The Global Structure of Totally Disconnected Locally Compact Polish Groups

BY

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THESIS

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LIST OF ABBREVIATIONS

AMS  American Mathematical Society
CTAN  Comprehensive \TeX\ Archive Network
TUG  \TeX\ Users Group
UIC  University of Illinois at Chicago
UICTHESI  Thesis formatting system for use at UIC
t.d.  Totally disconnected
l.c.  Locally compact
s.c.  Second countable
SIN  Small invariant neighborhood
Locally compact groups arise frequently and are quite useful in many fields of mathematics. Certainly, Lie groups play an important role in geometry and topology. In number theory, Lie groups over the field of $p$-adics are essential tools. In geometric group theory, the isometry groups of proper metric spaces, e.g. locally finite trees, are locally compact. In mathematical logic, a striking application of locally compact groups occurs in the work of Z. Chatzidakis and E. Hrushovski on the model theory of difference fields \cite{14}.

It is well known that a locally compact group is connected by totally disconnected. Indeed, letting $G_0$ denote the connected component of the identity of a locally compact group $G$, there is a short exact sequence of topological groups $1 \to G_0 \to G \to G/G_0 \to 1$ where $G_0$ is connected and $G/G_0$ is totally disconnected. The study of locally compact groups thus reduces to the study of connected locally compact groups and totally disconnected locally compact (t.d.l.c.) groups.

In the connected case, a rich and deep theory exists: the study of connected locally compact groups reduces to the study of inverse limits of Lie groups by the celebrated solution to Hilbert’s fifth problem.

**Theorem 0.1** (Gleason, Montgomery, Yamabe, Zippin). A connected locally compact group is pro-Lie.

One naturally next considers the totally disconnected quotient. It is immediately obvious that any theory for t.d.l.c. groups must account for the troublesome discrete groups. The class
SUMMARY (Continued)

of t.d.l.c. groups also contains a wilderness of non-discrete groups such as the isometry groups of
locally finite trees, Lie groups over the $p$-adics, and Neretin’s group of spheromorphisms. Worse
still, techniques from the study of connected locally compact groups do not generally apply in
the totally disconnected setting. For many years, discrete groups, the breath of non-discrete
examples, and lack of techniques made the general study of t.d.l.c. groups seem intractable.

In the mid-nineties, however, G. Willis’ breakthrough [40] suggested that, up to discrete
groups, t.d.l.c. groups may admit a general theory. Today, there is much more evidence for
this suspicion; indeed, a general theory up to discrete groups is emerging for t.d.l.c. groups.
We call this theory up to discrete groups the topological theory of t.d.l.c. groups.

This thesis contributes to the topological theory of t.d.l.c. Polish groups where

Definition 0.2. A topological group is Polish if the underlying topology is separable and
completely metrizable.

While much of the theory developed hitherto does not require additional assumptions on
the group topology, the Polish assumption is quite mild. Indeed, by classical results, a t.d.l.c.
group is Polish if and only if it is second countable. Alternatively, most natural examples of
t.d.l.c. groups are Polish. For example, the isometry groups of locally finite trees, most $p$-
adic Lie groups, and any compactly generated t.d.l.c. group modulo a certain compact normal
subgroup are t.d.l.c. Polish groups.

Chapter 2 begins our study of the topological structure of t.d.l.c. Polish groups by isolating
the class of elementary groups. This class captures the collection of groups “built by hand”
SUMMARY (Continued)

from profinite and discrete groups; such groups arise frequently in the study of t.d.l.c. Polish
groups.

**Definition 0.3.** The class of *elementary groups* is the smallest class $\mathcal{E}$ of t.d.l.c. Polish groups
such that

(i) $\mathcal{E}$ contains all second countable profinite groups and countable discrete groups.

(ii) $\mathcal{E}$ is closed under group extensions. I.e. if $G$ is a t.d.l.c. Polish group and $H \triangleleft G$ is a
closed normal subgroup with $H, G/H \in \mathcal{E}$, then $G \in \mathcal{E}$.

(iii) If $G$ is a t.d.l.c. Polish group and $G = \bigcup_{i \in \omega} O_i$ where $(O_i)_{i \in \omega}$ is an $\subseteq$-increasing
sequence of open subgroups of $G$ with $O_i \in \mathcal{E}$ for all $i$, then $G \in \mathcal{E}$. We say $\mathcal{E}$ is closed under
*countable increasing unions.*

We show $\mathcal{E}$ satisfies strong permanence properties:

**Theorem 0.4.** $\mathcal{E}$ enjoys the following permanence properties:

1. If $G \in \mathcal{E}$, $H$ is a t.d.l.c. Polish group, and $\psi : H \to G$ is a continuous, injective homomor-
   phism, then $H \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under taking closed subgroups.

2. $\mathcal{E}$ is closed under taking quotients by closed normal subgroups.

3. If $G$ is a residually elementary t.d.l.c. Polish group, then $G \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed
   under inverse limits.

4. $\mathcal{E}$ is closed under quasi-products.

5. $\mathcal{E}$ is closed under local direct products.
Via these properties, we arrive at a general structure theorem:

**Theorem 0.5.** Let $G$ be a t.d.l.c. Polish group. Then there is a sequence of closed characteristic subgroups $\{1\} \leq N \leq Q \leq G$ such that $N$ and $G/Q$ are elementary, $Q/N$ has no non-trivial closed elementary normal subgroups, and $Q/N$ has no non-trivial Hausdorff elementary quotients.

Theorems in later chapters of this thesis show Theorem 0.5 to be quite useful.

The next three chapters consider the following vague question:

**Question 0.6.** If a t.d.l.c. Polish group $G$ has a compact open subgroup with property $x$, what can we say about the structure of $G$?

When a t.d.l.c. Polish group has a compact open subgroup with property $x$, we say it is *locally $x$*. Question 0.6 follows naturally from a number of results in the literature showing the compact open subgroups in non-discrete compactly generated t.d.l.c. groups which are topologically simple cannot have certain properties; cf. [11], [3], [42].

Chapter 3 begins this study. We first consider locally solvable groups.

**Theorem 0.7.** If $G$ is a locally solvable t.d.l.c. Polish group, then $G$ is elementary.

We next explore locally pro-nilpotent groups:

**Theorem 0.8.** Suppose $G$ is a t.d.l.c. Polish group which is locally pro-nilpotent. Then there is $N \leq G$ open such that $N = \bigcup_{i \in \omega} H_i$ where $(H_i)_{i \in \omega}$ is an \(\subseteq\)-increasing sequence of compactly generated open subgroups and for each $i$ there is a finite set of primes $\pi(i)$ so that $H_i/\text{Rad}_{\mathcal{L}_E}(H_i)$ is locally pro-nilpotent and locally pro-$\pi(i)$.
SUMMARY (Continued)

**Theorem 0.9.** Suppose $G$ is a t.d.l.c. Polish group which is locally pro-nilpotent and locally pro-$\pi$ for some finite set of primes $\pi$. Then there is a sequence of closed normal subgroups $\{1\} = L_0 \leq L_1 \leq \ldots \leq L_k \leq G$ such that $0 \leq k \leq |\pi| - 1$ and

1. $G/L_k$ is locally pro-$p$ for some $p \in \pi$.
2. For each $1 \leq i \leq k$ there is $q(i) \in \pi$ such that $L_i/L_{i-1} = \bigcup_{j \in \omega} H_j$ where $(H_j)_{j \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups of $L_i/L_{i-1}$ with $H_j/\text{Rad}_{\mathcal{L}}(H_j)$ locally pro-$q(i)$ for every $j \in \omega$.

We conclude this chapter by considering groups with a more technical local condition; here we give a new proof of a result implicit in work of Y. Barnea, M. Ershov, and T. Weigel [3].

**Proposition 0.10** (cf. Barnea, Ershov, Weigel [3, Theorem 5.4]). Suppose $G$ is a non-elementary t.d.l.c. Polish group which is locally hereditarily just infinite. Then there is a characteristic series $G \supseteq D_1 \supseteq D_2 \supseteq \{1\}$ such that $D_1$ is open, $D_2$ is discrete, and $D_1/D_2$ is non-discrete and topologically simple.

Chapter 4 continues our study of Question 0.6 by considering $[A]$-regular t.d.l.c. Polish groups. These groups are identified in [12] and are defined by a technical condition on the compact open subgroups; moreover, $[A]$-regular groups are the barrier to applying the powerful new theory developed in [12]. We show

**Theorem 0.11.** If $G$ is a t.d.l.c. Polish group which is $[A]$-regular, then $G$ is elementary.

Our investigations of Question 0.6 conclude in Chapter 5. This chapter yields the most interesting of our results:
Theorem 0.12. Suppose $G$ is a non-elementary t.d.l.c. Polish group which is locally of finite rank and locally pro-$\pi$ for some finite set of primes $\pi$. Then there is a characteristic series $\{1\} \leq A_1 \leq A_2 \leq G$ such that $A_1$ is elementary, $G/A_2$ is finite, and $A_2/A_1 \simeq N_1 \times \cdots \times N_k$ for $0 < k < \infty$ where each $N_i$ is a non-elementary compactly generated topologically simple $p(i)$-adic Lie group of adjoint simple type.

We remark that l.c.s.c. $p$-adic Lie groups meet the hypotheses of the previous theorem.

The next thread of research, Chapter 6, investigates the collection of periodic $n$-tuples, $P_n(G)$, where a $n$-tuple $(g_1, \ldots, g_n)$ is periodic if the coordinates generate a relatively compact subgroup of $G$. We, in particular, consider the following question:

**Question 0.13** (Rosendal). For a t.d.l.c. Polish group $G$ and $n \geq 1$, is $P_n(G)$ closed in the $n$-th Cartesian power of $G$?

We show Question 0.13 has a positive answer in a large class of t.d.l.c. Polish groups:

**Theorem 0.14.** If $G \in E$, then $P_n(G)$ is closed for all $n$.

**Theorem 0.15.** Let $\mathcal{P}$ be the class of all l.c.s.c. $p$-adic Lie groups for all $p$. If $G$ is in $E \mathcal{P}$, the elementary closure of $\mathcal{P}$, then $P_n(G)$ is closed for all $n$.

These results show, if an example giving a negative answer to Question 0.13 exists, it is not easily found.

Our investigations conclude with Chapter 7. Here we answer a question of A. Kechris and C. Rosendal:
SUMMARY (Continued)

**Question 0.16** (Kechris, Rosendal). Can a non-trivial t.d.l.c. Polish group admit a comeagre conjugacy class?

We answer Question 0.16 in the negative via an analysis of t.d.l.c. Polish groups with topological conditions on a conjugacy class.

**Theorem 0.17.** If $G$ is a non-trivial t.d.l.c. Polish group and $g^G := \{hgh^{-1} \mid h \in G\}$ is dense, then $g^G$ is meagre and Haar null.

**Corollary 0.18.** A non-trivial t.d.l.c. Polish group does not admit a comeagre conjugacy class.
CHAPTER 1

INTRODUCTION

We begin with a brief overview of the necessary background. The notations, definitions, and facts discussed here are used frequently and, typically, without reference. This introduction, of course, falls well short being comprehensive. We direct the interested reader to the following excellent texts: [6], [20], [35], [44].

1.1 Notations and Conventions

All groups are taken to be Hausdorff topological groups. All subgroups are taken to be closed unless otherwise stated. We write $H \leq_o G$ and $H \leq_{cc} G$ to indicate $H$ is an open subgroup and cocompact subgroup of $G$, respectively. Recall $H \leq G$ is cocompact if the quotient space $G/H$ is compact in the quotient topology. For any subgroup $K \leq G$, $C_G(K)$ is the collection of elements of $G$ which centralize every element of $K$. $N_G(K)$ denotes the collection of elements of $G$ which normalize $K$. For $A, B \subseteq G$, we put $A^B := \{bab^{-1} \mid a \in A \text{ and } b \in B\}$, $[A, B] := \langle aba^{-1}b^{-1} \mid a \in A \text{ and } b \in B\rangle$, and $A^n := \{a_1 \ldots a_n \mid a_i \in A\}$.

$S(G)$ denotes the collection of closed subgroups of $G$; $\mathcal{U}(G)$ denotes the collection of compact open subgroups. We write $G^\times n$ to denote the $n$-th Cartesian power of $G$. We write $G \actson X$ when $G$ acts on a set $X$. If $x \in X$, then $G(x)$ denotes the stabilizer of $x$ in $G$. 
When used as an index, \( \omega \) denotes the first countable ordinal; i.e. \( \omega = \mathbb{N} \) as linear orders. \( \omega_1 \) denotes the first uncountable ordinal. Ordinal numbers are used frequently in this work; [24] contains a nice introduction to ordinal numbers and ordinal arithmetic.

1.2 History

The study of t.d.l.c. groups begins with an old theorem of D. van Dantzig.

**Theorem 1.1** (van Dantzig). A t.d.l.c. group admits a basis at 1 of compact open subgroups.

**Proof.** ([20]) Let \( G \) as hypothesized and let \( V \) be a neighborhood of 1. By classical results, \( G \) admits a basis of clopen sets at 1. We may thus find \( U \subseteq V \) a compact open neighborhood of 1.

We claim there is \( L \subseteq U \) a symmetric open neighborhood of 1 such that \( UL \subseteq U \). Indeed, for each \( x \in U \) there is \( W_x \ni 1 \) open with \( xW_x \subseteq U \) and an open symmetric set \( L_x \ni 1 \) with \( L_x^2 \subseteq W_x \). Plainly, \( U \subseteq \bigcup_{x \in U} xL_x \). So \( U \subseteq x_1L_{x_1} \cup \cdots \cup x_kL_{x_k} \) for some \( x_1, \ldots, x_k \) in \( U \) since \( U \) is compact. Putting \( L = \bigcap_{i=1}^k L_{x_i} \), we have

\[
UL \subseteq \bigcup_{i=1}^k x_iL_{x_i}L \subseteq \bigcup_{i=1}^k x_iL_{x_i}^2 \subseteq \bigcup_{i=1}^k x_iW_{x_i} \subseteq U
\]

It is easy to see that \( L^n \subseteq U \) for all \( n \geq 0 \). Indeed, if \( L^n \subseteq U \), then \( L^{n+1} = L^nL \subseteq UL \subseteq U \) by the previous paragraph. We conclude \( W := \bigcup_{n \geq 0} L^n \subseteq U \). Since \( L \) is symmetric, \( W \) is a clopen subgroup of the compact open set \( U \) and, therefore, a compact open subgroup. \( \square \)

We remark that the compact open subgroups need not be normal, e.g. there are many non-discrete t.d.l.c. groups which are topologically simple. Further, the presence of such subgroups
indicates non-discrete t.d.l.c. groups are wildly different from almost connected locally compact groups. Indeed, the general linear group over the $p$-adics, $GL_n(\mathbb{Q}_p)$, is a t.d.l.c. group and, therefore, admits a basis at 1 of compact open subgroups. On the contrary, $GL_n(\mathbb{R})$, having a finite index connected subgroup, does not admit such a basis!

The compact open subgroups do, however, allow us to realize all t.d.l.c. second countable (s.c.) groups as permutation groups; a fact which makes the study of t.d.l.c. Polish groups interesting to mathematical logicians.

**Corollary 1.2** (folklore). *Every t.d.l.c.s.c. group is isomorphic to a closed subgroup of Sym(\mathbb{N}).*

**Proof.** Let $G$ be a t.d.l.c.s.c. group and $(U_i)_{i \in \omega}$ a countable basis at 1 of compact open subgroups as given by Theorem 1.1. Setting $\mathcal{K}$ to be the collection of left cosets of the $U_i$, we have that $G$ acts on $\mathcal{K}$ continuously and faithfully by left multiplication.

Let $\psi : G \to Sym(\mathbb{N})$ be the continuous homomorphism induced via the above action. It suffices to show $\psi$ is a closed map. To that end, say $A \subseteq G$ is closed and $(\psi(a_i))_{i \in \omega} \subseteq \psi(A)$ is such that $\psi(a_i) \to h$ in $Sym(\mathbb{N})$. Fixing some $kU \in \mathcal{K}$, there is $N$ such that for all $i, j \geq N$, $\psi(a_j^{-1}a_i)$ stabilizes $kU$. We thus have that $a_j^{-1}a_i \in kUk^{-1}$ for all $i, j \geq N$. Fixing $j \geq N$, we have that $a_i \in a_jkUk^{-1}$ for all $i \geq N$. Since $a_jkUk^{-1}$ is compact, there is a subsequence $i_k$ such that $a_{i_k}$ converges to some $a \in A$. It is now clear that $\psi(a_{i_k}) \to h$ and, therefore, $\psi(a) = h$. We conclude that $\psi(A)$ is closed as desired. \qed
The compact open subgroups are also profinite - i.e. inverse limits of finite groups - and are subject to a rich theory. This fact is used heavily not only in this thesis but also in the topological theory of t.d.l.c. groups.

The next major step forward in the topological theory of t.d.l.c. groups arises in work of H. Abels from the 1970’s: he shows an analogue of the Cayley graph exists for all compactly generated t.d.l.c. groups.

**Theorem 1.3** (Abels [1]). Let $G$ be a compactly generated t.d.l.c. group. Then there is a locally finite connected graph $\Gamma$ on which $G$ acts continuously and vertex transitively by graph automorphisms such that for all $v \in V\Gamma$ the stabilizer of $v$ in $G$ is compact and open.

**Proof.** Let $G$ be as hypothesized, fix $U \in \mathcal{U}(G)$, and let $X$ be a compact generating set for $G$. Since $\{xU : x \in X\}$ is an open cover of $X$, we may find $B$ a finite symmetric set containing 1 such that $X \subseteq BU$. Of course, $UB$ is again a compact set. There is thus a finite symmetric $A$ such that $B \subseteq A \subseteq UBU$ and $UB \subseteq AU$. We now see that $UAU = UBU \subseteq AUU = AU$.

Thus, $UAU = AU$, and it is easy to verify $(UAU)^n = A^nU$ for all $n \geq 1$. We conclude

$$G = \langle BU \rangle = \langle UAU \rangle = \bigcup_{n \geq 1} (UAU)^n = \bigcup_{n \geq 1} A^nU = \langle A \rangle U$$

Define $\Gamma$ by $V\Gamma := G/U$ and

$$E\Gamma := \bigcup_{h \in A} \{gU, ghU \mid g \in G\}$$
We claim $\Gamma$ satisfies the theorem. It is clear $G$ acts vertex transitively on $\Gamma$ with compact open stabilizers. It remains to show $\Gamma$ is connected and locally finite. For connectivity, take $gU \in VT$. Since $G = (A)U$, we have that $g = h_{i_0} \ldots h_{i_n}u$ for $h_{i_j} \in A$. Thus,

$$U, h_{i_0}U, h_{i_0}h_{i_1}U, \ldots, h_{i_0} \ldots h_{i_n}U$$

is a path in $\Gamma$ connecting $U$ to $gU$ and $\Gamma$ is connected.

For local finiteness, we note that $B_1(U) = \{hU \mid h \in A\}$. Indeed, if $\{kU, khU\}$ is an edge with $U$ as an end point, then either $k \in U$ or $khU = U$. In the former case, $khU \in \{UhU \mid h \in A\} = \{hU \mid h \in A\}$ and we are done. For the latter, we have that $k = uh^{-1}$ for some $u \in U$ and $kU = uh^{-1}U$. Since $h^{-1} \in A$, there is $a \in A$ such that $uh^{-1}U = aU$. We conclude $kU \in \{hU \mid h \in A\}$ and $\Gamma$ is locally finite.

$$\square$$

Such a $\Gamma$ is called a Cayley-Abels graph for $G$. We remark that the Cayley-Abels graph for a compactly generated t.d.l.c. group is unique up to quasi-isometry. See [23] for a nice, self contained proof of the uniqueness up to quasi-isometry of a Cayley-Abels graph and a host of other results.

While the results of van Dantzig and Abels are general, they seem to have had little impact on the community. The modern research program on and interest in t.d.l.c. groups unequivocally begins with a powerful, general result due to G. Willis from the mid nineties:
Theorem 1.4 (Willis, [40]). Let $G$ be a t.d.l.c. group and $s : G \to \mathbb{Z}$ be defined by

$$s(g) := \min \{|U : U \cap g^{-1}Ug| : U \leq G \text{ is compact and open} \}$$

Then

(1) $s : G \to \mathbb{Z}$ is continuous. Indeed, if $U \leq G$ is a compact open subgroup which witnesses $s(g)$, then $s(h) = s(g)$ for all $h \in UgU$.

(2) $s(g^n) = s(g)^n$ for all $n \geq 1$.

(3) $\Delta(g) = \frac{s(g^{-1})}{s(g)}$ where $\Delta$ is the modular function for the left invariant Haar measure.

The function $s$ in Theorem 1.4 is called the scale function. The scale function and the surrounding theory has proved to be extremely useful. For example, Willis uses his theory of the scale function to settle a conjecture of K.H. Hofmann and A. Mukherjea [41]. Alternatively, via an elegant application of the scale function and the theory thereof, W.H. Previts and T.S. Wu give a new proof of an old question of P. Halmos [32]. The scale function is moreover a clear indication that a non-trivial t.d.l.c. group topology induces non-trivial structure.

The results of van Dantzig, Abels, and Willis are today accompanied by an abundance of theorems that show a non-trivial t.d.l.c. group topology has non-trivial structural consequences. This thesis is a contribution to this growing body of work.
1.3 Profinite Groups

The compact open subgroups given by van Dantzig’s theorem are profinite; i.e. inverse limits of finite groups. We remark that every compact t.d.l.c. group is profinite and vice versa. We therefore use the terms compact and profinite interchangeably in this work.

Profinite groups have a basis at 1 of compact open normal subgroups, and this basis is countable when the group is second countable. For a Polish profinite group $U$, we say $(W_i)_{i \in \omega}$ is a normal basis at 1 for $U$ if $(W_i)_{i \in \omega}$ is an $\subseteq$-decreasing sequence of open normal subgroups with $U = W_0$ and $\bigcap_{i \in \omega} W_i = \{1\}$.

A profinite group $U$ is said to be pro-$x$ for $x$ some property of finite groups if $U$ is an inverse limit of finite groups with property $x$. E.g. $U$ may be pro-$p$ for some prime $p$ or pro-nilpotent.

Profinite groups have a Sylow theory arising from the inverse limit construction. For a prime $p$, a $p$-Sylow subgroup of a profinite group $U$, denoted $U_p$, is a maximal pro-$p$ subgroup of $U$.

Analogous to the finite setting,

**Proposition 1.5** ([44, 2.2.2]). Let $U$ be a profinite group and $p$ a prime. Then

1. $U$ has $p$-Sylow subgroups.

