an estimate of $\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T}$. The aim is to choose specific values of R and T so that $\mathcal{A}$ is a contraction on $B_R(0)$. As the calculations are very similar as those appearing in the quadratic case presented in section 3.1.1, the details will be omitted. Using the definition of the operator $\mathcal{A}$, for $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v})$;

$$\begin{aligned}
\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T} =& \sup_{x\in\mathbb{R}} \left| \int_0^T \int_{-\infty}^{\infty} K(x-y) \Big[u(y,s) + \Pi(u,v) - \tilde{u}(y,s) - \Pi(\tilde{u}.\tilde{u})\Big]\, dyds \right| \\
&+ \sup_{x\in\mathbb{R}} \left| \int_0^T \int_{-\infty}^{\infty} K(x-y) \Big[v(y,s) + \Psi(u,v) - \tilde{v}(y,s) - \Psi(\tilde{u}.\tilde{u})\Big]\, dyds \right| \\
\leq& \sup_{x\in\mathbb{R}} \int_0^T \int_{-\infty}^{\infty} \big|K(x-y)\big| \Big[|u-\tilde{u}|\Big(A_{11}(|u|^2 + |u||\tilde{u}| + |\tilde{u}|^2) + A_{12}|\tilde{v}|(|u| + |\tilde{u}|) \\
&+ A_{13}|\tilde{v}|^2\Big) + |v-\tilde{v}|\Big(A_{12}|u|^2 + A_{13}|u|(|v| + |\tilde{v}|) + A_{14}(|v|^2 + |v||\tilde{v}| + |\tilde{v}|^2)\Big)\Big]\, dyds \\
&+ \sup_{x\in\mathbb{R}} \int_0^T \int_{-\infty}^{\infty} \big|K(x-y)\big| \Big[|v-\tilde{v}|\Big(A_{24}(|v|^2 + |v||\tilde{v}| + |\tilde{v}|^2) + A_{23}|\tilde{u}|(|v| + |\tilde{v}|) \\
&+ A_{22}|\tilde{u}|^2\Big) + |u-\tilde{u}|\Big(A_{23}|v|^2 + A_{22}|v|(|u| + |\tilde{u}|) + A_{21}(|u|^2 + |u||\tilde{u}| + |\tilde{u}|^2)\Big)\Big] dyds.
\end{aligned}$$

Furthermore, as $U, \tilde{U} \in B_R(0)$, it follows that the inequality can be continued, *viz.*

$$\left\|\mathcal{A}U - \mathcal{A}\tilde{U}\right\|_{X_T} \leq \left\|U - \tilde{U}\right\|_{X_T} C_R\ T, \tag{6.12}$$

where C_R is a constant only dependent on R. The choices of R and T are to be made such that $\mathcal{A}$ maps $B_R(0)$ into itself through and so that $\mathcal{A}$ is a contraction. In particular, one needs to have

$$C_R T < 1.$$

As before, fix the relation

$$C_R T = \frac{1}{2}.$$

Then, as before, for any $U \in B_R(0)$, $\|\mathcal{A}U\|_{X_T} \leq R$, provided $R = 2\|U_0\|_\infty$.

This implies that

$$T = \frac{1}{2\ C_R}.$$

The Contraction Mapping Theorem applies with this values of R and T to the integral equation (2.3). The result is existence of a unique U that is solution of the given system when Π and Ψ are cubics.

6.2.2 Local Existence in Other Spaces

Lemma 1, repeated below for reader's convenience, was stated and proved in paragraph 3.1.2 for the case when P and Q are quadratic polynomials and contains local well-posedness results for spaces embedded in L_∞. The proof for the cubic case is very similar thus we'll omit most of the calculations and only detail the results that are specific to the cubic polynomials.

Lemma 3. *Suppose that Z is a space embedded in L_∞ such that, for any $u, v \in Z$, there exist global constants C_1 and C_2 satisfying*

$$(i) \quad \|uv\|_Z \leq C_1 \|u\|_Z \|v\|_Z \qquad \textit{and} \qquad (ii) \quad \left\| \int_{-\infty}^{\infty} K(x-y)u(y)dy \right\|_Z \leq C_2 \|u\|_Z .$$

Then there is a unique local solution for the initial-value problem for (1.1) in the space $X_T = C\big(0, T; Z \times Z\big)$.