2. All $p$-Sylow subgroups are conjugate.

3. Every pro-$p$ subgroup is contained in a $p$-Sylow subgroup.

**Proposition 1.6.** Let $U$ be a profinite group, $K \leq U$, and $P$ a $p$-Sylow subgroup. Then $K \cap P$ is a $p$-Sylow subgroup of $K$ and $KP/K$ is a $p$-Sylow subgroup of $G/K$. 
1.4 Totally Disconnected Locally Compact Groups

The topological analogues of the familiar isomorphism theorems hold for t.d.l.c. Polish groups with some slight modifications:

**Theorem 1.7** ([20, (5.34)]). Let $G$ and $\tilde{G}$ be t.d.l.c. Polish groups and let $\phi : G \to \tilde{G}$ be a continuous homomorphism of $G$ onto $\tilde{G}$. Suppose further $\tilde{H}$ is a closed normal subgroup of $\tilde{G}$ and put $H := \phi^{-1}(\tilde{H})$. Then $G/H$, $\tilde{G}/\tilde{H}$, and $(G/\ker(\phi))/(H/\ker(\phi))$ are isomorphic as topological groups.

**Theorem 1.8** ([20, (5.33)]). Let $G$ be a t.d.l.c. Polish group, $A \subseteq G$ a closed subgroup, and $H \trianglelefteq G$ a closed normal subgroup. If $AH$ is closed, then $AH \simeq A/(A \cap H)$ as topological groups.

T.d.l.c. Polish groups also have a nice topological structure:

**Theorem 1.9** ([22, (5.3)]). The following are equivalent for a t.d.l.c. group $G$:

1. $G$ is Polish.
2. $G$ is second countable.
3. $G$ is metrizable and $K_{\sigma}$ - i.e. has a countable exhaustion by compact sets.

In the category of locally compact groups, care must be taken with infinite unions and direct products. Suppose $(G_i)_{i \in \omega}$ is a countable increasing union of t.d.l.c. Polish groups such that $G_i \leq_o G_{i+1}$ for each $i$. Then $G := \bigcup_{i \in \omega} G_i$ is a t.d.l.c. Polish group under the *inductive limit topology*: $A \subseteq G$ is defined to be open if and only if $A \cap G_i$ is open in $G_i$ for each $i$. 
Definition 1.10. Suppose \((G_i)_{i \in \omega}\) is a sequence of t.d.l.c. Polish groups and for each \(i\) distinguish \(U_i \in \mathcal{U}(G_i)\). Put

- \(S_0 := \prod_{i \in \omega} U_i\)
- \(S_{n+1} := G_0 \times \cdots \times G_n \times \prod_{i \geq n+1} U_i\)

The \textit{local direct product} of \((G_i)_{i \in \omega}\) over \((U_i)_{i \in \omega}\) is defined to be \(\bigoplus_{i \in \omega} (G_i, U_i) := \bigcup_{i \in \omega} S_i\) with the inductive limit topology.

We remark that the local direct product can similarly be defined for index set \(\mathbb{Z}\) by taking \(S_{n+1} := \prod_{i \leq -n-1} U_i \times G_{-n} \times \cdots \times G_n \times \prod_{i \geq n+1} U_i\).

Observation 1.11. Let \((G_i)_{i \in \omega}\) be a sequence of t.d.l.c. Polish groups and for each \(i \in \omega\) fix \(U_i \in \mathcal{U}(G_i)\). Then

1. Since \(S_n \sqsubseteq S_{n+1}\) for each \(n\), \(\bigoplus_{i \in \omega} (G_i, U_i)\) is again a t.d.l.c. Polish group.
2. \(\prod_{i \in \omega} U_i\) is a compact open subgroup of \(\bigoplus_{i \in \omega} (G_i, U_i)\).

We conclude this section by recalling a number of properties of t.d.l.c. Polish groups. A t.d.l.c. Polish group \(G\) is \textit{compactly generated} if there is a compact set \(K \subseteq G\), such that \(G = \langle K \rangle\). Note that if \(G\) is compactly generated and \(H \leq_{cc} G\), then \(H\) is compactly generated [27]. \(G\) is said to be \textit{residually} \(Q\) for \(Q\) some property of groups if for all \(g \in G \setminus \{1\}\) there is \(N \leq G\) such that \(G/N\) has property \(Q\) and the image of \(g\) in \(G/N\) is non-trivial. \(G\) is said to be a \textit{small invariant neighborhood}, denoted \(SIN\), group if \(G\) admits a basis at 1 of compact open normal subgroups. We note a characterization of compactly generated \(SIN\) groups.
**Theorem 1.12** (Caprace, Monod [10]). A compactly generated t.d.l.c. group is SIN if and only if it is residually discrete.

$G$ is called *locally elliptic* if every finite subset generates a relatively compact subgroup.

**Theorem 1.13** (Platonov [31]). A l.c. group is locally elliptic if and only if it is a directed union of compact open subgroups.

We remark that a proof of Theorem 1.13 in the case of t.d.l.c. Polish groups is given in Proposition 6.5.

$G$ is a *quasi-product* with quasi-factors $N_1, \ldots, N_k$ if $N_i \leq G$ for each $1 \leq i \leq k$ and the multiplication map $m : N_1 \times \cdots \times N_k \to G$ is injective with dense image. Two subgroups $M \leq G$ and $N \leq G$ are *commensurate*, denoted $M \sim_c N$, if $|M : M \cap N|$ and $|N : M \cap N|$ are finite. It is easy to check $\sim_c$ is an equivalence relation on $\mathcal{S}(G)$ and is preserved under the action by conjugation of $G$ on $\mathcal{S}(G)$. The commensurability relation allows us to define a *commensurator subgroup*: For $N \leq G$, the commensurator subgroup of $N$ in $G$ is defined to be $\text{Comm}_G(N) := \{g \in G \mid gNg^{-1} \sim_c N\}$. We say $N$ is commensurated if $\text{Comm}_G(N) = G$.

### 1.5 Normal Subgroups

A *filtering family* of subgroups of a group $G$, denoted $\mathcal{F}$, is a collection of subgroups such that for all $N, M \in \mathcal{F}$ there is $K \leq M \cap N$ such that $K \in \mathcal{F}$.

**Proposition 1.14** (Caprace, Monod [10]). Let $G$ be a compactly generated t.d.l.c. group. Then for any compact open subgroup $V$ the subgroup $Q_V := \bigcap_{g \in G} gVg^{-1}$ is such that any filtering family of non-discrete closed normal subgroups of $G/Q_V$ has non-trivial intersection.
Proof. Let $V$ be a compact open subgroup of $G$ and let $\Gamma$ be the Cayley-Abels graph on the coset space $G/V$. The kernel of the action of $G$ on $\Gamma$ is precisely $Q_V = \bigcap_{g \in G} gVg^{-1}$, and thus, $G/Q_V$ acts faithfully on $\Gamma$.

Suppose $\mathcal{N}$ is a filtering family of non-discrete closed normal subgroups of $G/Q_V$, fix $v \in V\Gamma$, and let $B(v)$ be the neighbors of $v$ in $\Gamma$. For $N \in \mathcal{N}$, certainly $N(v)$ acts on $B(v)$ and induces a homomorphism $N(v) \to \text{Sym}(B(v))$. Say this map has image $F_N$. If $F_N$ is trivial, then $N(gv) = gN(v)g^{-1}$ must act trivially on $B(gv)$ for all $g \in G$. Since $\Gamma$ is connected, it follows that $N(v)$ acts trivially on $\Gamma$. So $N(v) = (G/Q_V)(v) \cap N = \{1\}$ contradicting that $N$ is non-discrete. $F_N$ is thus non-trivial for every $N \in \mathcal{N}$.

Since $\{F_N : N \in \mathcal{N}\}$ is a filtering family and $\text{Sym}(B(v))$ is finite, $\bigcap_{N \in \mathcal{N}} F_N \neq \{1\}$. Take $\sigma \in \bigcap_{N \in \mathcal{N}} F_N$ non-trivial and let $N_\sigma$ be the pre-image of $\sigma$ in $N$ for each $N \in \mathcal{N}$. The $N_\sigma$ are closed subsets of $G(v)$, and further, the collection $(N_\sigma)_{N \in \mathcal{N}}$ has the finite intersection property. As $G(v)$ is compact, we conclude that $\bigcap_{N \in \mathcal{N}} N_\sigma$ is non-empty and, therefore, $\bigcap_{N \in \mathcal{N}} N$ is non-trivial. \qed

Corollary 1.15. Suppose $G$ is a compactly generated t.d.l.c. Polish group and $\mathcal{N}$ is a filtering family of closed normal subgroups of $G$. If $\bigcap \mathcal{N}$ is trivial, then some element of $\mathcal{N}$ is compact by discrete.

Proof. Suppose $\mathcal{N}$ is a filtering family of closed normal subgroups of $G$ such that $\bigcap \mathcal{N} = \{1\}$. It is easy to see $\{N^c : N \in \mathcal{N}\}$ is an open cover of $G \setminus \{1\}$. Since $G \setminus \{1\}$ is Lindelöf, there is a countable subcover. We may thus take $\mathcal{N}$ to be a countable filtering family of $G$ such that
$\bigcap \mathcal{N} = \{1\}$. Since $\mathcal{N}$ is countable, there is $(N_i)_{i \in \omega}$ an \(\subseteq\)-decreasing sequence of members of $\mathcal{N}$ such that $\bigcap_{i \in \omega} N_i = \{1\}$.

Fix $V \in \mathcal{U}(G)$ and let $Q_V$ be as given by Proposition 1.14. Since $Q_V$ is compact, $N_i Q_V$ is a closed normal subgroup of $G$. Furthermore, $\bigcap_{i \in \omega} N_i Q_V = Q_V$. Indeed, take $x \in \bigcap_{i \in \omega} N_i Q_V$.

So for each $i \in \omega$, we have that $x = n_i q_i \in N_i Q_V$. Since $Q_V$ is compact, there is a subsequence $q_{i_j}$ that converges to some $q \in Q_V$. It is now obvious that $n_{i_j}$ must converge to some $n$ as well. Taking $N_k$ for any fixed $k$, we conclude that, since $(N_i)_{i \in \omega}$ is decreasing, $n_{i_j} \in N_k$ for all $i_j \geq k$.

It follows $n \in N_k$ for all $k$ and must be trivial, so $x \in Q_V$ as desired.

Now $(N_i Q_V / Q_V)_{i \in \omega}$ is a filtering family in $G / Q_V$. If $x Q_V \in \bigcap_{i \in \omega} N_i Q_V / Q_V$, then $x Q_V = n_i Q_V$ for some $n_i \in N_i$ for each $i \in \omega$. We conclude that $x = n_i q_i$ for each $i$ and, therefore, $x \in \bigcap_{i \in \omega} N_i Q_V = Q_V$. Hence, $\bigcap_{i \in \omega} N_i Q_V / Q_V = \{1\}$. Applying Proposition 1.14, some $N_i Q_V / Q_V$ is discrete. Since $N_i Q_V / Q_V \simeq N_i / (N_i \cap Q_V)$, we have that $N_i$ is compact by discrete as desired.

We conclude this section by recalling a variety of canonical normal subgroups of a t.d.l.c. Polish group $G$. The quasi-center of $G$ is $QZ(G) := \{g \in G \mid C_G(g) \text{ is open}\}$ [9]. We note that $QZ(G)$ is characteristic but not necessarily closed. For example, consider the profinite group $F^\mathbb{N}$ for $F$ a non-abelian finite simple group [9]. The discrete residual of $G$, denoted $Res(G)$, is the intersection of all open normal subgroups.

**Proposition 1.16.** If $G$ is a t.d.l.c. Polish group, then $G / Res(G)$ is a countable increasing union of SIN groups.
Proof. Let $(g_i)_{i \in \omega}$ list a countable dense subset of $G$, fix $U \in \mathcal{U}(G)$, and set $O_i := \langle U, g_0, \ldots, g_i \rangle$. Certainly, $G = \bigcup_{i \in \omega} O_i$. Since $G/\text{Res}(G)$ is residually discrete, $O_i \text{Res}(G)/\text{Res}(G) \simeq O_i/O_i \cap \text{Res}(G)$ is residually discrete as well as compactly generated. We conclude $O_i/O_i \cap \text{Res}(G)$ is a SIN group, and the proposition is proven.

\textbf{Theorem 1.17} (Platonov [31]). For a l.c. group $G$ there is a unique maximal closed normal subgroup which is locally elliptic, denoted $\text{Rad}_{\text{LE}}(G)$.

$\text{Rad}_{\text{LE}}(G)$ is called the \textit{locally elliptic radical} of $G$.

\subsection*{1.6 Synthetic Subgroups}

The inchoate idea of a synthetic subgroup appears in the work of I.V. Protasov and V.S. Ćarın in [33] from the late seventies. Specifically, for a t.d.l.c. group $G$ and $P_1(G) := \{ g \in G \mid \text{cl}(\langle g \rangle) \text{ is compact}\}$, Protasov and Ćarın define the \textit{periodic radical} to be the collection of $g \in G$ such that $gP_1(G) \subseteq P_1(G)$. This collection is easily shown to be a normal subgroup of $G$.

We abstract this idea to show a general relationship between sets of closed subgroups and normal subgroups. The key idea we borrow from Protasov and Ćarın is to consider the elements which can be \textit{added} to the members of a collection of subgroups.

For a t.d.l.c. group $G$, a subset $\mathcal{A} \subseteq S(G)$ is \textit{conjugation invariant} if $\mathcal{A}$ is fixed setwise under the action by conjugation of $G$ on $S(G)$. We say $\mathcal{A}$ is \textit{hereditary} if for all $A \in \mathcal{A}$ and closed $B \leq A$, we have $B \in \mathcal{A}$. 
Definition 1.18. Suppose $\mathcal{A} \subseteq S(G)$ is conjugation invariant and hereditary. The **$\mathcal{A}$-core**, denoted $N_\mathcal{A}$, is the collection of $g \in G$ such that if $C \in \mathcal{A}$, then $cl(\langle g, C \rangle) \in \mathcal{A}$.

Proposition 1.19. If $\mathcal{A} \subseteq S(G)$ is conjugation invariant and hereditary, then $N_\mathcal{A}$ is a normal subgroup of $G$.

Proof. Certainly, $N_\mathcal{A}$ is closed under inverses. For multiplication, take $h, g \in N_\mathcal{A}$ and $C \in \mathcal{A}$. Plainly, $cl(\langle h, g, C \rangle) \in \mathcal{A}$ and $cl(\langle hg, C \rangle) \leq cl(\langle h, g, C \rangle)$. Since $\mathcal{A}$ is hereditary, we conclude that $cl(\langle hg, C \rangle) \in \mathcal{A}$ and $N_\mathcal{A}$ is a subgroup.

For normality, take $g \in N_\mathcal{A}$, $k \in G$, and $C \in \mathcal{A}$. Since $\mathcal{A}$ is conjugation invariant, $k^{-1}Ck \in \mathcal{A}$, whereby $cl(\langle g, k^{-1}Ck \rangle) \in \mathcal{A}$. Conjugating with $k$, $cl(\langle kgk^{-1}, C \rangle) \in \mathcal{A}$, $kgk^{-1} \in N_\mathcal{A}$, and $N_\mathcal{A}$ is normal. \qed

Definition 1.20. A subgroup of a t.d.l.c. group $G$ of the form $N_\mathcal{A}$ for some conjugation invariant and hereditary $\mathcal{A} \subseteq S(G)$ is called a **synthetic subgroup** of $G$.

We give four examples of synthetic subgroups. The latter three appear regularly in later chapters.

(i) Let $N \trianglelefteq G$ be a closed normal subgroup of a t.d.l.c. group $G$. So $S(N) \subseteq S(G)$ is conjugation invariant and hereditary. One checks that $N$ is the $S(N)$-core.

(ii) Let $G$ be a t.d.l.c. group and consider the quasi-center of $G$, $QZ(G)$. Putting $QZ := \{C \in S(G) \mid C_G(C) \text{ is open}\}$, we see that $QZ(G)$ is the $QZ$-core.

(iii) Our second example is Platonov’s locally elliptic radical in the Polish setting. Let $G$ be a t.d.l.c. Polish group and $\mathcal{C} \subseteq S(G)$ be the collection of compact subgroups. Certainly, $\mathcal{C}$
is conjugation invariant and hereditary. Form $N_C$. Since $\{1\} \in C$, $cl(\langle X \rangle)$ is compact for all $X \subseteq N_C$ finite. Via results of [31] or Proposition 6.5 here, $cl(N_C) \leq Rad_{LE}(G)$.

Conversely, take $r \in Rad_{LE}(G)$ and $C \in C$. Since $r^C \subseteq Rad_{LE}(G)$ is compact, Proposition 6.5 implies $r^C$ generates a compact subgroup. Thus, $cl(\langle r, C \rangle) \leq cl(\langle r^C \rangle)C$ is a compact subgroup, and $r \in N_C$. We conclude that $N_C = Rad_{LE}(G)$ and $Rad_{LE}(G)$ is the $C$-core.

(iv) Consider the following family of subgroups in a t.d.l.c. Polish group $G$:

$$SIN(G) := \{ C \in S(G) \mid \forall V \in U(G) \exists W \leq_o V : C \leq N_G(W) \}$$

It is easy to check $SIN(G)$ is conjugation invariant and hereditary. We call $SIN(G)$ the $SIN$ family and $SIN(G) := N_{SIN(G)}$ the $SIN$-core.

The SIN-core enjoys many nice properties. Immediately, $QZ(G) \leq SIN(G)$.

**Proposition 1.21.** If $G$ is a t.d.l.c. Polish group, then $SIN(G)$ is closed.

**Proof.** Take $C \in SIN(G)$ and let $U \in U(G)$ be such that $C \leq N_G(U)$. We claim $CU \in SIN(G)$. Indeed, fix $V \in U(G)$ and find $W \leq_o U$ such that $W \leq V$. Since $C \in SIN(G)$, there is $L \leq_o W$ such that $C \leq N_G(L)$. Thus,

$$\bigcap_{cu \in CU} cuWu^{-1}c^{-1} = \bigcap_{e \in C} eWe^{-1} \geq L$$

and $\bigcap_{cu \in CU} cuWu^{-1}c^{-1}$ is an open subgroup of $V$ normalized by $CU$. We conclude that $CU \in SIN(G)$. 
Say \( g_i \to g \) with \( g_i \in \text{SIN}(G) \) for all \( i \). Take \( C \in \text{SIN}(G) \), find \( U \in \mathcal{U}(G) \) such that \( CU \in \text{SIN}(G) \), and fix \( i \) with \( g_i^{-1} g \in U \). Since \( g_i \in \text{SIN}(G) \), we have that \( \langle g_i, CU \rangle \in \text{SIN}(G) \) and, by choice of \( i \), \( \langle g, CU \rangle \in \text{SIN}(G) \). The hereditary property implies \( \text{cl}(\langle g, C \rangle) \in \text{SIN}(G) \), and we have the proposition.

The SIN-core also has a very nice internal structure:

**Proposition 1.22.** If \( G \) is a t.d.l.c. Polish group, then \( \text{SIN}(G) \) is an increasing union of compactly generated relatively open SIN groups.

**Proof.** Fix \( U \in \mathcal{U}(G) \), let \( (n_i)_{i \in \omega} \) list a countable dense subset of \( N := \text{SIN}(G) \), and put \( H_i := \langle N \cap U, n_0, \ldots, n_i \rangle \). Certainly, \( N = \bigcup_{i \in \omega} H_i \) where \( (H_i)_{i \in \omega} \) is an \( \subseteq \)-increasing sequence of compactly generated relatively open subgroups of \( N \). It remains to show each \( H_i \) is SIN. To that end, fix \( i \in \omega \), take \( O \subseteq H_i \) a neighborhood of 1 in \( H_i \), and find \( W \in \mathcal{U}(G) \) such that \( W \cap H_i \subseteq O \). Since \( H_i \in \text{SIN}(G) \) by choice of \( n_0, \ldots, n_i \), there is \( L \leq_o W \) with \( H_i \in N_G(L) \).

Hence, \( L \cap H_i \leq_o H_i \) is a subset of \( O \cap H_i \), and \( H_i \) is a SIN group. \( \square \)
CHAPTER 2

ELEMENTARY GROUPS

In the study of t.d.l.c. Polish groups, groups “built” from profinite groups and discrete groups arise often. Certainly, there are a number of counterexamples built as such. Less obviously, the work of Platonov [31] demonstrates that a t.d.l.c. Polish group such that every finite set generates a relatively compact subgroup may be written as a countable increasing union of profinite groups. More subtly still, a residually discrete t.d.l.c. Polish group may be written as a countable increasing union of SIN groups by results of P-E. Caprace and N. Monod [10].

T.d.l.c. Polish groups built from profinite and discrete groups also seem to play an important role in the structure theory of arbitrary t.d.l.c. Polish groups. Indeed, [18] shows the kernel of the adjoint representation of a $p$-adic Lie group is such a group. The work [10] indicates groups built from profinite and discrete are the barrier to finding minimal non-trivial normal subgroups, an important step in decomposing a t.d.l.c. group into “basic” groups.

The frequency of occurrence of t.d.l.c. Polish groups built from profinite and discrete groups and the important role they appear to play in the structure of general t.d.l.c. Polish groups encourages further study. We here initiate this study and make a number of contributions to the theory.

We begin by considering a seemingly narrow class of t.d.l.c. Polish groups built from profinite and discrete groups:
**Definition 2.1.** The class of *elementary groups* is the smallest class $\mathcal{E}$ of t.d.l.c. Polish groups such that

(i) $\mathcal{E}$ contains all second countable profinite groups and countable discrete groups.

(ii) $\mathcal{E}$ is closed under group extensions. I.e. if $G$ is a t.d.l.c. Polish group and $H \trianglelefteq G$ is a closed normal subgroup with $H, G/H \in \mathcal{E}$, then $G \in \mathcal{E}$.

(iii) If $G$ is a t.d.l.c. Polish group and $G = \bigcup_{i \in \omega} O_i$ where $(O_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of open subgroups of $G$ with $O_i \in \mathcal{E}$ for all $i$, then $G \in \mathcal{E}$. We say $\mathcal{E}$ is *closed under countable increasing unions*.

**Remark 2.2.** The class $\mathcal{E}$ could more accurately but less efficiently be called the class of *topologically elementary* groups. We omit “topologically” as it is redundant in light of the fact we are interested in the topological theory. We apologize for reducing the interesting and complicated classes of discrete groups and profinite groups to the status of “elementary” groups. Of course, such definitional slights are common in mathematics: the rich and deep theory of finite groups concerns virtually trivial groups.

In this chapter, we show $\mathcal{E}$ satisfies strong permanence properties indicating this class captures our intuitive idea of a group “built” from profinite groups and discrete groups:

**Theorem 2.3.** $\mathcal{E}$ enjoys the following permanence properties:

(1) If $G \in \mathcal{E}$, $H$ is a t.d.l.c. Polish group, and $\psi : H \to G$ is a continuous, injective homomorphism, then $H \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under taking closed subgroups.
(2) $\mathcal{E}$ is closed under taking quotients by closed normal subgroups.

(3) If $G$ is a residually elementary t.d.l.c. Polish group, then $G \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under inverse limits.

(4) $\mathcal{E}$ is closed under quasi-products.

(5) $\mathcal{E}$ is closed under local direct products.

We go on to prove a general structure theorem which plays an important role in the structure theory of t.d.l.c. Polish groups as evidenced by its many applications in later chapters.

**Theorem 2.4.** Let $G$ be a t.d.l.c. Polish group. Then there is a sequence of closed characteristic subgroups $\{1\} \leq N \leq Q \leq G$ such that $N$ and $G/Q$ are elementary, $Q/N$ has no non-trivial elementary normal subgroups, and $Q/N$ has no non-trivial elementary quotients.

We remark that Theorem 2.4 indeed follows from either of two more precise theorems proved below.

The chapter concludes with a second structure theorem and a variety of examples.

### 2.1 The Construction Rank

For $G \in \mathcal{E}$, define

- $G \in \mathcal{E}_0$ if and only if $G$ is profinite or discrete.

- Suppose $\mathcal{E}_\alpha$ is defined. Put $G \in \mathcal{E}_\alpha^e$ if and only if there exists $N \leq G$ such that $N \in \mathcal{E}_\alpha$ and $G/N \in \mathcal{E}_\alpha$. Put $G \in \mathcal{E}_\alpha^l$ if and only if $G = \bigcup_{i \in \omega} H_i$ where $(H_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of open subgroups of $G$ and $H_i \in \mathcal{E}_\alpha$ for all $i \in \omega$. Define $\mathcal{E}_{\alpha+1} := \mathcal{E}_\alpha^e \cup \mathcal{E}_\alpha^l$. 
• For $\lambda$ a limit ordinal, $E_\lambda := \bigcup_{\beta < \lambda} E_\beta$.

We set $rk(G) := \min\{\alpha \mid G \in E_\alpha\}$; we call this rank the construction rank of $G$.

**Observation 2.5.** Let $G$ be elementary.

1. If $rk(G) = \alpha > 0$ and $G$ is compactly generated, then $G$ is a group extension of a group of strictly lower rank by a group of strictly lower rank.

2. $rk(G)$ is always a successor ordinal less then $\omega_1$, the first uncountable ordinal.

3. If $G \simeq H$ as topological groups, then $H$ is elementary and $rk(G) = rk(H)$.

The proofs of the above observations are elementary and are omitted.

**Proposition 2.6.** If $G \in E$ and $O \leq_G G$, then $O \in E$ and $rk(O) \leq rk(G)$.

**Proof.** We induct on the rank of $G$. As the base case is immediate, suppose the lemma holds for groups with rank at most $\alpha$, $G$ has rank $\alpha + 1$, and $O \leq_G G$. If $G$ is built by group extension of groups with lower rank, there is $N \trianglelefteq G$ such that $rk(N) \leq \alpha$ and $rk(G/N) \leq \alpha$.

So $ON/N \leq G/N$, and by induction, $rk(ON/N) \leq \alpha$. We conclude that $O/O \cap N \simeq ON/N$ has rank at most $\alpha$. Since $O \cap N \leq_G N$, the induction hypothesis implies $O \cap N$ also has rank at most $\alpha$. Hence, $rk(O) \leq \alpha + 1$.

If $G = \bigcup_{i \in \omega} W_i$ where $(W_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of open subgroups with $rk(W_i) \leq \alpha$, then $O = \bigcup_{i \in \omega} O \cap W_i$, and $rk(O \cap W_i) \leq \alpha$ for all $i$ by the induction hypothesis. So $rk(O) \leq \alpha + 1$, and we have completed the induction. \qed

**Proposition 2.7.** If $G \in E$ and $K \trianglelefteq G$ is compact, then $G/K \in E$ and $rk(G/K) \leq rk(G)$.
Proof. We induct on \( rk(G) \) for the proposition. As the base case is obvious, suppose \( rk(G) = \alpha + 1 \). If \( G \) is given by a group extension, find \( H \trianglelefteq G \) such that \( H \in \mathcal{E}_\alpha \) and \( G/H \in \mathcal{E}_\alpha \). Observe that \( HK \trianglelefteq G \) is a closed normal subgroup, and therefore, \( KH/H \trianglelefteq G/H \) is a compact normal subgroup. The induction hypothesis implies \( G/KH \) is elementary and \( rk(G/KH) \leq rk(G/H) \).

Applying the induction hypothesis again, we have \( HK/K \cong H/(H \cap K) \) is elementary of rank at most \( rk(H) \). We now have a short exact sequence of topological groups:

\[
1 \rightarrow HK/K \rightarrow G/K \rightarrow G/HK \rightarrow 1
\]

\( G/K \) is thus rank at most \( \alpha \) by rank at most \( \alpha \), whereby \( rk(G/K) \leq \alpha + 1 \) as desired.

Suppose \( G = \bigcup_{i \in \omega} H_i \) with \( (H_i)_{i \in \omega} \) an \( \subseteq \)-increasing sequence of open subgroups with \( rk(H_i) \leq \alpha \). Plainly, \( G/K = \bigcup_{i \in \omega} H_i K/K \). Since \( H_i K/K \cong H_i/H_i \cap K \), the induction hypothesis implies that \( rk(H_i/H_i \cap K) \leq rk(H_i) \) and, thus, \( rk(G/K) \leq \alpha + 1 \).

\[\square\]

2.2 Permanence Properties

Proposition 2.8. Residually discrete t.d.l.c. Polish groups are elementary.

Proof. Let \( G \) be a residually discrete t.d.l.c. Polish group, so \( Res(G) = \{1\} \). Via Proposition 1.16, we see \( G \in \mathcal{E} \).

\[\square\]

Proposition 2.9. Suppose \( G \) is a t.d.l.c. Polish group. Suppose further \( cl(\bigcup_{i \in \omega} L_i) = G \) and \( (L_i)_{i \in \omega} \) is an \( \subseteq \)-increasing sequence of closed elementary subgroups of \( G \) such that there is \( U \in U(G) \) with \( U \leq N_G(L_i) \) for all \( i \). Then \( G \in \mathcal{E} \).
Proof. Certainly, $UL_i \leq_o G$ is elementary. Since $G = \bigcup_{i \in \omega} UL_i$, we conclude that $G$ is elementary.

We now set about proving the main permanence properties. In the results below, we additionally keep track of how the elementary rank increases. It is important to remember that the arithmetic on the construction ranks is ordinal arithmetic. Arithmetic for transfinite ordinals has a number of peculiarities; see [24].

**Theorem 2.10.** If $G \in \mathcal{E}$, $H$ is a t.d.l.c. Polish group, and $\psi : H \to G$ is a continuous, injective homomorphism, then $H \in \mathcal{E}$ with $rk(H) \leq rk(G) + 2$. In particular, if $H \leq G$ is a closed subgroup, then $H \in \mathcal{E}$ with $rk(H) \leq rk(G) + 2$

Proof. We induct on $rk(G)$ for the theorem. For the base case, $rk(G) = 0$, $G$ is either discrete or profinite. The former case plainly implies $H$ is discrete and, therefore, has rank zero. For the latter case, let $(U_i)_{i \in \omega}$ be a normal basis at 1 for $G$. Consider $(\psi^{-1}(U_i))_{i \in \omega}$. Certainly, $\psi^{-1}(U_i) \leq_o H$ for each $i \in \omega$, and $\bigcap_{i \in \omega} \psi^{-1}(U_i) = \{1\}$; i.e. $H$ is residually discrete. Via Proposition 2.8, we conclude $H$ is elementary, in particular an increasing union of $SIN$ groups. Hence, $rk(H) \leq 2$.

Suppose $rk(G) = \beta + 1$. If $G = \bigcup_{i \in \omega} O_i$ with $(O_i)_{i \in \omega}$ an $\subseteq$-increasing sequence of open subgroups of rank at most $\beta$, then $H = \bigcup_{i \in \omega} \psi^{-1}(O_i)$. Each $\psi^{-1}(O_i)$ is elementary by the induction hypothesis, and therefore, $H$ is elementary. Since the induction hypothesis gives $rk(\psi^{-1}(O_i)) \leq \beta + 2$ of each $i$, we have that $rk(H) \leq \beta + 3$. 

If $G$ is a group extension, take $M \trianglelefteq G$ such that $rk(M), rk(G/M) \leq \beta$. By the induction hypothesis, we have that $\psi^{-1}(M) \trianglelefteq H$ is elementary with rank at most $\beta + 2$. Additionally, $\psi$ induces a continuous, injective homomorphism $\tilde{\psi} : H/\psi^{-1}(M) \to G/M$. So the induction hypothesis implies $H/\psi^{-1}(M)$ is also elementary with rank at most $\beta + 2$. We conclude $H$ is elementary with rank at most $\beta + 3$, and the induction is finished. \hfill \Box

The class of elementary groups is also closed under quotients. Towards demonstrating this permanence property, a lemma is required.

**Lemma 2.11.** Let $G$ be a t.d.l.c. Polish group. If $P \trianglelefteq G$ is elementary, $L \trianglelefteq G$, and $[P, L] = \{1\}$, then $cl(PL)/L$ is elementary with $rk(cl(PL)/L) \leq 3(rk(P) + 1)$.

**Proof.** We induct on $rk(P)$ for the lemma. For the base case, $rk(P) = 0$, $P$ is either compact or discrete. For the former case $PL$ is closed. So $PL/L \simeq P/P \cap L$ and $PL/L$ is elementary with rank zero. For the latter case, every element of $P$ has an open centralizer since $P$ discrete and normal. Plainly, every element of $P$ in $cl(PL)$ also has an open centralizer. We thus have that $SIN(cl(PL)/L) = cl(PL)/L$ and $cl(PL)/L$ is elementary. Since the SIN-core is an increasing union of SIN groups, $rk(cl(PL)/L) \leq 2$.

Suppose $rk(P) = \beta + 1$. Fix $U \in U(cl(PL))$, fix $V \in U(P)$, and let $(p_i)_{i \in \omega}$ list a countable dense subset of $P$. Setting $P_i := \langle V^U, p_0^U, \ldots, p_i^U \rangle$, we have that each $P_i$ is open in $P$, compactly generated, and normalized by $U$. Additionally, $(P_iUL)_{i \in \omega}$ forms an $(\subseteq)$-increasing sequence of open subgroups which exhausts $cl(PL)$. We thus see it suffices to show $P_iUL/L$ is elementary.
for each $i$. Since $cl(P_i L)/L \leq_{cc} P_i UL/L$, it indeed suffices to show $cl(P_i L)/L$ is elementary for each $i$.

Fix $i \in \omega$, put $Q := P_i$, and set $R = cl(QL)$. Since $Q \leq_o P$ and $Q$ is compactly generated, $rk(Q) \leq \beta + 1$ and there is $M \leq Q$ that witnesses the rank of $Q$. Furthermore, $Q \leq R$ and $M \leq R$ because $L$ centralizes $P$. Passing to $R/M$, we have that $Q/M \leq R/M$ and $cl(ML)/M \leq R/M$. Additionally, $Q/M$ is elementary with $rk(Q/M) \leq \beta$, and $[Q/M, cl(ML)/M] = \{1\}$. By the induction hypothesis and construction of $R$, we have $(R/M)/(cl(ML)/M)$ is elementary with rank at most $3(\beta + 1)$. Thus, $R/cl(ML)$ is elementary and $rk(R/cl(ML)) \leq 3(\beta + 1)$.

On the other hand, $rk(M) \leq \beta$, $[M, L] = \{1\}$, and $M, L \leq R$. The induction hypothesis implies $cl(ML)/L$ is also elementary with $rk(cl(ML)/L) \leq 3(\beta + 1)$. Hence, $cl(ML)/L \leq R/L$ is elementary, and $(R/L)/(cl(ML)/L) \simeq R/cl(ML)$ is elementary. We conclude $R/L$ is elementary with $rk(R/L) \leq 3\beta + 4$.

Thus, $rk(URL/L) \leq 3\beta + 5$, and since $cl(PL)/L$ is an increasing union of such subgroups, $rk(cl(PL)/L) \leq 3\beta + 6 = 3(\beta + 2)$. This completes the induction, and we conclude the lemma.

**Theorem 2.12.** If $G \in \mathcal{E}$ and $L \leq G$ is a closed normal subgroup, then $G/L \in \mathcal{E}$ with $rk(G/L) \leq 8^{rk(G)}$.

**Proof.** We induct on $rk(G)$ for the theorem. As the base case is obvious, we suppose $rk(G) = \beta + 1$. Suppose $G = \bigcup_{i \in \omega} H_i$ with $(H_i)_{i \in \omega}$ an $\subseteq$-increasing sequence of open subgroups with
rank at most $\beta$. So $G/L = \bigcup_{i \in \omega} H_i L / L$, and since $H_i L / L \simeq H_i / H_i \cap L$ for each $i$, the induction hypothesis implies $G/L$ is elementary. Further, $rk(H_i / H_i \cap L) \leq 8^{\beta}$, and

$$rk(G/L) \leq 8^\beta + 1 \leq 8^{\beta+1}$$

Suppose $G$ is a group extension of groups of lower rank. Say $H \trianglelefteq G$ is such that $rk(H) \leq \beta$ and $rk(G/H) \leq \beta$. Pass to $\tilde{G} := G/L \cap H$ and let $\tilde{H}, \tilde{L}$ be the images of $H, L$ respectively in $\tilde{G}$. Since $rk(H) \leq \beta$, we have that $\tilde{H}$ is elementary with $rk(\tilde{H}) \leq 8^\beta$ by the induction hypothesis and, additionally, $[\tilde{H}, \tilde{L}] = \{1\}$. Lemma 2.11 now implies $cl(\tilde{H} \tilde{L})/\tilde{L}$ is elementary with rank at most $3(8^\beta + 1)$. Since $cl(\tilde{H} \tilde{L})/\tilde{L} \simeq cl(HL)/L$, we conclude $cl(HL)/L$ is elementary with rank at most $3(8^\beta + 1) = 3 \cdot 8^\beta + 3$.

On the other hand, $cl(HL)/H \trianglelefteq G/H$ and $rk(G/H) \leq \beta$. The induction hypothesis gives that $(G/H)/(cl(HL)/H) \simeq G/cl(HL) \simeq (G/L)/(cl(HL)/L)$ is elementary with rank at most $8^\beta$. We now see that $G/L$ is a group extension of elementary groups and, thus, elementary. Further, $rk(G/L) \leq 3 \cdot 8^\beta + 4 \leq 4 \cdot 8^\beta + 4 \cdot 8^\beta$. If $\beta$ is finite, $4 \cdot 8^\beta + 4 \cdot 8^\beta = 8^{\beta+1}$. If $\beta$ is transfinite, then $1 + \beta = \beta$ and, thereby,

$$4 \cdot 8^\beta + 4 \cdot 8^\beta \leq 8 \cdot 8^\beta + 8 \cdot 8^\beta = 8^{1+\beta} \cdot 2 = 8^\beta \cdot 2 \leq 8^{\beta+1}$$

This completes the induction, and we conclude the theorem. \qed

We remark that the bounds on the rank obtained in Theorem 2.12 is probably quite coarse.
Theorem 2.13. If $G$ is a t.d.l.c. Polish group that is residually $\mathcal{E}$, then $G \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under inverse limits.

Proof. Suppose $G$ is a t.d.l.c. Polish group that is residually elementary and let $\mathcal{F}$ be the collection of closed normal subgroups of $G$ with elementary quotient.

We first claim $\mathcal{F}$ is a filtering family; it suffices to show $\mathcal{F}$ is closed under intersection. Take $N, M \in \mathcal{F}$ and consider $M/M \cap N$. Certainly, the restriction of the usual projection gives a continuous, injective homomorphism $M/M \cap N \to G/N$. Since $G/N$ is elementary, Theorem 2.10 implies $M/M \cap N$ is elementary, so $M/M \cap N$ is a closed normal elementary subgroup of $G/N \cap M$ with elementary quotient. We conclude that $G/M \cap N$ is elementary and $\mathcal{F}$ is closed under intersection.

Fix $O \leq_o G$ a compactly generated open subgroup of $G$ and put $\mathcal{F}_O := \{ N \cap O \mid N \in \mathcal{F} \}$. Since $O$ is open, for each $N \in \mathcal{F}$ we have that $ON/N \simeq O/N \cap O$ and, therefore, $O/N \cap O$ is elementary. So $\mathcal{F}_O$ is a filtering family of closed normal subgroups of $O$ with elementary quotient. Since $G$ is residually elementary, $\bigcap \mathcal{F}_O = \{1\}$. Corollary 1.15 now implies there is $N \in \mathcal{F}_O$ which is profinite-by-discrete - i.e. elementary. Since $O/N$ is also elementary, we conclude $O$ is elementary. Since $G$ has a countable $\leq$-increasing exhaustion by such $O$, $G$ is elementary.

Suppose $(C_\alpha, \phi_\alpha)_{\alpha \in I}$ is an inverse system of elementary groups. We have that $H := \lim_{\alpha \in I}(C_\alpha, \phi_\alpha)$ is a closed subgroup of $\prod_{\alpha \in I} C_\alpha$. Further, the projection onto each coordinate $\alpha \in I$, $\pi_\alpha$, is a continuous homomorphism into an elementary group. Theorem 2.10 thus
implies $H/\ker(\pi_\alpha|_H)$ is elementary for all $\alpha \in I$. Since $\bigcap_{\alpha \in I} \ker(\pi_\alpha|_H) = \{1\}$, we conclude $H$ is residually elementary and, by the above, elementary. 

The reader familiar with the theory of elementary amenable discrete groups notes that many of our closure properties above mirror those of the class of elementary amenable groups. Theorem 2.13, however, is surprisingly disanalogous. Indeed, the analogous statement is patently false in the class of elementary amenable groups: the free group on two generators is residually nilpotent but, certainly, is not elementary amenable.

**Proposition 2.14.** $\mathcal{E}$ is closed under quasi-products.

**Proof.** Suppose $G$ is a t.d.l.c. Polish group which is a quasi-product of $N_1, \ldots, N_k$. Via an easy induction argument, it suffices to check the case when $G$ is a quasi-product of two groups $N, M$.

Fixing $U \in U(G)$, we have that $UNM = G$ and, further,

$$G/M = UNM/M \simeq UN/(UN \cap M)$$

since $UN$ is open. Certainly, $UN \in \mathcal{E}$, so Theorem 2.12 implies $UN/(UN \cap M) \in \mathcal{E}$. We conclude $G/M$ is elementary, and the theorem follows. 

**Proposition 2.15.** Suppose $(G_i)_{i \in \omega}$ consists of elementary groups and for each $i$ we have $U_i \in U(G_i)$. Then $\bigoplus_{i \in \omega}(G_i, U_i) \in \mathcal{E}$ and

$$\sup_{i \in \omega}(rk(G_i)) \leq rk\left(\bigoplus_{i \in \omega}(G_i, U_i)\right)$$
Proof. It is immediate that $\bigoplus_{i \in \omega} (G_i, U_i)$ is elementary. Indeed, $\bigoplus_{i \in \omega} (G_i, U_i)$ is an increasing union of groups of the form $G_0 \times \cdots \times G_n \times \prod_{i \geq n+1} U_i$. These groups are easily seen to be built via repeated group extension from profinite groups and $G_0, \ldots, G_n$.

By construction of $\bigoplus_{i \in \omega} (G_i, U_i)$, we have

$$
(U_1 \times \cdots \times U_{n-1} \times G_n \times \prod_{i \geq n+1} U_i) \leq_{n} \bigoplus_{i \in \omega} (G_i, U_i)
$$

Furthermore,

$$
(U_1 \times \cdots \times U_{n-1} \times \{1\} \times \prod_{i \geq n+1} U_i) \leq (U_1 \times \cdots \times U_{n-1} \times G_n \times \prod_{i \geq n+1} U_i)
$$

whose quotient isomorphic to $G_n$. Applying Proposition 2.7 and Proposition 2.6, we have

$$
\text{rk}(G_n) \leq \text{rk} \left( U_1 \times \cdots \times U_{n-1} \times G_n \times \prod_{i \geq n+1} U_i \right) \leq \text{rk} \left( \bigoplus_{i \in \omega} (G_i, U_i) \right)
$$

The proposition now follows. \hfill \Box

Collecting our results above, we have proved the following theorem:

**Theorem 2.16.** $\mathcal{E}$ enjoys the following properties:

(1) If $G \in \mathcal{E}$, $H$ is a t.d.l.c. Polish group, and $\psi : H \to G$ is a continuous, injective homomorphism, then $H \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under taking closed subgroups.

(2) $\mathcal{E}$ is closed under taking quotients by closed normal subgroups.
(3) If $G$ is a residually $\mathcal{E}$ t.d.l.c. Polish group, then $G \in \mathcal{E}$. In particular, $\mathcal{E}$ is closed under inverse limits.

(4) $\mathcal{E}$ is closed under taking quasi-products.

(5) $\mathcal{E}$ is closed under local direct products.

**Definition 2.17.** Given a class of t.d.l.c. Polish groups $\mathcal{G}$, the *elementary closure* of $\mathcal{G}$, denoted $\mathcal{E}\mathcal{G}$, is the smallest class of t.d.l.c. Polish groups such that

(i) $\mathcal{E}\mathcal{G}$ contains $\mathcal{G}$, all second countable profinite groups, and countable discrete groups.

(ii) $\mathcal{E}\mathcal{G}$ is closed under group extensions.

(iii) If $G$ is a t.d.l.c. Polish group and $G = \bigcup_{i \in \omega} G_i$ where $(G_i)_{i \in \omega}$ is a countable increasing sequence of open subgroups of $G$ with $G_i \in \mathcal{E}\mathcal{G}$ for each $i$, then $G \in \mathcal{E}\mathcal{G}$. I.e. $\mathcal{E}\mathcal{G}$ is closed under countable increasing unions.

The next observation follows from the proof of Theorem 2.16.

**Theorem 2.18.** Suppose $\mathcal{G}$ is a class of t.d.l.c. Polish groups which is closed under subgroups and quotients which satisfies the following:

(a) If $H$ is a t.d.l.c. Polish group and $\psi : H \to G$ is a continuous homomorphism with $G \in \mathcal{G}$, then $H \in \mathcal{E}\mathcal{G}$.

(b) If $G$ is a t.d.l.c. Polish group, $H, L \subseteq G$, $[H, L] = \{1\}$, and $H \in \mathcal{G}$, then $\text{cl}(HL)/L \in \mathcal{E}\mathcal{G}$.