Proof. Using the same Banach space $Z_T = C(0, T; Z \times Z)$ with the norm

$$\left\| U - \tilde{U} \right\|_{Z_T} := \sup_{t\in[0,T]} \|u(\cdot, t) - \tilde{u}(\cdot, t)\|_Z + \sup_{t\in[0,T]} \|v(\cdot, t) - \tilde{v}(\cdot, t)\|_Z , \tag{6.13}$$

where $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v})$, restrict the operator $\mathcal{A}$ from (2.5) to

$$B_{R,Z}(0) := \{U \in Z_T : \|U\|_{Z_T} \leq R\}.$$

Consider $U, \tilde{U} \in B_{R,Z}(0)$. The aim is to choose specific values of R and T so that $\mathcal{A}$ is a contraction on $B_{R,Z}(0)$. To estimate $\left\| U - \tilde{U} \right\|_{Z_T}$ apply (ii) from (3.6), the triangle inequality and (3.6) part (i), and similarly with the L_∞ case from the preceding section, yields

$$\|\mathcal{A}U - \mathcal{A}V\|_{Z_T} \leq \; TC_1C_2C_R \|U - V\|_{Z_T} .$$

Choosing $R = 2\|U_0\|_Z$ and T, for example, so that

$$T = \frac{1}{2C_1C_2^2C_R},$$

the Contraction Mapping Theorem can be applied to the integral equation (2.3) derived from the initial system and the result is a unique U that is solution of the given system. Moreover, this solution found above is seen to be continuously dependent on the initial data.

From the regularity perspective, the smoothing associated with the temporal derivative is analogous with the L_∞ case presented in the previous section, specifically for $U \in H^k \times H^k$, U_t is spatially smoother and it lies in $C(0, \infty, ; H^{k+1} \times H^{k+1})$. As for the spatial regularity, the system solution has at most the same regularity as the initial data. $\square$

This Lemma provides the local well-posedness on H^1 , H^2 and in general in H^k, for $k \geq 1$.

6.2.3 Smoothness and Regularity

The argument presented in Chapter 3 is independent of the polynomial degree, therefore the result applies to the cubic case as well. The strong regularity results for the flow map, the temporal smoothing and the spacial regularity being at most as good as the initial data are still valid.

CHAPTER 7

CONCLUSION

We have considered coupled systems

$$\begin{cases} u_t + u_x - u_{xxt} + P(u,v)_x = 0, \\ v_t + v_x - v_{xxt} + Q(u,v)_x = 0, \end{cases} \tag{7.1}$$

of two evolution equations of generalized BBM-type, posed for $x \in \mathbb{R}$ and $t \geq 0$ and with specified initial data at $t = 0$, *viz*.

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \mathbb{R}. \tag{7.2}$$

These systems were shown to be locally well-posed in the L_2-based Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 0$ when P and Q are homogeneous quadratic polynomials. Moreover, the solution (u,v) depends continuously on the initial data (u_0, v_0). The spacial regularity of the solution (u,v) is exactly as good as the initial data's. However, there is some smoothing in the temporal variable.

Furthermore, for those polynomial coefficients $A, B, \cdots, F$ such that the system

$$\begin{cases} 2Ba + \Big(E - 2A\Big)b - 4Dc = 0, \\ 4Ca + \Big(2F - B\Big)b - 2Ec = 0 \end{cases}$$

has solutions a, b and c satisfying $4ac > b^2$, or, what is the same, the matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ is positive definite, then the well-posedeness is global in time.

Similar results are sketched for the homogeneous cubic polynomials.

In future work, we plan to consider the periodic initial-value problem and the more practical case of non-homogeneous boundary conditions. We will also be interested in the question of global well-posedness in case the nonlinearities are not homogeneous.

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