Then the permanence properties in Theorem 2.16 hold of $\mathcal{E}\mathcal{G}$.

Classes $\mathcal{G}$ which satisfy (a) and (b) above are called *elementarily robust*. In Chapter 5, we show $\mathcal{P}$, the collection of all l.c.s.c. $p$-adic Lie groups for all primes $p$, is elementarily robust.
2.3 General Structure Theorems

It is easy to see elementary groups do not exhaust the category of t.d.l.c. Polish groups; a variety of examples of non-elementary groups are given in the examples section of this chapter. As such, we consider elementary groups appearing as normal subgroups or as quotients of an arbitrary t.d.l.c. Polish group.

Let $G$ be a t.d.l.c. Polish group and put $\mathcal{S}_E(G) := \{ N \trianglelefteq G \mid N \text{ is elementary} \}$. Suppose $(N_\alpha)_{\alpha<\lambda}$ is an $\subseteq$-increasing chain in $\mathcal{S}_E(G)$. Fix $U \in \mathcal{U}(G)$ and consider $(UN_\alpha)_{\alpha<\lambda}$. Since $\bigcup_{\alpha<\lambda} UN_\alpha$ is open, it is a Lindelöf space, so there is a countable subcover $(UN_\alpha)_{i \in \omega}$. Thus, $\bigcup_{i \in \omega} UN_\alpha \in \mathcal{E}$. In view of Theorem 2.16, $\text{cl}(\bigcup_{\alpha<\lambda} N_\alpha) \in \mathcal{E}$ and, therefore, an element of $\mathcal{S}_E$.

We conclude $\subseteq$-chains in $\mathcal{S}_E$ have upper bounds. Zorn’s lemma now implies $\mathcal{S}_E(G)$ has maximal elements.

**Claim 1.** $\mathcal{S}_E(G)$ has a unique $\subseteq$-maximal element.

**Proof.** Suppose $M, N$ are two $\subseteq$-maximal elements. Form $\text{cl}(MN)$ and consider $\text{cl}(MN)/M \cap N$.

It is easy to see $\text{cl}(MN)/M \cap N$ is a quasi-product of $M/M \cap N$ and $N/M \cap N$ and both of these groups are elementary via Theorem 2.16. Hence, $\text{cl}(MN)/M \cap N \in \mathcal{E}$, and since $M \cap N \in \mathcal{E}$, $\text{cl}(MN)$ is elementary. We conclude that $M = \text{cl}(MN) = N$ as required. 

We denote the unique maximal element of $\mathcal{S}_E(G)$ by $\text{Rad}_E(G)$. $\text{Rad}_E(G)$ is called the elementary radical of $G$. It is clear $\text{Rad}_E(G)$ is characteristic and contains every element of $\mathcal{S}_E(G)$. 

We now explore the dual notion to the elementary radical: Let \( G \) be a t.d.l.c. Polish group and put \( \mathcal{Q}_E(G) := \{ N \trianglelefteq G \mid G/N \text{ is elementary} \} \).

**Claim 2.** \( \subseteq \)-decreasing chains in \( \mathcal{Q}_E(G) \) have \( \subseteq \)-minimal elements in \( \mathcal{Q}_E(G) \).

**Proof.** Let \((N_\alpha)_{\alpha \in I}\) be an \( \subseteq \)-decreasing chain in \( \mathcal{Q}_E(G) \) and put \( N := \bigcap_{\alpha \in I} N_\alpha \). Plainly, \( G/N \) is a residually elementary group and, via the permanence properties of \( \mathcal{E} \), \( G/N \) is elementary. So \( N \in \mathcal{Q}_E(G) \), and we have the claim. \( \square \)

By Claim 2 and Zorn’s lemma, \( \mathcal{Q}_E(G) \) has minimal elements.

**Proposition 2.19.** \( \mathcal{Q}_E(G) \) has a unique \( \subseteq \)-minimal element.

**Proof.** Suppose \( M,N \) are \( \subseteq \)-minimal elements of \( \mathcal{Q}_E(G) \). Certainly, \( M/M \cap N \to G/N \) is a continuous, injective group homomorphism. Theorem 2.16 thus implies \( M/M \cap N \) is elementary. Since \((G/M \cap N)/(M/M \cap N) \simeq G/M \) and \( G/M \) is elementary, we conclude that \( G/M \cap N \) is elementary, whereby \( M = M \cap N = N \) as required. \( \square \)

We call the unique element of \( \mathcal{Q}_E(G) \) the elementary residual of \( G \), denoted \( \text{Res}_E(G) \). Note that \( \text{Res}_E(G) \) is characteristic and is contained in every element of \( \mathcal{Q}_E(G) \).

Via the elementary radical and elementary residual, we obtain three general structure results. For the first two, we require a few definitions.

**Definition 2.20.** Given a t.d.l.c. Polish group \( G \) the descending elementary series, is defined by \( D_0 := G, D_1 := \text{Res}_E(D_0), D_2 := \text{Rad}_E(D_1), \) and \( D_3 := \{1\} \).
Definition 2.21. Given a t.d.l.c. Polish group $G$ the ascending elementary series, is defined by $A_0 := \{1\}, A_1 := \text{Rad}_\mathcal{E}(G), A_2 := \pi^{-1}(\text{Res}_\mathcal{E}(G/A_1))$ where $\pi : G \to G/A_1$ is the usual projection, and $A_3 := G$.

It is worth noting the relationship between the descending and ascending elementary series. The diagram below follows from the permanence properties of $\mathcal{E}$; the arrows indicate inclusion.

\[
\begin{array}{c}
\text{D}_1 A_1 = A_2 \\
\text{D}_1 \quad A_1 \quad \text{D}_2 \quad \text{D}_1 \cap \text{D}_1
\end{array}
\]

Definition 2.22. A t.d.l.c. Polish group is elementary-free if it has no non-trivial elementary normal subgroups and no non-trivial elementary quotients.

Theorem 2.23. Let $G$ be a t.d.l.c. Polish group. Then the descending elementary series

\[ G \geq D_1 \geq D_2 \geq \{1\} \]

is a series of characteristic subgroups with $G/D_1$ elementary, $D_1/D_2$ elementary-free, and $D_2$ elementary.

Proof. It is immediate that $D_1$, $D_2$ are characteristic and $G/D_1$ and $D_2$ are elementary. For the remaining claim, it suffices to show $\text{Rad}_\mathcal{E}(D_1/D_2) = \{1\}$ and $\text{Res}_\mathcal{E}(D_1/D_2) = D_1/D_2$. The former is immediate since $D_2 = \text{Rad}_\mathcal{E}(D_1)$. For the latter, let $\pi : D_1 \to D_1/D_2$ be the usual
projection, and put $H := \pi^{-1}(\text{Res}_E(D_1/D_2))$. It is easy to see that $H$ is the intersection of all normal subgroups of $D_1$ extending $D_2$ whose quotient is elementary. Since $D_2$ is characteristic in $D_1$, $H$ is characteristic in $D_1$ and, thus, is normal in $G$.

We now have a short exact sequence of topological groups:

$$1 \to D_1/H \to G/H \to G/D_1 \to 1$$

This sequence shows that $G/H$ is a group extension of elementary groups and, therefore, elementary. Since $D_1$ is minimal with respect to normal subgroups with elementary quotient $D_1 = H$ and $\text{Res}_E(D_1/D_2) = D_1/D_2$ as required. \hfill \square

**Theorem 2.24.** Let $G$ be a t.d.l.c. Polish group. Then the ascending elementary series

$$\{1\} \leq A_1 \leq A_2 \leq G$$

is a series of characteristic subgroups with $A_1$ elementary, $A_2/A_1$ elementary-free, and $G/A_2$ elementary.

**Proof.** It is immediate that $A_1$, $A_2$ are characteristic and $A_1$ and $G/A_2$ are elementary. For the remaining claim, it again suffices to show $\text{Rad}_E(A_2/A_1) = \{1\}$ and $\text{Res}_E(A_2/A_1) = A_2/A_1$. For the former, $\text{Rad}_E(A_2/A_1)$ is characteristic in $A_2/A_1$ and, therefore, normal in $G/A_1$. Letting $\pi : G \to G/A_1$ be the usual projection, we see that $\pi^{-1}(\text{Rad}_E(A_2/A_1))$ is a elementary normal
subgroup of $G$. By choice of $A_1$, it must be the case that $\pi^{-1}(\text{Rad}_G(A_2/A_1)) = A_1$ and, thus, $\text{Rad}_G(A_2/A_1) = \{1\}$.

For the residual, put $H := \text{Res}_G(A_2/A_1)$. Since, $H$ is characteristic in $A_2/A_1$, we have that $H$ is normal in $G/A_1$. This gives a short exact sequence of topological groups:

$$1 \to (A_2/A_1)/H \to (G/A_1)/H \to (G/A_1)/(A_2/A_1) \to 1$$

This sequence shows that $(G/A_1)/H$ is a group extension of elementary groups and, therefore, elementary. Since $A_2/A_1$ is minimal with respect to normal subgroups of $G/A_1$ whose quotient is elementary, $A_2/A_1 = H$ as required.

**Remark 2.25.** Theorem 2.23 and Theorem 2.24 give different decompositions. It is also worth noting that $D_1/D_2$ and $A_2/A_1$ need not be compactly generated when $G$ is compactly generated. Examples are presented in 2.4.4 demonstrating these phenomena.

For our last structure result in this section, we require a fact from the literature:

**Theorem 2.26** (Caprace, Monod [10]). Let $G$ be a compactly generated t.d.l.c. group. Then one of the following holds:

(1) $G$ has an infinite discrete normal subgroup;

(2) $G$ has a non-trivial compact normal subgroup;

(3) $G$ has exactly $0 < n < \infty$ many minimal non-trivial closed normal subgroups.
Theorem 2.27. Let $G$ be a compactly generated t.d.l.c. Polish group. Then there is a series of closed characteristic subgroups $\{1\} = H_0 \leq H_1 \leq \ldots \leq H_\lambda \leq G$ such that $G/H_\lambda \in \mathcal{E}$ and $(H_{\beta+1}/H_\beta) / \text{Res}_\mathcal{E}(H_{\beta+1}/H_\beta)$ is a quasi-product of $0 < n_{\beta+1} < \infty$ many topologically characteristically simple non-elementary groups.

Proof. Consider $G/A_1$. In view of Theorem 2.26, there are exactly $0 < n < \infty$ many minimal non-trivial normal subgroups of $G/A_1$; let $N_1, \ldots, N_n$ list these subgroups. Letting $\pi : G \to G/A_1$, we define $H_1 := \pi^{-1}(\text{cl}(\langle N_1, \ldots, N_n \rangle))$.

We now inductively define the remaining $H_\beta$. Suppose we have $H_\beta$ a characteristic subgroup of $G$. Pass to $\tilde{G} := G/H_\beta$. If $\tilde{G} \in \mathcal{E}$, we stop; else, let $\pi : G \to \tilde{G}/\text{Rad}_\mathcal{E}(\tilde{G})$ be the usual projection. Applying Theorem 2.26, $\tilde{G}/\text{Rad}_\mathcal{E}(\tilde{G})$ has exactly $0 < m < \infty$ many minimal non-trivial normal subgroups. Put $H_{\beta+1} := \pi^{-1}(\text{cl}(\langle N_1, \ldots, N_m \rangle))$. If we have $H_\beta$ for all $\beta < \gamma$, we put $H_\gamma := \text{cl}(\bigcup_{\beta < \gamma} H_\beta)$.

This construction halts at some $\lambda < \omega_1$ which satisfies the hypotheses of the theorem. □

2.4 Examples

We conclude this chapter by giving a number of examples of elementary groups, non-elementary groups, elementary-free groups, and the elementary series.

2.4.1 First Examples and Non-Examples

Proposition 2.28. The following are elementary groups:

(1) T.d.l.c. Polish SIN groups; in particular, abelian t.d.l.c. Polish groups.

(2) T.d.l.c. Polish solvable groups.
(3) Locally elliptic t.d.l.c. Polish groups.

Proof. (1) and (2) follow immediately since SIN groups are profinite by discrete and solvable groups are build via group extension from abelian groups. (3) follows from Theorem 1.13. □

For non-examples, recall there are many non-discrete compactly generated t.d.l.c. Polish groups which are topologically simple. Indeed, let $T_n$ denote the $n$-regular tree for any $n \geq 3$. By work of Tits [37], there is an index two non-discrete compactly generated t.d.l.c. Polish subgroup of $Aut(T_n)$, denoted $Aut(T_n)^+$, that is topologically simple. Alternatively, $PSL_n(\mathbb{Q}_p)$ for $n \geq 3$ is also a non-discrete compactly generated topologically simple t.d.l.c. Polish group, cf. [4] [16].

Proposition 2.29. If a group $G$ is compactly generated, topologically simple, and elementary, then $G$ is discrete. In particular, $PSL_n(\mathbb{Q}_p)$ and $Aut(T_n)^+$ for $n \geq 3$ are non-elementary.

Proof. Let $G$ be as hypothesized and consider $rk(G)$. If $rk(G) = 0$, then the theorem obviously holds. Suppose for contradiction, $rk(G) = \beta + 1$. If $G = \bigcup_{i \in \omega} H_i$ with $(H_i)_{i \in \omega}$ a $\subseteq$-increasing sequence of open subgroups of rank at most $\beta$, then the $H_i$ form an open cover of the generating set of $G$. So $H_i = G$ for all sufficiently large $i$, and $rk(G) < \beta + 1$ contradicting the rank of $G$.

Suppose $G$ is built by group extension. Say $N \trianglelefteq G$ is such that $rk(N) \leq \beta$ and $G/N \leq \beta$. Since $G$ is topologically simple, either $N = G$ or $N = \{1\}$. In either case, $rk(G) \leq \beta$ again arriving at a contradiction. □
2.4.2 A Non-discrete Topologically Simple Elementary Group

The class $\mathcal{E}$ does, however, contain non-discrete topologically simple groups. We here give an example due to Willis [42] of such a group. For $n \geq 0$, make the following definitions

- $G_n$ is the group of $\sigma \in Sym(\mathbb{N})$ such that $\sigma |_{[0, 2^{n+1}]}$ is an even permutation of $[0, 2^{n+1}]$ and $\sigma$ fixes all $k > 2^{n+1}$
- $Q_n$ is the group of $\sigma \in Sym(\mathbb{N})$ which act as the group of cyclic permutations on $[2^n + 1, 2^{n+1}]$ and fix all $k \in \mathbb{N} \setminus [2^n + 1, 2^{n+1}]$.

Plainly, $G_n$ and $Q_n$ are finite subgroups of $Sym(\mathbb{N})$. Furthermore, $G_n \times Q_{n+1} \simeq G_nQ_{n+1} \leq G_{n+1}$. Setting $\tilde{G}_n := G_n \times \prod_{k \geq n+1} Q_k$, it is now easy to see we may take $(\tilde{G}_n)_{n \geq 0}$ to be an $\subseteq$-increasing sequence of compact groups each open in the next. Under the inductive limit topology, $G := \bigcup_{n \in \omega} \tilde{G}_n$ is a t.d.l.c. Polish group. Further, $G$ is elementary with rank 1 and has a compact open abelian subgroup $\prod_{n \geq 0} Q_n$.

**Proposition 2.30** (Willis [42]). $G$ is topologically simple.

*Proof.* Fix a non-trivial $\sigma \in G$ and consider the normal closure of $\sigma$ in $G$, denoted $N(\sigma)$. We claim $N(\sigma)$ is dense in $G$. To that end, find a large $n$ such that $\sigma \in \tilde{G}_n$ and there is $m \leq 2^{n+1}$ such that $\sigma(m) \neq m$. So $N(\sigma) \cap \tilde{G}_n$ is a non-trivial normal subgroup of $\tilde{G}_n$. Letting $\pi_1$ be the projection onto $G_n$, we have that $\pi(N(\sigma) \cap \tilde{G}) = G_n$ since $G_n$ is a finite simple group. It now follows that $N(\sigma)$ is dense as desired. \qed
2.4.3 Elementary Groups of Rank up to $\omega + 1$

We describe a family of elementary groups with members of arbitrarily large finite rank. Using these, we produce an elementary group of rank $\omega + 1$.

For the construction herein we require a technical result:

**Theorem 2.31.** Let $F$ be a non-abelian finite simple group, $D$ a discrete group, $G$ a t.d.l.c. Polish group and $\Omega'$ a countable set. Suppose $G$ acts on $\Omega'$ continuously and $\Omega \subseteq \Omega'$ is a $G$-invariant subset. Then

1. Form $H := F \wr_{D^{\omega}} D^{<\omega} \Omega$ where $D^{\omega} \acts D^{<\omega} \Omega'$ via left multiplication. Then $G \acts H$ by $g.(\alpha,f) = (g.\alpha, g.f)$ where $g.f$ is the usual shift and $g.\alpha$ is the shift induced by the action of $G$ on $D^{<\omega} \Omega'$ by shift. Further, this is a continuous action by topological group automorphisms.

2. Put $C := \{(f,s) \mid f \in D^{<\omega} \Omega'$ and $s \in F\}$ and $B := \{(f,s) \mid f \in D^{<\omega} \Omega$ and $s \in F\}$. Then $H \acts C$ continuously via the coset action: $(\alpha,g).(f,s) = (gf, \alpha(gf)s)$. Further, $B$ is invariant under this action and the action is transitive on $B$.

3. The coset action lifts to a continuous action of $H \rtimes G$ on $C$ via

$$(\alpha,g,k).(f,s) = \left(g \cdot (k.f), \alpha(g \cdot (k.f))s\right)$$

Further, $B$ is an invariant set under this action on which $G$ acts transitively, and

(a) $\ker(H \rtimes G \acts C) \leq \ker(G \acts \Omega')$, 


(b) \( \ker(H \ltimes G \rhd B) = K(D^{<\Omega}) \times \ker(G \rhd \Omega) \) where

\[
K(D^{<\Omega}) := \{ \alpha \in F^{D^{<\Omega}} : \alpha \upharpoonright_{D^{<\Omega}} = 1 \}
\]

and

(c) If \( g \in H \ltimes G \) centralizes \( H \), then \( g \) is an element of \( \ker(G \rhd \Omega') \).

**Proof.** See Appendix A Theorem A.12. \( \square \)

We now set about building our examples: Let \( A_5 \) be the alternating group on five letters; recall \( A_5 \) is a non-abelian finite simple group. Let \( S \) denote the infinite four generated simple group built by Higman [21]. Define

(i) \( L_1 := A_5 \wr S \)

(ii) \( C_1 := \{ (f, s) \mid f \in S \text{ and } s \in A_5 \} \) where \( L_1 \rhd C_1 \) by the coset action.

Suppose we have built the pairs \( (L_i, C_i) \) up to stage \( n \). Put

(i) \( L_{n+1} := A_5 \wr S^{<C_n} \)

(ii) \( C_{n+1} := \{ (f, s) \mid f \in S^{<C_n} \text{ and } s \in A_5 \} \) where \( L_{n+1} \rhd C_{n+1} \) by the coset action.

We now inductively define a second sequence of groups \( M_i \): For the base case, set \( M_1 := L_1 \). By construction, \( M_1 \) acts continuously, transitively, and faithfully on \( C_1 \). Suppose we have built \( M_n \) such that \( M_n \) acts continuously, transitively, and faithfully on \( C_n \). Applying Theorem 2.31, \( M_n \) acts on \( L_{n+1} \) by group automorphisms. We may thus form \( M_{n+1} := L_{n+1} \rtimes \).
$M_n$. Theorem 2.31 implies that $M_{n+1}$ acts continuously on $C_{n+1}$ by the coset action; it is obvious the action is also transitive. Further,

$$\ker(M_{n+1} \curvearrowright C_{n+1}) = \ker(M_n \curvearrowright C_n) = \{1\}$$

So the action is also faithful.

**Theorem 2.32.** For $n \geq 1$, $rk(M_{4^n}) \geq n$.

*Proof.* See Appendix A Theorem A.19. □

**Corollary 2.33.** For each $n \geq 0$, fix $U_n \in \mathcal{U}(M_{4^n})$. Then $\omega + 1 \leq rk\left(\bigoplus_{i \in \omega} (M_{4^n}, U_n)\right)$.

*Proof.* By Proposition 2.15, we have

$$\omega \leq \sup_{n \geq 1} rk\left(M_{4^n}\right) \leq rk\left(\bigoplus_{i \in \omega} (M_{4^n}, U_n)\right)$$

Since all elementary groups have successor rank, $\omega + 1 \leq rk\left(\bigoplus_{i \in \omega} (M_{4^n}, U_n)\right)$. □

### 2.4.4 Elementary-free Groups and Decompositions

For this last subsection, we give an example of a non-topologically simple elementary-free group and compute the descending and ascending elementary series in examples.

Fix a prime $p$, fix $U \in \mathcal{U}(PSL_3(\mathbb{Q}_p))$, and let $(G_i)_{i \in \mathbb{Z}}$ list countably many copies of $PSL_3(\mathbb{Q}_p)$ with $U_i$ the copy of $U$ in $G_i$. Form $G := \bigoplus_{i \in \mathbb{Z}} (G_i, U_i)$. $G$ has non-trivial proper normal subgroups, is non-elementary, and is not compactly generated. Indeed, $G$ has a non-trivial proper normal subgroup: $\prod_{i \leq -1} \{1\} \times G_0 \times \prod_{i \geq 1} \{1\} \leq G$. The presence of this subgroup
also indicates $G$ is non-elementary via Theorem 2.16 since $G_0$ is non-elementary. Lastly, if $G$ is compactly generated, then for some $n$ we have $\prod_{i \leq -n-1} U_i \times G_{-n} \times \cdots \times G_n \times \prod_{i \geq n+1} U_i = G$ which is plainly absurd.

**Lemma 2.34.** If $N \trianglelefteq G$, then there is $I \subseteq \mathbb{Z}$ such that $N = \bigoplus_{i \in I} (G_i, U_i) \times \prod_{i \notin I} \{1\}$.

**Proof.** Suppose $N \trianglelefteq G$ is proper and non-trivial. Let $\pi_n$ be the projection of $G$ onto the $n$-th coordinate. Define $J \subseteq \mathbb{Z}$ to be the collection of $n \in \mathbb{Z}$ such that $\pi_n(N) = 1$ and put $I := \mathbb{Z} \setminus J$.

Fix $n \in I$ and let $\alpha \in N$ be such that $h := \alpha(n) \neq 1$. Since $G_i$ has a trivial center, we may find $g \in G_n$ such that $g \alpha(n) g^{-1} \neq \alpha(n)$. Define $\beta_{n,g} \in \bigoplus_{i \in \mathbb{Z}} (G_i, U_i)$ to be such that $\beta$ is one except at $n$ where $\beta_{n,g}(n) = g$. Since $N$ is normal, $\beta_{n,ghg^{-1}h^{-1}} = \beta_{n,g} \alpha \beta_{n,g}^{-1} \alpha^{-1} \in N$ where $\beta_{n,ghg^{-1}h^{-1}}$ is defined analogously to $\beta_{n,g}$.

By conjugating with $\beta_{n,k}$ for $k \in G_n$, we see $L := \text{cl}(\langle \beta_{n,l} \mid l \in (ghg^{-1}h^{-1} G) \rangle) \subseteq N$. Since $G_n$ is topologically simple and $ghg^{-1}h^{-1} \neq 1$, we conclude that $L = \prod_{i \leq n-1} \{1\} \times G_n \times \prod_{i \geq n+1} \{1\}$. For all $n \in I$ and $g \in G_n$, we have shown $\beta_{n,g} \in N$, and the lemma follows. \hfill \Box

**Proposition 2.35.** $G$ is elementary-free.

**Proof.** Take $N \trianglelefteq G$ a normal subgroup. Lemma 2.34 implies that $N$ is either trivial or non-elementary. On the other hand, via Lemma 2.34, $G/N \cong \bigoplus_{i \in J} (G_i, U_i)$ for some $J \subseteq \mathbb{Z}$. So either $G/N$ is trivial or non-elementary. We conclude $G$ is elementary-free as desired. \hfill \Box

Using $G$, we produce a t.d.l.c. Polish group such that all terms of the descending elementary series are non-trivial. Let $V := \prod_{i \in \omega} U_i$ and let $D$ be the countable collection of left cosets of
$V$ in $G$. Certainly, $G \curvearrowleft D$ transitively by left multiplication. This action is also faithful since $G$ has no non-trivial compact normal subgroups via Lemma 2.34. Define $H := A_5 \rtimes_G G \times \mathbb{Z}$.

We now compute the descending elementary series for $H$: It is clear that $D_1 \leq A_5 \rtimes_G G$ and is non-trivial since $G$ is non-elementary. Via Lemma A.5, any non-trivial closed normal subgroup of $A_5 \rtimes_G G$ must contain $A_5^D$. It is easy to see $D_1/A_5^D \leq G$ and has a elementary quotient. Since $G$ is elementary-free, we conclude $D_1/A_5^D = G$ and $A_5 \rtimes_G G = D_1$. A similar argument gives $D_2 = A_5^D$. The descending elementary series for $H$ is thus $H \geq A_5 \rtimes_G G \geq A_5^D \geq \{1\}$.

$H$ also demonstrates the descending and ascending elementary series may differ. By a similar argument as above, it may be shown $A_1 = A_5^D \times \mathbb{Z}$ and $A_2 = H$ for $H$.

We lastly use $G$ to show a compactly generated t.d.l.c. Polish group $L$ need not have $D_1/D_2$ or $A_2/A_1$ compactly generated. For this, observe that $\mathbb{Z} \curvearrowleft G$ by shift and form $L := G \rtimes \mathbb{Z}$. Letting $X$ be a compact generating set for $\text{PSL}_3(\mathbb{Q}_p)$ containing 1, we see that $X \times \{1\} \subseteq L$ is a compact generating set for $L$. On the other hand, similar to the previous example, $D_1 = G$, $D_2 = \{1\}$ and $A_1 = \{1\}$ $A_2 = G$. Hence, $D_1/D_2 = G = A_2/A_1$ are not compactly generated while $L$ is compactly generated.
CHAPTER 3

FIRST LOCAL TO GLOBAL STRUCTURE RESULTS

A priori, the structure of the compact open subgroups of a t.d.l.c. Polish group have little affect on the global structure. Over the last decade, however, a number of results have uncovered highly non-trivial relationships between the compact open subgroups and global structure, cf. [3], [12], [13].

In this chapter and the two subsequent chapters, we explore the relationship between the compact open subgroups and global structure of a t.d.l.c. Polish groups further.

Definition 3.1. We say a t.d.l.c. Polish group $G$ is locally $x$ for $x$ a property of profinite groups if there is $U \in \mathcal{U}(G)$ with property $x$.

We specifically study the structure of locally $x$ t.d.l.c. Polish groups for a variety of natural properties $x$. We first consider locally solvable groups.

Theorem 3.2. If $G$ is a locally solvable t.d.l.c. Polish group, then $G$ is elementary.

We next consider locally pro-nilpotent groups.

Theorem 3.3. Suppose $G$ is a t.d.l.c. Polish group which is locally pro-nilpotent. Then there is $N \triangleleft G$ open such that $N = \bigcup_{i \in \omega} H_i$ where $(H_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups and for each $i$ there is a finite set of primes $\pi(i)$ so that $H_i/\text{Rad}_{\mathcal{L}(G)}(H_i)$ is locally pro-nilpotent and locally pro-$\pi(i)$. 43
**Theorem 3.4.** Suppose $G$ is a t.d.l.c. Polish group which is locally pro-nilpotent and locally pro-$\pi$ for some finite set of primes $\pi$. Then there is a sequence of closed normal subgroups 

$$\{1\} =: L_0 \leq L_1 \leq \ldots \leq L_k \leq G$$

such that $0 \leq k \leq |\pi| - 1$ and

1. $G/L_k$ is locally pro-$p$ for some $p \in \pi$.
2. For each $1 \leq i \leq k$ there is $q(i) \in \pi$ such that $L_i/L_{i-1} = \bigcup_{j \in \omega} H_j$ where $(H_j)_{j \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups of $L_i/L_{i-1}$ with $H_j/\text{Rad}_{L_{L_k}}(H_j)$ locally pro-$q(i)$ for every $j \in \omega$.

We conclude this chapter by considering locally hereditarily just infinite groups; here we give a new proof of a result implicit in [3].

**Proposition 3.5** (cf. Barnea, Ershov, Weigel [3, Theorem 5.4]). Suppose $G$ is a non-elementary t.d.l.c. Polish group which is locally hereditarily just infinite. Then the descending elementary series $G \geq D_1 \geq D_2 \geq \{1\}$ is such that $D_1$ is open, $D_2$ is discrete, and $D_1/D_2$ is non-discrete and topologically simple.

### 3.1 Locally Solvable and Locally Nilpotent Groups

The **derived series** of a topological group $G$ is the sequence of closed normal subgroups $(G^{(k)})_{k \geq 0}$ defined by $G^{(0)} := G$ and $G^{(n+1)} := \text{cl}([G^{(n)}, G^{(n)}])$. $G$ is called *solvable* if the derived series stabilizes at $\{1\}$ after finitely many steps. For solvable $G$ the **derived length**, $l(G)$, is the least $k$ such that $G^{(k)} = \{1\}$.

The **lower central series** of a topological group $G$ is the sequence of closed normal subgroups $(G_k)_{k \geq 0}$ defined by $G_1 := G$ and $G_{n+1} := \text{cl}([G, G_n])$. $G$ is called *nilpotent* if the lower central
series stabilizes at \( \{1\} \) after finitely many steps. The *nilpotence class* of nilpotent \( G \), \( n(G) \), is the least \( k \) so that \( G_k = \{1\} \). Note the last non-trivial term in the lower central series of a nilpotent topological group is central.

For a locally solvable t.d.l.c. Polish group \( G \), define

\[
rs(G) := \min \{ l(U) \mid U \in \mathcal{U}(G) \text{ solvable} \}
\]

We call \( rs(G) \) the *solvable rank* of \( G \). All countable discrete groups \( G \) are considered to be locally solvable with \( rs(G) = 1 \).

**Theorem 3.6.** If \( G \) is a locally solvable t.d.l.c. Polish group, then \( G \) is elementary and \( rk(G) \leq 3rs(G) \).

**Proof.** We induct on \( rs(G) \). For the base case, \( rs(G) = 1 \), let \( U \in \mathcal{U}(G) \) be abelian and consider \( SIN(G) \).

**Claim 1.** \( U \subseteq SIN(G) \)

*Proof of claim.* Take \( u \in U \), \( C \in SIN(G) \), and \( V \in \mathcal{U}(G) \). It suffices to show there is \( W \leq_o V \) such that \( cl(\langle u, C \rangle) \leq N_G(W) \). By definition of \( SIN(G) \), there is \( W \leq_o U \cap V \) such that \( C \leq N_G(W) \). As \( U \) is abelian, we have that \( u \in N_G(W) \), so \( cl(\langle u, C \rangle) \leq N_G(W) \). \( \square \)

Since \( SIN(G) \) is open and normal in \( G \), \( rk(G/SIN(G)) = 0 \). On the other hand, \( SIN(G) = \bigcup_{i \in \omega} H_i \) where \( (H_i)_{i \in \omega} \) is an \( \subseteq \)-increasing sequence of compactly generated open SIN groups by
Proposition 1.22. As SIN groups are profinite by discrete, \( rk(H_i) \leq 1 \), \( rk(SIN(G)) \leq 2 \), and \( rk(G) \leq 3 \).

Suppose \( r_s(G) = k + 1 \), let \( U \in \mathcal{U}(G) \) have derived length \( k + 1 \), and put

\[
\mathcal{A}_{k+1} := \left\{ C \in S(G) \mid \forall V \in \mathcal{U}(G) \exists \text{solvable } W \leq_o V : C \leq N_G(W^{(k)}) \right\}
\]

Certainly, \( \mathcal{A}_{k+1} \) is hereditary. To see conjugation invariance, take \( C \in \mathcal{A}_{k+1} \), \( V \in \mathcal{U}(G) \), and \( g \in G \). Find \( W \leq_o g^{-1}Vg \) such that \( C \leq N_G(W^{(k)}) \). Now \( gCg^{-1} \leq N_G(gW^{(k)}g^{-1}) = N_G((gWg^{-1})^{(k)}) \) and \( gWg^{-1} \leq_o V \). It follows \( gCg^{-1} \in \mathcal{A}_{k+1} \), and \( \mathcal{A}_{k+1} \) is invariant under conjugation.

Form \( N_{k+1} \) the \( \mathcal{A}_{k+1} \)-core.

**Claim 2.** \( U^{(k)} \leq N_{k+1} \)

*Proof of claim.* Take \( u \in U^{(k)} \), \( C \in \mathcal{A}_{k+1} \), and \( V \in \mathcal{U}(G) \). It suffices to show there is \( W \leq_o V \) such that \( cl(\langle u, C \rangle) \leq N_G(W^{(k)}) \). By definition of \( \mathcal{A}_{k+1} \), there is \( W \leq_o U \cap V \) such that \( C \leq N_G(W^{(k)}) \). We see that \( W^{(k)} \leq U^{(k)} \) and, since \( U^{(k)} \) is abelian, \( u \in N_G(W^{(k)}) \). So \( cl(\langle u, C \rangle) \leq N_G(W^{(k)}) \) as required. \( \square \)

Put \( N = cl(N_{k+1}) \). By Claim 2, \( r_s(G/N) \leq k \), and the induction hypothesis implies \( rk(G/N) \leq 3k \). Consider \( r_s(N) \). If \( r_s(N) \leq k \), then the theorem follows by induction. Else, say \( r_s(N) = k + 1 \) and let \( (n_i)_{i \in \omega} \) list a countable dense subset of \( N_{k+1} \). Certainly, \( N \) is an increasing union of \( M_i := \langle U \cap N, n_0, \ldots, n_i \rangle \). Fixing \( i \in \omega \), we have \( M_i \in \mathcal{A}_{k+1} \) by choice of \( n_0, \ldots, n_i \).
We may thus find $W \leq_o U$ such that $M_i$ normalizes $W^{(k)}$. So $W^{(k)} \cap N \leq \text{Rad}_{\mathcal{L}_E}(M_i)$. Since $(W \cap N)^{(k)} \leq W^{(k)} \cap N$, we have $r_s(M_i/\text{Rad}_{\mathcal{L}_E}(M_i)) \leq k$.

The induction hypothesis now implies $rk(M_i/\text{Rad}_{\mathcal{L}_E}(M_i)) \leq 3k$. As $rk(\text{Rad}_{\mathcal{L}_E}(M_i)) \leq 1$ via Theorem 1.13, $rk(M_i) \leq 3k + 1$, $rk(N) \leq 3k + 2$, and $rk(G) \leq 3k + 3 = 3(k + 1)$.

The proof of the above theorem gives a procedure for decomposing a locally solvable t.d.l.c. Polish group into groups which are profinite, discrete, or have smaller solvable rank. Further, the base case, a group with a compact open abelian subgroup, has a nice structure. We record these observations here:

**Proposition 3.7.** If $G$ is a locally abelian t.d.l.c. Polish group, then $\text{SIN}(G) \leq_o G$.

**Theorem 3.8.** Suppose a t.d.l.c. Polish group $G$ is locally solvable and $r_s(G) > 1$. Then there is $1 \leq k \leq r_s(G) - 1$ and a sequence of closed normal subgroups $\{N_i\} =: N_0 \leq N_1 \leq \ldots \leq N_k \leq G$ such that

1. $r_s(G) > r_s(G/N_1) > \cdots > r_s(G/N_k) = 1$

2. For all $1 \leq j \leq k$, $N_j/N_{j-1} = \bigcup_{i \in \omega} H_i$ where $(H_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups of $N_j/N_{j-1}$ for which $r_s(H_i/\text{Rad}_{\mathcal{L}_E}(H_i)) < r_s(G/N_{j-1})$

for all $i \in \omega$.

In the special case that $G$ is locally nilpotent, we obtain a less complicated structure theorem. To this end, define the **nilpotence rank** of a locally nilpotent t.d.l.c. Polish group $G$ by

$$r_n(G) := \min\{n(U) \mid U \in \mathcal{U}(G) \text{ nilpotent}\}$$
where \( n(U) \) is the nilpotence class of \( U \).

**Theorem 3.9.** Suppose a t.d.l.c. Polish group \( G \) is locally nilpotent. Then there is \( 1 \leq k \leq r_n(G) \) and a sequence of closed normal subgroups

\[
\{1\} =: N_0 \leq N_1 \leq \ldots \leq N_k \leq_o G
\]

such that \( N_i/N_{i-1} \simeq SIN(G/N_{i-1}) \) for \( 1 \leq i \leq k \).

**Proof.** Let \( G \) be a t.d.l.c. Polish group which is locally nilpotent and let \( U \in \mathcal{U}(G) \) witness \( r_n(G) = k \). We may assume \( G \) is non-discrete since else the result is trivial.

We proceed by induction on \( i \) to build the \( N_i \). For the base case, put \( N_1 := SIN(G) \) and let \( \pi_1 : G \to G/N_1 \) be the usual projection. By the non-discreteness of \( G \), we have that \( Z(U) \) is non-trivial. It follows that \( Z(U) \leq SIN(G) \) and, since \( Z(U) \leq N_1 \), \( n(\pi_1(U)) < n(U) \) or \( n(\pi_1(U)) = 2 \).

Suppose we have defined \( N_l \) for \( l \geq 1 \) and let \( \pi_l : G \to G/N_l \) be the usual projection. If \( n(\pi_l(U)) = 2 \), we stop. Else, put \( N_{l+1} := \pi_l^{-1}(SIN(G/N_l)) \). We thus have \( N_{l+1}/N_l \simeq SIN(G/N_l) \) and \( G/N_{l+1} \simeq (G/N_l)/SIN(G/N_l) \). Noting \( SIN(G/N_l) \) contains \( Z(\pi_l(U)) \), we see \( n(\pi_{l+1}(U)) < n(\pi_l(U)) \).

Certainly, this procedure halts at some stage \( k < n(U) \). If \( G/N_k \) is discrete, then \( N_1 \leq N_2 \leq \ldots \leq N_k \leq_o G \) satisfy the theorem. Else, \( G/N_k \) has a compact open abelian subgroup. Via Proposition 3.7, we have that \( SIN(G/N_k) \) is open in \( G/N_k \). Thus, \( N_{k+1} := \pi_k^{-1}(SIN(G/N_k)) \) along with \( N_1, \ldots, N_k \) satisfy the theorem.
Remark 3.10. The results in this section demonstrate a relationship between local and global structure in t.d.l.c. Polish groups. Furthermore, these results show that local assumptions affect global structure even in the setting of elementary groups where, ostensibly, one can build whatever one pleases.

We conclude this section with examples illustrating Proposition 3.7 and Theorem 3.8. For the former, let $F := Z_3$ be the three element cyclic group and let $(F_i)_{i \in \mathbb{Z}}$ list countably many copies of $F$. Set $U_i = Z_3$ for $i \leq 0$ and $U_i = \{1\}$ for $i > 0$ and form $\bigoplus_{i \in \mathbb{Z}} (F_i, U_i)$. Since $Z \trianglelefteq \bigoplus_{i \in \mathbb{Z}} (F_i, U_i)$ by shift, we may form $G := \bigoplus_{i \in \mathbb{Z}} (F_i, U_i) \rtimes Z$, and since $\prod_{i \leq 0} F_i$ is a compact open subgroup, we have that $G$ is locally abelian. We now set about computing the decomposition given by Proposition 3.7 for $G$.

It is easy to see $\bigoplus_{i \in \mathbb{Z}} (F_i, U_i) \leq SIN(G)$. On the other hand, elements of $Z$ do not normalize any compact open subgroup. Thus, $SIN(G) = \bigoplus_{i \in \mathbb{Z}} (F_i, U_i)$. Furthermore, $SIN(G)$ is locally elliptic and, thereby, an increasing union of compact subgroups; a fortiori, $SIN(G)$ is an increasing union of SIN groups.

For Theorem 3.8, let $P$ be the simple locally elliptic group with a compact open abelian subgroup built by Willis - i.e. the group described in Proposition 2.30. Let $V \in \mathcal{U}(P)$ be abelian, define $\mathcal{D}$ to be the collection of left cosets of $V$ in $P$, and let $P \trianglelefteq \mathcal{D}$ by left multiplication. Take $S$ to be the finitely generated simple group built by Higman [21] and form $Z_3 \wr S^{<\mathcal{D}}$. Applying Proposition A.8, we may further form $H := (Z_3 \wr S^{<\mathcal{D}}) \rtimes P$.

Consider the compact open subgroup $U := Z_3^{S^{<\mathcal{D}}} \times \{1\} \times V$ of $H$. Plainly, $H$ acts on the set of left cosets of $U$, denoted $\mathcal{F}$, transitively. One checks $Z_3^{S^{<\mathcal{D}}}$ is in the kernel of the action. We
may thus form $G := S^F \times H$ where $H \triangleleft S^F$ by shift. Since $U$ is a compact open subgroup of $G$, we have that $G$ is locally solvable with $r_s(G) = 2$. We now compute the decomposition given by Theorem 3.8.

Since $U^{(1)} = \mathbb{Z}_3^{S^D}$, we have $\mathbb{Z}_3^{S^D} \subseteq N_1$. Since $S^F$ centralizes $\mathbb{Z}_3^{S^D}$, we also have $S^F \leq N_1$. We now consider the quotient $G/(S^F \times \mathbb{Z}_3^{S^D})$. Letting $\pi$ be the usual projection, $\pi(N_1)$ is a normal subgroup of $S^D \rtimes P$. Via Lemma A.5, $\pi(N_1)$ is either trivial or extends $S^D$ since $P \triangleleft D$ transitively and faithfully. In the first case we are done. For the latter case, $S^D$ is a subgroup of $N_1$. This is absurd since elements of $S^D$ do not normalize arbitrarily small open subgroups of $\mathbb{Z}_3^{S^D}$. We conclude that $N_1 = S^F \times \mathbb{Z}_3^{S^D}$.

It is now obvious that $G/N_1$ is locally abelian. Further, since $\mathbb{Z}_3^{S^D}$ acts trivially on $S^F$, we have $N_1 = S^F \times \mathbb{Z}_3^{S^D}$. Setting $H_n := S^n \times \mathbb{Z}_3^{S^D}$, we see $N_1 = \bigcup_{n \geq 0} H_n$ and $H_n / \text{Rad}_{\text{LE}}(H_n)$ is discrete and, thereby, locally abelian.

3.2 Locally Pro-nilpotent Groups

Definition 3.11. A profinite group is pro-nilpotent if it is an inverse limit of finite nilpotent groups.

Fact 3.12 ([35, Proposition 2.3.8]). Suppose $U$ is pro-nilpotent. Then

(1) For each prime $p$, $U$ has a unique normal $p$-Sylow subgroup, $U_p$.

(2) For primes $p \neq q$, $U_p \leq C_U(U_q)$.

Definition 3.13. A profinite group is pro-$\pi$ for $\pi$ a set of primes if it is an inverse limit of finite $\pi$-groups - i.e. groups whose order is divisible by no primes outside of $\pi$. 
We begin by considering groups which are locally pro-nilpotent and also locally pro-$\pi$ for $\pi$ some finite set of primes. Observe that discrete groups are locally pro-$\pi$ for any finite set of primes $\pi$.

**Theorem 3.14.** Suppose $G$ is a t.d.l.c. Polish group which is locally pro-nilpotent and locally pro-$\pi$ for some finite set of primes $\pi$. Then there is a sequence of closed normal subgroups

\[ \{1\} =: L_0 \leq L_1 \leq \ldots \leq L_k \leq G \]

such that $0 \leq k \leq |\pi| - 1$ and

(1) $G/L_k$ is locally pro-$p$ for some $p \in \pi$.

(2) For each $1 \leq i \leq k$ there is $q(i) \in \pi$ such that $L_i/L_{i-1} = \bigcup_{j \in \omega} H_j$ where $(H_j)_{j \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups of $L_i/L_{i-1}$ with $H_j/\text{Rad}_{LE}(H_j)$ locally pro-$q(i)$ for every $j \in \omega$.

**Proof.** We proceed by induction on $|\pi|$. Since the base case, $|\pi| = 1$, is obvious, we suppose $G$ is as hypothesized with $\pi = \{p_1, \ldots, p_{n+1}\}$. Consider

\[ \mathcal{D} := \left\{ C \in S(G) \mid \forall V \in \mathcal{U}(G) \exists \text{ pro-nilpotent, pro-$\pi$ } W \leq_o V : C \leq \bigcap_{k=2}^{n+1} N_G(W_{p_k}) \right\} \]

It is easy to check $\mathcal{D}$ is hereditary and conjugacy invariant. We may thus form $N_\mathcal{D}$, the $\mathcal{D}$-core, and set $L_1 := cl(N_\mathcal{D})$. 

Fix $U \in \mathcal{U}(G)$ pro-nilpotent and pro-$\pi$. Letting $(n_j)_{j \in \omega}$ be a countable dense subset of $N_D$, form the subgroups $H_j := \langle U \cap L_1, n_0, \ldots, n_j \rangle$. Certainly, $(H_j)_{j \in \omega}$ is a $\subseteq$-increasing sequence of compactly generated open subgroups of $L_1$ which exhausts $L_1$. By construction of $H_j$, we have that $H_j \in \mathcal{D}$ and there is a pro-nilpotent, pro-$\pi$ $W \leq_o U$ such that $H_j \leq \bigcap_{k=2}^{n+1} N_G(W_{p_k})$.

So $H_j \cap W_{p_k} \leq Rad_{LE}(H_j)$ for each $2 \leq k \leq n+1$. By the uniqueness of $W_{p_k}$ in $W$, we see that $(H_j \cap W)_{p_k} \leq H_j \cap W_{p_k}$ and conclude $H_j/Rad_{LE}(H_j)$ is locally pro-$p_1$.

We now set about producing the other $L_i$. Observe $U_{p_1} \leq N_D$. Indeed, fix $C \in \mathcal{D}$, $u \in U_{p_1}$, and $V \in \mathcal{U}(G)$. Find $W \leq_o U \cap V$ such that $C \leq \bigcap_{k=2}^{n+1} N_G(W_{p_k})$. Since $W_{p_k} \leq U_{p_k}$, we have that $u$ centralizes $W_{p_k}$ for $2 \leq k \leq n+1$ via Fact 3.12. Thus, $cl(\langle u, C \rangle) \leq \bigcap_{k=2}^{n+1} N_G(W_{p_k})$ as required.

So $G/L_1$ is locally pro-nilpotent and locally pro-$\{p_2, \ldots, p_{n+1}\}$. There is thus $\{1\} =: M_0 \leq M_1 \ldots \leq M_{k'} \leq G/L_1$ which satisfy the theorem by the induction hypothesis. Letting $\chi : G \rightarrow G/L_1$ be the usual projection and putting $L_{i+1} := \chi^{-1}(M_i)$, we conclude $L_1, \ldots, L_{k'+1}$ satisfy the theorem for $G$.

This result may be extended to any locally pro-nilpotent t.d.l.c. Polish group.

**Theorem 3.15.** Suppose a t.d.l.c. Polish group $G$ is locally pro-nilpotent. Then there is $N \leq_o G$ such that $N = \bigcup_{i \in \omega} H_i$ where $(H_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups and for each $i$ there is a finite set of primes $\pi(i)$ so that $H_i/Rad_{LE}(H_i)$ is locally pro-nilpotent and locally pro-$\pi(i)$.
Proof. Suppose \( G \) is a locally pro-nilpotent t.d.l.c. Polish group, fix \( U \in \mathcal{U}(G) \) pro-nilpotent, and let \( \{p_i\}_{i \in \omega} \) list the primes. For \( C \in S(G) \), define \( C \in \mathcal{D} \) if and only if for all \( V \in \mathcal{U}(G) \) there is a pro-nilpotent \( W \leq_o V \) and \( n \in \omega \) such that \( C \leq \bigcap_{j \geq n} N_G(W_{p_j}) \). It is easy to check \( \mathcal{D} \) is conjugation invariant and hereditary. We may thus form the \( \mathcal{D} \)-core \( N_D \).

For all \( i \in \omega \), we have \( U_{p_i} \leq N_D \). Indeed, take \( u \in U_{p_i} \), \( C \in \mathcal{D} \), and \( V \in \mathcal{U}(G) \). We may find \( W \leq_o V \cap U \) and \( n \in \omega \) such that \( C \leq \bigcap_{j \geq n} N_G(W_{p_j}) \); without loss of generality, \( n > i \). Since \( u \) centralizes \( U_{p_j} \geq W_{p_j} \) for any \( j \geq n \), we have that \( u \in N_G(W_{p_j}) \) and \( cl(\langle u, C \rangle) \leq \bigcap_{j \geq n} N_G(W_{p_j}) \), so \( u \in N_D \) as desired. We conclude \( U \leq N := cl(N_D) \), whereby \( N \) is open and normal in \( G \).

We now argue \( N \) satisfies the theorem. Let \( (n_i)_{i \in \omega} \) list a countable dense subset of \( N_D \) and put \( H_i := \langle U, n_0, \ldots, n_i \rangle \). Certainly, we have that \( N \) is the increasing union of the \( H_i \) each of which is compactly generated and open. Further, \( H_i \in \mathcal{D} \) since \( U \in \mathcal{D} \). We may thus find \( W \leq_o U \) and \( n \in \omega \) such that \( H_i \leq \bigcap_{j \geq n} N_G(W_{p_j}) \). So \( W_{p_j} \leq Rad_{\mathcal{E}}(H_i) \) for all \( j \geq n \). If \( n = 0 \), then \( H_i/Rad_{\mathcal{E}}(H_i) \) is discrete and, thereby, trivially pro-\( \pi \) for some finite set of primes \( \pi \). Else, we have that \( H_i/Rad_{\mathcal{E}}(H_i) \) is locally pro-\( \{p_0, \ldots, p_{n-1}\} \). We have thus proved the theorem.

\[ \square \]

Remark 3.16. Theorem 3.14 and Theorem 3.15 together show every locally pro-nilpotent t.d.l.c. polish group is built from locally pro-\( p \) groups and elementary groups via group extension and countable increasing union.

We conclude this section by computing the decompositions given by Theorem 3.14 and Theorem 3.15 in examples.
**Example 1:** We compute the decomposition given by Theorem 3.14. Fix \( p \neq q \) primes, let \( \hat{\mathbb{Z}}_q \) be the \( q \)-adic integers, and fix \( U \in \mathcal{U}(PSL_3(\mathbb{Q}_p)) \). Let \((G_i)_{i \in \omega}\) list countably many copies of \( PSL_3(\mathbb{Q}_p) \), take \( U_i \) to be the copy of \( U \) in \( G_i \), and form

\[
P := PSL_3(\mathbb{Q}_q) \times \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i)
\]

Fixing \( V \in \mathcal{U}(PSL_3(\mathbb{Q}_q)) \), we have that \( W := V \times \hat{\mathbb{Z}}_q \times \prod_{i \in \omega} U_i \) is a compact open subgroup of \( P \). \( P \) is thereby locally pro-nilpotent and locally pro-\( \{p, q\} \).

Taking \( p = p_1 \) and \( q = p_2 \), we now compute \( L_1 \) as given by Theorem 3.14. We first observe that \( \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i) \leq L_1 \). Indeed, take \( g \in \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i) \), \( C \in D \) where \( D \) is as defined in the proof of Theorem 3.14, and \( Y \in \mathcal{U}(P) \). By definition of \( D \), there is \( K \leq_o W \cap Y \) such that \( C \leq N(K_q) \). Since \( W \) is pro-nilpotent, \( K_q \leq \hat{\mathbb{Z}}_q \times V \), whereby \( g \) centralizes \( K_q \). So \( cl(\langle g, C \rangle) \leq N_G(K_q) \) and \( g \in N_D \leq L_1 \).

We now consider \( P / \left( \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i) \right) \simeq PSL_3(\mathbb{Q}_q) \). Letting \( \pi \) be the usual projection, we have that \( \pi(L_1) \) is a closed normal subgroup of \( PSL_3(\mathbb{Q}_q) \). Since \( PSL_3(\mathbb{Q}_q) \) is topologically simple, \( \pi(L_1) \) is either trivial or the entire group. Suppose for contraction that \( \pi(L_1) = PSL_3(\mathbb{Q}_q) \). \( \pi(N_D) \) is then a dense subgroup of \( \pi(L_1) \); moreover, each \( g \in \pi(N_D) \) normalizes a compact open subgroup of \( PSL_3(\mathbb{Q}_q) \). We thus have that \( \pi(L_1) \) is uniscalar via Theorem 1.4. Results of Willis and Glöckner, however, imply \( \pi(L_1) = PSL_3(\mathbb{Q}_q) \) is elementary contradicting Proposition 2.29. We conclude \( L_1 = \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i) \).
Setting $H_j := \hat{\mathbb{Z}}_q \times G_0 \times \cdots \times G_j \times \prod_{i \geq j+1} U_i$, we have $L_1 = \bigcup_{j \in \omega} H_j$ with $(H_j)_{j \in \omega}$ an $\subseteq$-increasing sequence of compactly generated open subgroups of $L_1$. It is also easy to check $\text{Rad}_{\mathcal{L}\mathcal{E}}(H_j) = \hat{\mathbb{Z}}_q \times \{1\}^{j+1} \times \prod_{i \geq j+1} U_i$ and, therefore, $H_j/\text{Rad}_{\mathcal{L}\mathcal{E}}(H_j)$ is locally pro-$p$. We have thus exhibited Theorem 3.14 for $P$: We have $\{1\} \leq L_1 = \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i) \leq P$ with

$$L_1 = \hat{\mathbb{Z}}_q \times \bigoplus_{i \in \omega} (G_i, U_i) = \bigcup_{j \in \omega} \left( \hat{\mathbb{Z}}_q \times G_0 \times \cdots \times G_j \times \prod_{i \geq j+1} U_i \right) = \bigcup_{j \in \omega} H_j$$

We remark that the above example shows we may not assume $L_1$ is locally pro-$p_1$.

**Example 2:** We here compute the decomposition given by Theorem 3.15 in an example. Let $(p_i)_{i \in \omega}$ list the primes and for each $i$ fix $U_i \leq U(\text{PSL}_3(\mathbb{Q}_{p_i}))$. Form

$$P := \bigoplus_{i \in \omega} (\text{PSL}_3(\mathbb{Q}_{p_i}), U_i)$$

Observe $P$ is locally pro-nilpotent and fails to be locally pro-$\pi$ for any finite set of primes $\pi$. It is easy to see, via the same argument as in Lemma 2.34, that the only open normal subgroup of $P$ is $P$. So $N$ as given by Theorem 3.15 is $P$.

Setting $H_j := \text{PSL}_3(\mathbb{Q}_{p_0}) \times \cdots \times \text{PSL}_3(\mathbb{Q}_{p_j}) \times \prod_{i \geq j+1} U_i$, we have $N = \bigcup_{j \in \omega} H_j$ with $(H_j)_{j \in \omega}$ an $\subseteq$-increasing sequence of compactly generated open subgroups. It is easy to check $\text{Rad}_{\mathcal{L}\mathcal{E}}(H_j) = \{1\}^{j+1} \times \prod_{i \geq j+1} U_i$ and, therefore, $H_j/\text{Rad}_{\mathcal{L}\mathcal{E}}(H_j)$ is locally pro-$\{p_0, \ldots, p_j\}$. We have thus exhibited Theorem 3.15 for $P$. 
3.3 Locally Hereditarily Just Infinite Groups

In this section, we give an alternative proof of a result which is implicit in the work of Barnea, Ershov, and Weigel [3].

Definition 3.17. A profinite group is hereditarily just infinite, denoted h.j.i., if every non-trivial subgroup with an open normalizer is open.

It is equivalent to say that every open subgroup has no infinite index normal subgroup.

Proposition 3.18 (cf. Barnea, Ershov, Weigel [3, Theorem 5.4]). Suppose $G$ is a non-elementary t.d.l.c. Polish group which is locally h.j.i.. Then the descending elementary series is such that $D_1$ is open, $D_2$ is discrete, and $D_1/D_2$ is non-discrete and topologically simple.

Proof. Let $G \supseteq D_1 \supseteq D_2 \supseteq \{1\}$ be the descending elementary series and let $U \in \mathcal{U}(G)$ be h.j.i.. Since $D_1 \unlhd G$, we have $D_1 \cap U \unlhd U$. Thus, $D_1 \cap U = \{1\}$ or $D_1 \cap U$ is open. The former case implies $D_1$ is discrete, whereby $G$ is elementary contradicting our assumption on $G$.

Since $D_1$ is open in $G$, it is a locally h.j.i. t.d.l.c. Polish group. It follows just as in the previous paragraph that $D_2$ is either open or discrete. The former case is absurd since then $D_1$ is elementary and, thereby, $G$ is elementary. So $D_2$ is discrete as required.

We now consider $D_1/D_2$. Since $D_1/D_2$ is locally h.j.i., any normal subgroup of $D_1/D_2$ must be open or discrete. We have that $D_1/D_2$ is elementary-free by Theorem 2.23 and, thus, the only discrete normal subgroup of $D_1/D_2$ is $\{1\}$ and the only open normal subgroup is $D_1/D_2$. $D_1/D_2$ is hence topologically simple and non-discrete. □
Remark 3.19. The obvious analogue of Proposition 3.18 obtained by using the ascending elementary series also holds.
CHAPTER 4

[A]-REGULAR GROUPS

For this chapter, we consider a somewhat technical local assumption which plays an important role in the new, deep theory developed in [12] and [13].

**Definition 4.1.** Let \( [A] \) be the smallest class of profinite groups such that the following conditions hold:

(i) \( [A] \) contains all abelian profinite groups and all finite simple groups.

(ii) If \( U \in [A] \) and \( K \trianglelefteq U \), then \( K \in [A] \) and \( U/K \in [A] \).

(iii) If \( U = V_1 \ldots V_n \) with \( V_i \trianglelefteq U \) and \( V_i \in [A] \) for each \( 1 \leq i \leq n \), then \( U \in [A] \).

It is an easy application of Fitting’s theorem to see that \( [A] \) consists of virtually nilpotent profinite groups. For a profinite group \( U \), we let \( [A](U) \) denote the closed subgroup generated by all normal subgroups of \( U \) which are members of the class \( [A] \).

**Definition 4.2.** Let \( G \) be a t.d.l.c. Polish group. A compact \( K \trianglelefteq G \) is **locally normal** if \( N_G(K) \) is open.

**Definition 4.3.** Let \( U \) be a profinite group and \( G \) t.d.l.c. group.

(i) For \( K \trianglelefteq U \) a closed subgroup, \( K \) is **\( [A] \)-regular** if for every \( L \trianglelefteq U \) closed which does not contain \( K \), the image of \( K \) in \( U/L \) contains a non-trivial locally normal \( [A] \)-subgroup of \( U/L \).
(ii) For $H \leq G$ a closed subgroup, $H$ is $[A]$-regular if $U \cap H$ is $[A]$-regular in $U$ for all $U \in \mathcal{U}(G)$.

(iii) $G$ is $[A]$-semisimple if $G$ has trivial quasi-center and the only locally normal subgroup of $G$ belonging to $[A]$ is $\{1\}$

In this chapter, we prove

**Theorem 4.4.** If $G$ is a t.d.l.c. Polish group which is $[A]$-regular, then $G$ is elementary.

**Corollary 4.5.** If $G$ t.d.l.c. Polish group, then $G/\text{Rad}_e(G)$ is $[A]$-semisimple. In particular, if $G$ is a elementary-free t.d.l.c. Polish group, then $G$ is $[A]$-semisimple.

As a second corollary, we obtain a result of Caprace, Reid, and Willis:

**Corollary 4.6** (Caprace, Reid, Willis [13]). A non-discrete compactly generated topologically simple t.d.l.c. group is $[A]$-semisimple.

### 4.1 Preliminaries

The quasi-centralizer of $K$ modulo $L$ for $K, L \leq G$ with $G$ a t.d.l.c. group is

$$QC_G(K/L) := \{ g \in G \mid \exists U \in \mathcal{U}(G) \text{ such that } [g, U \cap K] \leq L \}$$

We remark that quasi-centralizers are not necessarily subgroups.

**Proposition 4.7** (Caprace, Reid, Willis [12]). Let $G$ be a t.d.l.c. group, let $K, L$ be non-trivial compact subgroups of $G$ such that $L \leq K$, and suppose $L, K$ are commensurated. Then $QC_G(K/L)$ is a normal subgroup of $G$ which contains $L$. 
For $G$, $K$, and $L$ as in Proposition 4.7, $QC_G(K/L)$ is not necessarily closed. It is convenient to define $\overline{QC}_G(K/L) \seteq cl(QC_G(K/L))$. We require a useful fact about the compact open subgroups of $\overline{QC}_G(K/L)$ when $K = U$ with $U \in \mathcal{U}(G)$: For $V := U \cap \overline{QC}_G(U/L)$, we have that $V \in \mathcal{U}(\overline{QC}_G(U/L))$, $L \mid V$, and since $U$ is open, $U \cap QC_G(U/L)$ is dense in $V$. For each $g \in U \cap QC_G(U/L)$, there is $W \leq_o U$ such that $[g, w] \in L$ for all $w \in W$. We conclude that $QZ(V/L)$ is dense in $V/L$.

We require a number of additional facts about locally normal $[A]$-subgroups. The next proposition is implicit in [12].

**Proposition 4.8.** If a profinite group $U$ contains an infinite locally normal $[A]$-subgroup, then $U$ contains an infinite normal $[A]$-subgroup.

**Proof.** Suppose $L \leq U$ is an infinite locally normal $[A]$-subgroup. Find $W \leq_o U$ such that $W \leq N_U(L)$. Since $L \cap W \leq_o L$ and $L$ is infinite, $L \cap W$ is an infinite $[A]$-subgroup. Further, $L \cap W \leq W$. Take $u_1, \ldots, u_n$ left coset representatives for $W$ in $U$. Since $L \cap W$ is an $[A]$-subgroup and $u_i(L \cap W)u_i^{-1} \leq W$, we have that $K := u_1(L \cap W)u_1^{-1} \ldots u_n(L \cap W)u_n^{-1}$ is an $[A]$-subgroup. Further, for each $u \in U$ and $1 \leq j \leq n$, there is $1 \leq i \leq n$ such that $wu_j(L \cap W)u_j^{-1}u^{-1} = u_i(L \cap W)u_i^{-1}$. Hence, $K \leq U$ giving an infinite $[A]$-subgroup which is normal in $U$. \hfill \Box

**Proposition 4.9.** If $G$ is a t.d.l.c. group, then $[A](U) \sim_c [A](V)$ for all $U, V \in \mathcal{U}(G)$.

**Proof.** Fix $U \in \mathcal{U}(G)$ and let $W \leq_o U$. We first show $[A](W) \sim_c [A](U)$. Indeed, let $A \leq W$ be an $[A]$-subgroup of $W$ and fix $u_1, \ldots, u_n$ left coset representatives for $W$ in $U$. As in the
previous proof, \( L := u_1 Au_1^{-1} \ldots u_n Au_n^{-1} \) is a normal \([A]\)-subgroup in \( U \). Thus, \( A \leq L \leq [A](U) \), and we conclude \([A](W) \leq [A](U)\).

On the other hand, take \( a \in [A](U) \cap W \). It suffices to assume \( a \in A \leq U \) with \( A \) some \([A]\)-subgroup since \( W \) is open. Since \( W \cap A \leq A \), we have that \( W \cap A \) is a normal \([A]\)-subgroup of \( W \) and \( W \cap A \leq [A](W) \). It now follows \([A](U) \cap W \leq [A](W) \leq [A](U)\), whereby \([A](U) \sim_c [A](W)\).

Now choose \( V \in \mathcal{U}(G) \) and find \( W \leq_o V \) such that \( W \leq_o U \cap V \). By our argument above, we have \([A](V) \sim_c [A](W) \sim_c [A](U \cap V)\). Similarly, \([A](U) \sim_c [A](U \cap V)\) proving the proposition. \( \square \)

For our last observation, we require a result from the literature.

**Theorem 4.10** (Caprace, Reid, Willis [12, Theorem 6.10]). If \( G \) is a t.d.l.c. Polish group, then \( G \) has a closed normal subgroup \( R_{[A]}(G) \), the \([A]\)-regular radical of \( G \), such that

1. \( R_{[A]}(G) \) is the unique largest closed normal subgroup of \( G \) that is \([A]\)-regular.

2. \( G/R_{[A]}(G) \) is \([A]\)-semisimple, and given any closed normal subgroup \( N \) of \( G \) such that \( G/N \) is \([A]\)-semisimple, \( R_{[A]}(G) \leq N \).

**Proposition 4.11.** If \( G \) is an \([A]\)-regular t.d.l.c. Polish group and \( N \leq G \), then \( G/N \) is \([A]\)-regular.

**Proof.** Let \( R := R_{[A]}(G/N) \leq G/N \) be the \([A]\)-regular radical of \( G/N \) as given by Theorem 4.10 and let \( \pi : G \to G/N \) be the usual projection. Since \( G/\pi^{-1}(R) \simeq (G/N)/R \), we have that
$G/\pi^{-1}(R)$ is $[A]$-semisimple. Theorem 4.10 implies $G = R_{[A]}(G) \leq \pi^{-1}(R)$, so $G/N$ is $[A]$-regular.

\[ \square \]

4.2 Quasi-centralizers and Commensurators

We now prove a series of technical lemmas which are used in the next section to show $QC_G(U/[A](U))$ is elementary in certain t.d.l.c. Polish groups $G$ with $U \in U(G)$.

**Lemma 4.12.** Suppose $U$ is a profinite group, $B \trianglelefteq U$, and $U/B$ has a dense quasi-center. If $A \trianglelefteq U$ and $A \preceq_o B$, then $U/A$ has a dense quasi-center.

**Proof.** Let $\psi : U/A \to U/B$ be defined by $uA \mapsto uB$; note that $\psi$ is open and continuous. Fix $\Sigma := (g_i)_{i \in I} \subseteq U$ coset representatives for the elements of $QZ(U/B)$ in $U/B$.

Consider the set of elements of $U/A$ with the form $g_ibA$ for $i \in I$ and $b \in B$. Since $\psi$ is open, the set $g_ibA$ is dense in $U/A$. Further, by choice of the $g_i$, for each $i \in I$ there is $W_i \preceq_o U$ such that $[g_i, W_i] \subseteq B$. In fact, more is true: for any $b \in B$ and $w \in W_i$ we have $[g_i b, w] = [b, w]^{g_i} [g_i, w] \in B$.

We now consider the map $\xi : W_i \to U/A$ by $w \mapsto [g_i b, w]A$. Certainly, $\xi$ is continuous, and by the above, it has image in $B/A$. Since $A \in im(\xi)$ and $B/A$ is finite, there is $V_i \preceq_o W_i$ such that $\xi(V_i) = A$. We conclude $C_{U/A}(g_i b A)$ is open for each $i$ and $b \in B$, proving the lemma. \[ \square \]

**Lemma 4.13.** Suppose $G$ is a t.d.l.c. Polish group. Suppose further there is $U \in U(G)$ with a virtually nilpotent $B \trianglelefteq U$ such that $Comm_G(B) = G$ and $QZ(U/B)$ is dense in $U/B$. Then $G$ is elementary.


Proof. We may find \( W \trianglelefteq_o U \) such that \( A := W \cap B \) is nilpotent. Plainly, \( A \trianglelefteq U \), \( A \) is nilpotent, and \( \text{Comm}_G(A) = G \). We induct on the nilpotence class of \( A \), \( n(A) \), for the lemma.

For the base case, \( n(A) = 2 \), \( A \) is abelian. Define \( \mathcal{H} \subseteq S(G) \) by \( C \in \mathcal{H} \) if and only if for all \( V \in \mathcal{U}(G) \) there exists \( N_W \trianglelefteq W \trianglelefteq_o V \) and \((W_i)_{i \in \omega}\) a normal basis at 1 for \( W \) such that

(i) \( N_W \sim_c A \)

(ii) \( C \leq \bigcap_{i \in \omega} N_G(N_W \cap W_i) \)

Plainly, \( \mathcal{H} \) is hereditary.

**Claim 1.** \( \mathcal{H} \) is conjugation invariant.

**Proof of claim.** Take \( C \in \mathcal{H} \) and \( g \in G \). Fix \( V \in \mathcal{U}(G) \) and find \( W \trianglelefteq_o g^{-1}Vg \) as given by the definition of \( C \in \mathcal{H} \). So there is \((W_i)_{i \in \omega}\) a normal basis at 1 for \( W \) and \( N_W \trianglelefteq W \) with \( N_W \sim_c A \) such that \( C \leq \bigcap_{i \in \omega} N_G(N_W \cap W_i) \).

We claim \( gWg^{-1} \trianglelefteq_o V \) satisfies the conditions for \( gCg^{-1} \in \mathcal{H} \). Certainly, \( gN_Wg^{-1} \trianglelefteq gWg^{-1}, (gW_ig^{-1})_{i \in \omega} \) is a normal basis at 1 for \( gWg^{-1} \), and

\[
gCg^{-1} \leq \bigcap_{i \in \omega} N_G(gN_Wg^{-1} \cap gW_ig^{-1})
\]

Additionally, since \( gN_Wg^{-1} \sim_c gAg^{-1} \) and \( gAg^{-1} \sim_c A \), it is the case that \( gN_Wg^{-1} \sim_c A \). We conclude \( \mathcal{H} \) is conjugation invariant as desired.

We now form \( N_\mathcal{H} \), the \( \mathcal{H} \)-core.
Claim 2. $A \leq N_H$.

Proof of claim. Take $g \in A$ and $C \in H$. Fix $V \in \mathcal{U}(G)$ and find $W \leq_o V$ as given by the definition of $C \in H$. So there is $(W_i)_{i \in \omega}$ a normal basis at 1 for $W$ with $N_W \sim_c A$ and $C \leq \bigcap_{i \in \omega} N_G(N_W \cap W_i)$. Since $N_W \sim_c A$, there is $i \in \omega$ such that $N_W \cap W_i \leq A \cap N_W$. Fix such an $i$ and put $N_{W_i} := N_W \cap W_i$. We now have that $N_{W_i} \subseteq W_i$, $N_{W_i} \sim_c A$, and $(W_j)_{j \geq i}$ is a normal basis at 1 for $W_i$. We further have that $N_{W_i} \leq A$ and, therefore, $cl((C,g)) \leq \bigcap_{j \geq i} N_G(N_{W_i} \cap W_j)$. So $cl((C,g)) \in H$, and therefore, $g \in N_H$. 

Put $N := cl(N_H)$ and let $\pi : G \to G/N =: \tilde{G}$ be the usual projection. Plainly, $\pi(U) \in \mathcal{U}(\tilde{G})$, and $\pi(U)$ is a quotient of $U/A$. Additionally, $\pi(U)$ has a dense quasi-center since $U/A$ has a dense quasi-center via Lemma 4.12. We conclude $\pi(U) \leq SIN(\tilde{G})$, $SIN(\tilde{G})$ is open in $\tilde{G}$, and $G/N$ is elementary.

We claim $N$ is also elementary. Indeed, let $(n_i)_{i \in \omega}$ list a countable dense set of $N_H$ and define $P_i := \langle U, n_0, \ldots, n_i \rangle$. Since $U \in H$ and $n_0, \ldots, n_i \in N_H$, we have that $P_i \in H$ for each $i$. Following from the definition of $H$, we may find $L \leq W \leq_o U$ such that $L \leq_o A$ and $P_i \leq N_G(L)$.

Now $W/(A \cap W) \leq_o U/A$ and, thereby, has a dense quasi center. Lemma 4.12 implies $W/L$ also has a dense quasi-center since $L \leq_o A \cap W$. Putting $\hat{P}_i = P_i/L$ and letting $\pi : P_i \to \hat{P}_i$ be the usual projection, we see that $\pi(W) \leq SIN(\hat{P}_i)$ and $SIN(\hat{P}_i) \leq_o \hat{P}_i$. We conclude $P_i$ is elementary, and since $N \leq \bigcup_{i \in \omega} P_i$ is closed, Theorem 2.16 implies $N$ is elementary.

Suppose the lemma holds for all triples $(G,U,B)$ as in the hypotheses such that there is a nilpotent $A \leq_o B$ with $A \leq U$ and $n(A) \leq k$ where $n(A)$ denotes the nilpotence class. Suppose
$G, U, B$ is as in the hypotheses, but every $A \leq_o B$ with $A \leq U$ and $A$ nilpotent has $n(A) = k + 1$. Fix such an $A$ and let $\mathcal{H} \subseteq S(G)$ be defined as in the base case.

Via the same argument as in the base case, we see that $Z(A) \leq N_\mathcal{H}$. Put $N := \text{cl}(N_\mathcal{H})$ and let $\pi : G \to G/N$ be the usual projection. We now have that $\text{Comm}_{G/N}(\pi(B)) = G/N$, so $(G/N, \pi(U), \pi(B))$ satisfies the hypotheses of the lemma. Moreover, $\pi(A) \leq_o \pi(B)$, $\pi(A) \subseteq \pi(U)$, and $n(\pi(A)) \leq k$. The induction hypothesis now implies $G/N$ is elementary. $N$ elementary follows as in the base case. We conclude $G$ is elementary finishing the induction. \qed

**Lemma 4.14.** Suppose $G$ is a t.d.l.c. Polish group and $U \in \mathcal{U}(G)$ contains a non-trivial virtually nilpotent $B \leq U$ such that $\text{Comm}_G(B) = G$. Then $QC_G(U/B)$ is a normal elementary subgroup of $G$ containing $B$.

*Proof.* Put $N := QC_G(U/B)$. Via Proposition 4.7, we have that $N$ is a normal subgroup containing $B$. Furthermore, $B \leq V := U \cap N \in \mathcal{U}(N)$, $QZ(V/B)$ is dense in $V/B$, and $\text{Comm}_N(B) = N$. Lemma 4.13 now implies $N$ is elementary. \qed

We now present a slight adaptation of a lemma due to Caprace, Reid, and Willis from their work [13].

Let $G$ be a t.d.l.c. group, $U \in \mathcal{U}(G)$, and $K \leq U$ be infinite. Take $g \in G$ and consider $K^g := gKg^{-1}$. Certainly, $K^g \leq U^g$. Since $U$ is compact and open, there is $u_1, \ldots, u_n$ in $U$ such that $U \subseteq \bigcup_{i=1}^n u_iU^g$. So for all $u \in U$ there is some $1 \leq i \leq n$ such that $uK^g u^{-1} = u_iK^g u_i^{-1}$.

Putting

$$(K^g)^U := \{u_1K^g u_1^{-1}, \ldots, u_nK^g u_n^{-1}\}$$
we see \((K^g)^U\) is permuted by \(U\) under the action by conjugation.

Suppose we have \(g_1, \ldots, g_n \in G\) and let \(K^{h_1}, \ldots, K^{h_m}\) list \(\{K\} \cup \bigcup_{i=1}^n (K^{g_i})^U\). Form \(H := cl(\langle K^{h_1}, \ldots, K^{h_m} \rangle)\), put \(\tilde{U} = U \cap H\), and define \(V := \bigcap_{j=1}^m N_{\tilde{U}}(K^{h_j})\). Since \(N_G(K^{h_j})\) is open in \(G\) for each \(1 \leq j \leq m\), we have \(V \in \mathcal{U}(H)\). Further, \(V \cap K^{h_j} \subseteq V\) for each \(1 \leq j \leq m\). So \(L := (V \cap K^{h_1}) \ldots (V \cap K^{h_m})\) is an infinite normal subgroup of \(V\).

We now consider \(Comm_H(L)\). Observe \(L \leq N_H(K^{h_j})\), and thus, \(LK^{h_j}\) is a compact subgroup. Moreover, since \((K^{h_j} \cap V) \leq L\) and \(LK^{h_j}/L \leftrightarrow K^{h_j}/L \cap K^{h_j}\), we have that \(LK^{h_j}/L\) is finite. So \(L\) is a finite index subgroup of \(LK^{h_j}\), and therefore, \(K^{h_j} \leq Comm_H(L)\). Since \(V \leq Comm_H(L)\), we see that \(Comm_H(L)\) is dense open subgroup of \(H\) and, therefore, must equal \(H\).

We have demonstrated the following lemma:

**Lemma 4.15** (Caprace, Reid, Willis). Suppose \(G\) is a t.d.l.c. Polish group, \(U \in \mathcal{U}(G)\), \(K \leq U\) is infinite, and \(g_1, \ldots, g_n \in G\). Let \(K^{h_1}, \ldots, K^{h_m}\) list \(\{K\} \cup \bigcup_{i=1}^n (K^{g_i})^U\) and put \(H := cl(\langle K^{h_1}, \ldots, K^{h_m} \rangle)\). Then

(1) \(U \leq N_G(H)\)

(2) For \(V := \bigcap_{j=1}^m N_{\tilde{U}\cap V}(K^{h_j})\) and \(L := (V \cap K^{h_1}) \ldots (V \cap K^{h_m})\), it is the case that \(L \leq V\), \(L\) is infinite, and \(Comm_H(L) = H\).

4.3 **Structure Theorems**

We are now ready to prove the key lemma.
Lemma 4.16. Suppose $G$ is a t.d.l.c. Polish group with trivial quasi-center. If $G$ has a non-trivial locally normal abelian subgroup and $U \in \mathcal{U}(G)$, then $\overline{QC}_G(U/[A](U))$ is a non-trivial elementary normal subgroup.

Proof. Since $QZ(G)$ is trivial and $G$ has a non-trivial locally normal abelian subgroup, $G$ has an infinite abelian locally normal subgroup. It follows $U$ has an infinite locally normal abelian subgroup. Via Proposition 4.8, we have that $U$ has an infinite normal $[A]$-subgroup and $[A](U)$ is non-trivial. Form $N := \overline{QC}_G(U/[A](U))$ and note Proposition 4.9 implies $N$ is a normal subgroup which contains $[A](U)$ since $Comm_G([A](U)) = G$.

Let $(A_j)_{j \in J}$ list all normal $[A]$-subgroups of $U$ and let $(W_i)_{i \in \omega}$ be a normal basis at 1 for $[A](U)$. Since for each $i$ there are finitely many $A_j$ which cover $[A](U)$ by $W_i$ cosets, we may find $\Omega_i := \{A_{j_1}, \ldots, A_{j_n}\}$ where $\bigcup \Omega_i$ contains coset representatives for $[A](U)/W_i$. Put $\Omega := \bigcup_{i \in \omega} \Omega_i$, list $\Omega$ as $(B_i)_{i \in \omega}$, and for each $i \in \omega$ define $C_i := B_0B_1 \ldots B_i$. So $(C_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of infinite closed normal virtually nilpotent subgroups of $U$. Furthermore, $\bigcup_{i \in \omega} C_i$ is dense in $[A](U)$.

Fix $\Sigma = \{g_i \mid i \in \omega\}$ a countable dense subgroup of $G$ and for $i \in \omega$ let $C_i^{h_1}, \ldots, C_i^{h_{m(i)}}$ list $C_i, (C_i^{g_0})^U, \ldots, (C_i^{g_{t_i}})^U$. We claim each $P_i := cl((C_i^{h_1}, \ldots, C_i^{h_{m(i)}}))$ is elementary. Indeed, by the construction of $P_i$, there is $V \in \mathcal{U}(P_i)$ and $L \subseteq V$ as given by Lemma 4.15. So $L \cap C_i^{h_j}$ is finite index in $C_i^{h_j}$ for all $1 \leq j \leq m(i)$, and $Comm_{P_i}(L) = P_i$. Since each $C_i$ is virtually nilpotent, we conclude $L$ is also virtually nilpotent. Setting $N_i := \overline{QC}_{P_i}(V/L)$, Lemma 4.14 implies $N_i$ is a normal elementary subgroup.
We now consider $P_i/N_i$; let $\pi : P_i \to P_i/N_i =: \tilde{P}_i$ be the usual projection. By construction, we have that $N_{P_i}(C_i^{h_j})$ is open in $P_i$ and, thus, $N_{\tilde{P}_i}(\pi(C_i^{h_j}))$ is open in $\tilde{P}_i$. However, $\pi(C_i^{h_j})$ is finite as $L \cap C_i^{h_j}$ is finite index in $C_i^{h_j}$, so $\pi(C_i^{h_j}) \subseteq QZ(\tilde{P}_i) \subseteq SIN(\tilde{P}_i)$. We conclude $SIN(\tilde{P}_i) = \tilde{P}_i$, $\tilde{P}_i$ is elementary, and $P_i$ is elementary. Further, since $(P_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence and $U \subseteq N_G(P_i)$ for each $i$, Proposition 2.9 gives that $P := cl(\bigcup_{i \in \omega} P_i)$ is also elementary.

Claim 1. $P \subseteq G$.

Proof of claim. It suffices to show the countable dense subgroup $\Sigma$ of $G$ used in the definition of $P$ normalizes $P$. Fixing $P_j$ from the construction of $P$ and $g \in \Sigma$, it is indeed enough to show $gP_jg^{-1} \subseteq P$.

Take $C_j^{ag_k}$ one of the generating subgroups from the construction of $P_j$. We may assume $u \in \Sigma$ since the normalizer of $C_j^{ag_k}$ is open and $\Sigma$ is dense. So $gug_k \in \Sigma$. There is thus some sufficiently large $l$ such that $C_j \subseteq C_l$ and $gug_k$ is in the first $l$ elements of the enumeration of $\Sigma$. So $C_j^{ag_k} \subseteq P_l$. Since $P_j$ is topologically generated by finitely many groups of the form $C_j^{ag_k}$, it follows $gP_jg^{-1} \subseteq P \subseteq P_{l'} \subseteq P$ for some large $l'$ as required.

By the construction of $P$, we have that $[A](U) \subseteq P \subseteq N$. Let $\pi : N \to N/P$ be the usual projection and put $W := N \cap U$. So $\pi(W)$ is a quotient of $W/[A](U)$, and $W/[A](U)$ has a dense quasi-center. We conclude $\pi(W) \subseteq SIN(N/P)$, $SIN(N/P)$ is open in $N/P$, and $N = QC_G(U/[A](U))$ is elementary.

**Theorem 4.17.** If $G$ is an $[A]$-regular t.d.l.c. Polish group, then $G$ is elementary.
Proof. Suppose $G$ is an $[A]$-regular group. By Proposition 4.11, we have that $G/Rad_E(G)$ is also $[A]$-regular. Suppose for contradiction $G/Rad_E(G)$ is non-trivial. So $G/Rad_E(G)$ is a non-discrete t.d.l.c. Polish group which has trivial quasi-center and a non-trivial locally normal abelian subgroup. Fixing $U \in U(G/Rad_E(G))$, we apply Lemma 4.16 to conclude $QC_{G/Rad_E(G)}(U/\{A\}(U))$ is a non-trivial elementary normal subgroup of $G/Rad_E(G)$. This is plainly absurd since $Rad_E(G)$ is maximal with respect to closed normal elementary subgroups of $G$. \hfill \Box

Corollary 4.18. If $G$ t.d.l.c. Polish group, then $G/Rad_E(G)$ is $[A]$-semisimple. In particular, if $G$ is a elementary-free t.d.l.c. Polish group, then $G$ is $[A]$-semisimple.

Proof. Certainly, $G/Rad_E(G)$ must have a trivial quasi-center. Since $G/Rad_E(G)$ has no non-trivial elementary normal subgroups, Lemma 4.16 implies that $G/Rad_E(G)$ is $[A]$-semisimple. \hfill \Box

Corollary 4.19. If $G$ is t.d.l.c. Polish group, then $R_{[A]}(G)$ is elementary.

Proof. By Corollary 4.18 and Theorem 4.10, we have that $R_{[A]}(G) \leq Rad_E(G)$. Applying Theorem 2.16, $R_{[A]}(G)$ is elementary. \hfill \Box

Corollary 4.20 (Caprace, Reid, Willis [13]). A non-discrete compactly generated topologically simple t.d.l.c. group is $[A]$-semisimple.

Proof. Given $G$ as hypothesized, $G$ must be non-elementary by Proposition 2.29. We conclude $G$ is elementary free, so $G$ is $[A]$-semisimple via Corollary 4.18. \hfill \Box
Remark 4.21. In general, \( \text{Rad}_{[A]}(G) \subseteq \text{Rad}_{[\varphi]}(G) \).

4.4 An Example

It is, of course, easy to make examples of \([A]\)-regular groups with solvable compact open subgroups. We build an example of an infinite profinite Polish group which is \([A]\)-regular but not virtually solvable. Let \((K_i)_{i \in \mathbb{N}}\) be a sequence of infinite profinite Polish groups which are solvable such that \(l(K_i) < l(K_{i+1})\); such a sequence may be produced by taking semi-direct products of abelian profinite groups. Put \(G := \prod_{i \in \mathbb{N}} K_i\).

Since the derived lengths of the \(K_i\) are unbounded, we have that \(G\) is not solvable. We now show every non-trivial quotient of \(G\) contains a non-trivial normal abelian subgroup. Take \(N \trianglelefteq G\) and let \(\pi : G \to G/N\) be the usual projection. Set \(n(i)\) to be such that \(K_i^{(n(i))}\) is the last term of the derived series of \(K_i\) with non-trivial image in \(G/N\). Certainly, \(\pi(\prod_{i \in \mathbb{N}} K_i^{(n(i))})\) is a normal subgroup of \(G/N\). By choice of \(n(i)\), we also have that \(\pi(\prod_{i \in \mathbb{N}} K_i^{(n(i))})\) is non-trivial and abelian.
CHAPTER 5

LOCALLY OF FINITE RANK GROUPS

Definition 5.1. A profinite group has rank $r < \infty$ if any closed subgroup may be topologically generated by $r$ elements where a profinite group is topologically $r$ generated if it admits a dense $r$ generated subgroup.

Remark 5.2. When a t.d.l.c. Polish group $G$ has a compact open subgroup of finite rank, we say $G$ is locally of finite rank to avoid overloading “locally finite” any further.

Remark 5.3. The main source of examples of locally of finite rank t.d.l.c. Polish groups are l.c.s.c. $p$-adic Lie groups.

We begin by considering the case of locally of finite rank groups that are also locally pro-$\pi$ for some finite set of primes $\pi$. Under this extra assumption, we obtain:

Theorem 5.4. Suppose $G$ is a non-elementary t.d.l.c. Polish group which is locally of finite rank and locally pro-$\pi$ for some finite set of primes $\pi$. Then the ascending elementary series is such that $G/A_2$ is finite and $A_2/A_1 \simeq N_1 \times \cdots \times N_k$ for $0 < k < \infty$ where each $N_i$ is a non-elementary compactly generated topologically simple $p(i)$-adic Lie group of adjoint simple type.

The adjoint simple type property is a useful technical condition coming from the theory of algebraic groups.
We go on to use synthetic subgroups to obtain a decomposition result for all locally of finite rank groups.

**Corollary 5.5.** Let $G$ be a non-trivial t.d.l.c. Polish group which is elementary-free and locally of finite rank. Then $G = \bigcup_{i \in \omega} P_i$ where $(P_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups and for each $P_i$ the ascending elementary series $A_1(i) \leq A_2(i) \leq P_i$ is such that $G/A_2(i)$ is finite and $A_2(i)/A_1(i) \cong N_1(i) \times \cdots \times N_k(i)$ for $0 < k(i) < \infty$ where each $N_j(i)$ is a non-elementary compactly generated topologically simple $p(i,j)$-adic Lie group of adjoint simple type.

We conclude by computing the decompositions in examples.

### 5.1 Preliminaries

The **Fitting subgroup** of a profinite group $U$, denoted $F(U)$, is the closed subgroup generated by all normal pro-nilpotent subgroups. It is easy to see $F(U)$ is pro-nilpotent and normal. The **Frattini subgroup** of a profinite group $U$, denoted $\Phi(U)$, is the intersection of all maximal proper open subgroups of $U$.

**Proposition 5.6** ([44, Proposition 2.5.1]). Let $U$ be a profinite group. Then

1. If $H \leq U$ and $H\Phi(U) = U$, then $H = U$.

2. If $K \trianglelefteq U$, $H \leq U$, and $K \leq \Phi(H)$, then $K \leq \Phi(U)$. In particular, $\Phi(H) \leq \Phi(U)$ for $H \trianglelefteq U$.

3. If $\Phi(G) \leq K \leq G$ and $K/\Phi(G)$ is pro-nilpotent, then $K$ is pro-nilpotent. In particular, $\Phi(G)$ is pro-nilpotent.
(4) $G$ is pro-nilpotent if and only if $G/\Phi(G)$ is abelian.

**Fact 5.7** ([35, Proposition 2.8.11]). *If $U$ is pro-supersolvable and pro-$\pi$ for some finite set of primes $\pi$, then $G$ is topologically finitely generated if and only if $\Phi(G)$ is open.*

The $\pi$-core of a profinite group $U$, $O_\pi(U)$, is the subgroup generated by all subnormal pro-$\pi$ subgroups of $U$ where $\pi$ is a possibly infinite set of primes. In [34], $O_\pi(U)$ is shown to be pro-$\pi$ and normal. Under certain assumptions on $U$, the $p'$-core behaves nicely where $p'$ denotes the collection of primes different from $p$.

**Fact 5.8** (Reid [34, Corollary 5.11]). *If $U$ is a profinite group such that $U$ has a topologically finitely generated $p$-Sylow subgroup, then $U/O_{p'}(U)$ is virtually pro-$p$."

**Fact 5.9** ([44, Theorem 8.4.1]). *If $U$ is a profinite group with finite rank, then $U$ has a series of normal subgroups $\{1\} \trianglelefteq C \trianglelefteq N \trianglelefteq U$ such that $C$ is pro-nilpotent, $N/C$ is solvable, and $U/N$ is finite.*

By Fact 5.9, $U/F(U)$ is solvable by finite for $U$ a finite rank profinite group.

Associated to a profinite group $U$ is the group of continuous automorphisms of $U$, denoted $Aut(U)$. There is a natural topology on $Aut(U)$: For $K \trianglelefteq U$ put

$$A_U(K) := \{ \phi \in Aut(U) \mid \phi(g)g^{-1} \in K \text{ for all } g \in U \}$$

$Aut(U)$ becomes a topological group by declaring the sets $A_U(K)$ as $K$ varies over open normal subgroups of $U$ to be a basis at 1. Following L. Ribes and P. Zalesskii [35], we call this topology the congruence subgroup topology of $Aut(U)$. 
Fact 5.10 ([35, Corollary 4.4.4]). If $U$ is a topologically finitely generated profinite group, then $\text{Aut}(U)$ is a profinite group when equipped with the congruence subgroup topology.

We conclude our preliminary discussion with a few basic lemmas.

Lemma 5.11. Suppose $U$ is a profinite group such that $U$ has finite rank, is solvable, and is pro-$\pi$ for some finite set of primes $\pi$. If $QZ(U)$ is dense in $U$, then $QZ(U) = U$.

Proof. Fix a profinite $U$ as hypothesized. We induct on $l(U) \geq n \geq 0$ for the hypothesis $U^{(n)} \subseteq QZ(U)$ where $U^{(0)} = U$ and $U^{(l(U))}$ is the first trivial term in the topological derived series. The base case is immediate since $U^{(l(U))} = \{1\}$.

For the successor case, $U^{(k-1)}$ is the closure of a verbal subgroup of $U$. Say

$$U^{(k-1)} = \text{cl}(\langle \gamma(u_1, \ldots, u_l) : u_i \in U \rangle)$$

where $\gamma$ is a word in $l$-variables. For any $u_1, \ldots, u_l \in U$, we may find sequences $u^j_i \to u_i$ as $j \to \infty$ with $u^j_i \in QZ(U)$ for all $j$. Plainly, $\gamma(u^j_1, \ldots, u^j_l) \to \gamma(u_1, \ldots, u_l)$ and $\gamma(u^j_1, \ldots, u^j_l) \in QZ(U)$ since $QZ(U)$ is a subgroup. It follows $QZ(U) \cap U^{(k-1)}$ is dense in $U^{(k-1)}$.

The image of $QZ(U) \cap U^{(k-1)}$ is also dense in $U^{(k-1)}/U^{(k)}$. Since $U^{(k-1)}/U^{(k)}$ is pro-$\pi$, abelian, and topologically finitely generated, Fact 5.7 implies the Frattini subgroup is open in $U^{(k-1)}/U^{(k)}$. We may thus find $r_1, \ldots, r_n \in QZ(U)$ such that $r_1, \ldots, r_n$ topologically generate $U^{(k-1)}$ modulo $U^{(k)}$. Since $U^{(k)} \leq QZ(U)$ by the induction hypothesis and is topologically finitely generated, it follows $U^{(k-1)} \leq QZ(U)$.

This finishes the induction, and we conclude $QZ(U) = U$. \qed
Lemma 5.12. If $G$ is a t.d.l.c. Polish group and $U, V \in \mathcal{U}(G)$, then $F(U) \sim_c F(V)$. In particular, $F(U)$ is commensurated by $G$.

Proof. Fix $U \in \mathcal{U}(G)$ and suppose $V \leq_o U$. Plainly, $F(U) \cap V \leq F(V) \leq F(U)$, so $F(V) \sim_c F(U)$. Suppose $V \in \mathcal{U}(G)$ is arbitrary. Take $W \leq_o U$ such that $W \leq U \cap V$; by the previous, $F(U) \sim_c F(W)$. On the other hand, we have that $W \leq_o U \cap V$, so $F(W) \sim_c F(U \cap V)$. Hence, $F(U) \sim_c F(U \cap V)$. Reversing the roles of $U$ and $V$, we conclude $F(U \cap V) \sim_c F(V)$, and the lemma is proved. \qed

Lemma 5.13. Suppose $G$ is a t.d.l.c. Polish group that is locally of finite rank and locally pro-$\pi$ for some finite set of primes $\pi$. If $U \in \mathcal{U}(G)$ has finite rank and is pro-$\pi$, then $\Phi(U)$ is commensurated.

Proof. Put $N = F(U)$. By Proposition 5.6 and the definition of $N$, we have that $\Phi(N) \leq \Phi(U) \leq N$. Fact 5.7 implies $\Phi(N)$ has finite index in $N$, so $\Phi(U) \sim_c N$. Applying Lemma 5.12, we conclude $\text{Comm}_G(\Phi(U)) = G$. \qed

5.2 Elementary-free Groups

When considering a t.d.l.c. Polish group, we may always reduce to a elementary-free normal section via Theorem 2.23. As such, we begin by examining locally of finite rank and locally pro-$\pi$ groups which are elementary-free.

Lemma 5.14. Suppose $G$ is a t.d.l.c. Polish group which is locally of finite rank and locally pro-$\pi$ for some finite set of primes $\pi$. If $G$ is elementary-free, then $G$ is locally pro-nilpotent.
Proof. Fix $U \in \mathcal{U}(G)$ with finite rank such that $U/F(U)$ is solvable. By Lemma 5.13, $\Phi(U)$ is commensurated. We may thus form $\overline{QC}_G(U/\Phi(U))$. Since $U/\Phi(U)$ is solvable and $\Phi(U) \leq \overline{QC}_G(U/\Phi(U))$, we have that $G/\overline{QC}_G(U/\Phi(U))$ is locally solvable. Theorem 3.6 implies $G/\overline{QC}_G(U/\Phi(U))$ is elementary, whereby $\overline{QC}_G(U/\Phi(U)) = G$.

So $U \cap QC_G(U/\Phi(U))$ is dense in $U$ and every element in $U \cap QC_G(U/\Phi(U))$ is quasi-central in the quotient $U/\Phi(U)$. Thus, $U/\Phi(U)$ is a profinite group which has finite rank, is solvable, is pro-$\pi$, and has a dense quasi-center. Applying Lemma 5.11, every element of $U/\Phi(U)$ is quasi-central. Letting $u_1, \ldots, u_n$ topologically generate $\tilde{U} := U/\Phi(U)$, we have that

$$C_{\tilde{U}}(u_1 \cap \cdots \cap C_{\tilde{U}}(u_n)$$

is open and central in $\tilde{U}$. So $Z(\tilde{U}) \leq_o \tilde{U}$. Proposition 5.6 implies $\pi^{-1}(Z(\tilde{U})) \leq_o U$ is pro-nilpotent, and we have the lemma.

We now adapt a lemma due to Caprace and Monod from [10] to our setting.

**Lemma 5.15.** Suppose $G$ is a non-trivial t.d.l.c. Polish group which is locally of finite rank and locally pro-$\pi$ for some finite set of primes $\pi$. If $G$ is elementary-free, then $G$ has exactly $0 < k < \infty$ non-trivial $\leq$-minimal normal subgroups.

**Proof.** Put $\mathcal{N} := \{N \leq G \mid \{1\} \neq N\}$. We claim filtering families in $\mathcal{N}$ have $\leq$-minimal elements in $\mathcal{N}$. Indeed, suppose $(N_\alpha)_{\alpha \in I}$ is a filtering family in $\mathcal{N}$ and put $N := \bigcap_{\alpha \in I} N_\alpha$. If $N$ is non-trivial, then $N$ is an $\leq$-minimal element for the family $(N_\alpha)_{\alpha \in I}$ in $\mathcal{N}$ and we are done. For contradiction suppose $N = \{1\}$. 
Fix $U \in \mathcal{U}(G)$ such that $U$ has finite rank and is pro-$\pi$. Via Lemma 5.14, we may take $U$ to be pro-nilpotent. Let $(g_n)_{n \in \omega}$ list a countable dense subset of $G$ and for each $n \geq 0$ put $P_n := \langle U, g_0, \ldots, g_n \rangle$. Certainly, $G = \bigcup_{n \in \omega} P_n$; since $Rad_\pi(G) = \{1\}$, we have that $P_n$ is non-elementary for all but finitely many $n$. By passing to a subsequence, we may assume every $P_n$ is non-elementary.

By Proposition 1.14, the normal core of $U$ in $P_n$, $K_n := \bigcap_{h \in P_n} hUh^{-1}$, is such that every filtering family of non-discrete closed normal subgroups of $P_n/K_n$ has non-trivial intersection. Observe that all $K_n$ must be non-trivial. Indeed, else fix $n$ such that $K_n$ is trivial. So $(N_\alpha \cap P_n)_{\alpha \in I}$ is a filtering family of closed normal subgroups for $P_n$ with trivial intersection. Some $N_\alpha \cap P_n$ must then be discrete. We conclude $N_\alpha$ is discrete since $P_n$ is open contradicting that $G$ is elementary-free.

The action of $P_n$ on $K_n$ by conjugation is a continuous action by automorphisms, and thus, the induced map $P_n \to Aut(K_n)$ is a continuous homomorphism when $Aut(K_n)$ is given the congruence subgroup topology. Since $K_n$ is topologically finitely generated, we have that $Aut(K_n)$ is profinite by Fact 5.10. Theorem 2.16 thus implies $P_n/C_{P_n}(K_n)$ is elementary.

Since $(K_n)_{n \in \omega}$ is $\subseteq$-decreasing, $(C_{P_n}(K_n))_{n \in \omega}$ forms an $\subseteq$-increasing sequence of subgroups. So $R := cl(\bigcup_{n \in \omega} C_{P_n}(K_n))$ is a normal subgroup of $G$. Plainly, $G/R = \bigcup_{n \in \omega} P_nR/R$, and since $P_nR/R \simeq P_n/P_n \cap R$ and $P_n/P_n \cap R$ is a quotient of $P_n/C_{P_n}(K_n)$, Theorem 2.16 implies $P_n/P_n \cap R$ is elementary. We conclude that $G/R$ is elementary, so $\bigcup_{n \in \omega} C_{P_n}(K_n)$ is dense in $G$ since $G$ is elementary-free.
Let \(u_1, \ldots, u_k\) topologically generate \(U\). Since \(\bigcup_{n \in \omega} C_{P_n}(K_n)\) is dense and \(\Phi(U)\) is open by Fact 5.7, there are \(g_1, \ldots, g_k \in C_{P_n}(K_n)\) for some sufficiently large \(n\) such that \(g_i \Phi(U) = u_i \Phi(U)\) for \(1 \leq i \leq k\). We conclude that \(cl(\langle g_1, \ldots, g_n \rangle) = U\) and \(C_{P_n}(K_n)\) is open for all sufficiently large \(n\). Each element of \(K_n\) for sufficiently large \(n\) thus has an open centralizer. This, however, contradicts that \(G\) is elementary-free since \(QZ(G) \leq \text{Rad}_E(G)\) and \(K_n\) is non-trivial. We conclude \(\bigcap_{\alpha \in I} N_\alpha\) is non-trivial and \((N_\alpha)_{\alpha \in I}\) has a \(\subseteq\)-minimal element in \(\mathcal{N}\).

As a particular case of the above paragraphs, decreasing \(\subseteq\)-chains have minimal elements in \(\mathcal{N}\). So \(\mathcal{N}\) admits \(\subseteq\)-minimal elements by Zorn’s lemma. It now remains to show there are finitely many minimal non-trivial normal subgroups.

Let \(\mathcal{M}\) list all \(\subseteq\)-minimal elements of \(\mathcal{N}\) and for each \(\mathcal{E} \subseteq \mathcal{M}\) let \(N_\mathcal{E} := cl(\langle N \mid N \in \mathcal{E} \rangle)\). Consider \(\mathcal{E} \subsetneq \mathcal{M}\) and take \(N \in \mathcal{M} \setminus \mathcal{E}\). We claim \(N_\mathcal{E}\) avoids the non-trivial elements in \(N\). Indeed, since \(N\) commutes with the generators of \(N_\mathcal{E}\), \(N_\mathcal{E} \cap N =: M\) is central in \(N_\mathcal{E}\). Abelian groups are elementary, whereby \(M \leq \text{Rad}_E(N_\mathcal{E})\). Since \(\text{Rad}_E(N_\mathcal{E})\) is characteristic, we have that \(\text{Rad}_E(N_\mathcal{E}) \leq \text{Rad}_E(G) = \{1\}\) and \(M = \{1\}\) as required.

Consider \(\mathcal{K} := \{N_{\mathcal{M}\setminus \mathcal{F}} \mid \mathcal{F} \subseteq \mathcal{M}\) is finite\}\) where, if needed, \(N_\emptyset := \{1\}\). Plainly, \(\mathcal{K}\) is a filtering family of closed normal subgroups, and when \(\mathcal{F} \neq \emptyset\), \(N_{\mathcal{M}\setminus \mathcal{F}}\) is a proper subgroup of \(G\) which avoids the non-trivial elements of each member of \(\mathcal{F}\). For contradiction, suppose \(\mathcal{M}\) is infinite. Every element of \(\mathcal{K}\) is thus non-trivial, so \(\mathcal{K} \subseteq \mathcal{N}\). Since filtering families in \(\mathcal{N}\) have lower bounds in \(\mathcal{N}\), \(K := \bigcap \mathcal{K}\) is non-trivial. However, \(K\) must contain a minimal non-trivial normal subgroup of \(G\). Fixing such a subgroup \(L\), the construction of \(K\) implies
\[ K \subseteq N_{\mathcal{M}\{L\}} \] which is absurd since \( N_{\mathcal{M}\{L\}} \) avoids \( L \setminus \{1\} \subseteq K \). Therefore, \( \mathcal{M} \) is finite proving the lemma.

For \( G \) as in the above lemma, we denote the collection of minimal non-trivial normal subgroups by \( \mathcal{M}(G) \).

**Lemma 5.16.** Suppose \( G \) is a non-trivial t.d.l.c. Polish group which is locally of finite rank and locally pro-\( \pi \) for some finite set of primes \( \pi \). If \( G \) is elementary-free, then \( \mathcal{M}(G) \) consists of topologically simple groups.

**Proof.** Fix \( N \in \mathcal{M}(G) \). Since the elementary radical and residual of \( N \) are characteristic subgroups, it follows that \( Rad_\varnothing(N) = \{1\} \) and \( Res_\varnothing(N) = N \); i.e. \( N \) is elementary-free.

Via Lemma 5.15, \( N \) has finitely many non-trivial minimal normal subgroups \( K_1, \ldots, K_n \). The action of \( G \) by conjugation permutes \( \{K_1, \ldots, K_n\} \) and induces a continuous homomorphism \( \psi : G \to Sym(n) \). Since \( ker(\psi) \) has finite index and \( Res_\varnothing(G) = G \), we see that \( \psi \) is the trivial homomorphism. So each of \( K_1, \ldots, K_n \) is normal in \( G \), and the minimality of \( N \) implies \( K_1 = \cdots = K_n = N \). We conclude that \( N \) is topologically simple.

Let \( G \) be a t.d.l.c. Polish group which is locally of finite rank, locally pro-\( \pi \) and is elementary-free. Let \( N_1, \ldots, N_k \) list the non-trivial minimal normal subgroups of \( G \) given by Lemma 5.15 and put \( N_{\text{min}}(G) := cl(\langle N_1, \ldots, N_k \rangle) \). So \( N_{\text{min}}(G) \) is a characteristic subgroup of \( G \) which is a quasi-product with \( 0 < k < \infty \) many non-elementary topologically simple quasi-factors.
5.3 Simple Normal Subgroups

We now examine the topologically simple normal subgroups we obtained in the previous section. We begin by showing they are indeed \( p \)-adic Lie groups.

**Definition 5.17.** A *Lie group* is a topological group with a \( \mathbb{K} \)-manifold structure such that the group operations are analytic where \( \mathbb{K} \) is either \( \mathbb{R} \), \( \mathbb{C} \), or some non-discrete complete ultrametric field.

In particular, a \( p \)-adic Lie group is a Lie group over \( \mathbb{Q}_p \), the \( p \)-adic numbers. For our approach, the following characterization of \( p \)-adic Lie groups is much more useful:

**Fact 5.18** (Lazard [25], Lubotzky, Mann [26]). A t.d.l.c. Polish group is a \( p \)-adic Lie group if and only if \( G \) has a compact open subgroup which is pro-\( p \) and has finite rank.

**Proposition 5.19.** A locally of finite rank and locally pro-\( \pi \) t.d.l.c. Polish group which is non-elementary and topologically simple is locally pro-\( p \) for some prime \( p \). In particular, such a group is a l.c.s.c. \( p \)-adic Lie group.

**Proof.** Suppose \( G \) is as hypothesized. Via Lemma 5.14, \( G \) is also locally pro-nilpotent. Theorem 3.14 implies there is \( p \) such that either \( G \) is locally pro-\( p \) or \( G = \bigcup_{n \in \omega} H_n \) such that \( (H_n)_{n \in \omega} \) is an \( \subseteq \)-increasing sequence of open subgroups of \( G \) for which \( H_n / \text{Rad}_{\mathcal{E}}(H_n) \) is locally pro-\( p \).

In the former case, we have the lemma in view of Fact 5.18.

For the latter, \( H_n / \text{Rad}_{\mathcal{E}}(H_n) \) is also locally pro-\( p \). Consider \( U \subseteq U(H_n / \text{Rad}_{\mathcal{E}}(H_n)) \) which is pro-nilpotent. Since \( U \) is pro-nilpotent, the Sylow subgroups centralize each other. Since \( U \) also has an open pro-\( p \) subgroup, we see that the \( q \neq p \) Sylow subgroups must be finite and,
therefore, have open centralizer. However, $H_n/Rad_\varphi(H_n)$ has trivial elementary radical, and elements with open centralizer fall in the elementary radical. Therefore, for every pro-nilpotent $U \in \mathcal{U}(H_n/Rad_\varphi(H_n))$, we have that $U$ is indeed pro-$p$.

For fixed $k \in \omega$, consider the sequence of subgroups $(H_k \cap Rad_\varphi(H_n))_{n \geq k}$. Put

$$L_k := \bigcap_{n \geq k} (H_k \cap Rad_\varphi(H_n))$$

Since $L_k \subseteq Rad_\varphi(H_k)$, Theorem 2.16 implies $L_k \leq H_k$ is elementary. Now $(L_k)_{k \in \omega}$ forms an $\subseteq$-increasing sequence of elementary groups. Fixing $V \in \mathcal{U}(H_0)$, we have that $V \leq N_G(L_k)$ for all $k \in \omega$, whereby Proposition 2.9 implies $cl(\bigcup_{k \in \omega} L_k)$ is an elementary normal subgroup of $G$. Hence, $\bigcup_{k \in \omega} L_k = \{1\}$, and a fortiori, $L_0 = \{1\}$.

Fix $W \in \mathcal{U}(H_0)$ which is pro-nilpotent and suppose $W$ has a non-trivial $q \neq p$ Sylow subgroup, $K_q$. By the previous paragraph, there must be $n$ such that $K_q/K_q \cap Rad_\varphi(H_n)$ is non-trivial. For such an $n$, we have that

$$K_q/K_q \cap Rad_\varphi(H_n) \leq W/W \cap Rad_\varphi(H_n) \in \mathcal{U}(H_n/Rad_\varphi(H_n))$$

is a non-trivial pro-$q$ subgroup of a pro-nilpotent compact open subgroup of $H_n/Rad_\varphi(H_n)$. This contradicts that every pro-nilpotent compact open subgroup of $H_n/Rad_\varphi(H_n)$ is pro-$p$.

We conclude $H_0$ is locally pro-$p$ and, therefore, $G$ is locally pro-$p$. Applying Fact 5.18, $G$ is a $p$-adic Lie group. $\square$
We now arrive at the deep theory of algebraic groups; Appendix B contains the necessary background material.

**Definition 5.20 ([15])**. A $p$-adic Lie group $G$ is said to be of *simple type* if $G \simeq S(\mathbb{Q}_p)^+$ where $S$ is an almost $\mathbb{Q}_p$-simple isotropic $\mathbb{Q}_p$-algebraic group. We say $G$ is of *adjoint simple type* if $S$ is also adjoint.

**Theorem 5.21.** If $G$ is a non-elementary topologically simple l.c.s.c. $p$-adic Lie group, then $G$ is of adjoint simple type.

**Proof.** See Theorem B.17

Theorem 5.21 is the bridge from $p$-adic Lie theory to algebraic group theory. With Theorem 5.21 in hand, we prove a number of useful lemmas. We begin by considering continuous homomorphisms from simple $p$-adic Lie groups.

**Fact 5.22 ([8, Lemma 5.3]).** Let $k$ be a local field, $G$ be a simply connected almost $k$-simple $k$-algebraic group, and $H$ a locally compact group. Then any non-trivial continuous homomorphism $\pi : G(k) \to H$ is proper - i.e. the preimage of a compact set is compact.

**Lemma 5.23.** Suppose $G$ is a t.d.l.c. Polish group, $N$ is non-elementary topologically simple $p$-adic Lie group, and $\psi : N \to G$ is a non-trivial continuous homomorphism. Then $\psi$ is a closed map and $N \simeq \psi(N)$.

**Proof.** Applying Theorem 5.21, we may assume $N = S(\mathbb{Q}_p)^+$ for $S$ some adjoint $\mathbb{Q}_p$-simple isotropic $\mathbb{Q}_p$-algebraic group. We now consider $\tilde{S}$, the simply connected covering of $S$. Since
isogenies carry almost $\mathbb{Q}_p$-simple factors to almost $\mathbb{Q}_p$-simple factors, $\tilde{S}$ is an almost $\mathbb{Q}_p$-simple group. Additionally, $\tilde{S}$ is isotropic; indeed, if $\tilde{S}$ is anisotropic, then $\tilde{S}(\mathbb{Q}_p)^+ = \{1\}$ via Proposition B.7, and Proposition B.8 implies $S(\mathbb{Q}_p)^+ = \{1\}$ which is absurd.

Theorem B.10 gives $\tilde{S}(\mathbb{Q}_p) = \tilde{S}(\mathbb{Q}_p)^+$. We thus have

$$\tilde{S}(\mathbb{Q}_p) = \tilde{S}(\mathbb{Q}_p)^+ \xrightarrow{\tilde{p}} S(\mathbb{Q}_p)^+ \xrightarrow{\psi} G$$

Via Proposition B.11, $\psi \circ \tilde{p} : \tilde{S}(\mathbb{Q}_p) \to G$ is a continuous homomorphism. In view of Fact 5.22, $\psi \circ \tilde{p}$ is proper. It follows that $\psi$ is also proper since $\tilde{p} \upharpoonright \tilde{S}(\mathbb{Q}_p)$ is surjective. We conclude $\psi : N \to G$ is a closed map and $N \simeq \psi(N)$. \hfill $\square$

As an easy consequence of Lemma 5.23, we have the following; the proof is omitted.

**Lemma 5.24.** Suppose $G$ is a t.d.l.c. Polish group and $G$ is a quasi-product with quasi-factors $N$ and $M$ where $N$ is a non-elementary topologically simple $p$-adic Lie group. Then $G \simeq N \times M$.

We additionally need to understand the structure of a t.d.l.c. Polish group $G$ when there is $N \trianglelefteq G$ a topologically simple non-elementary $p$-adic Lie group. To this end, we prove

**Theorem 5.25.** Let $G$ be a non-elementary topologically simple l.c.s.c. $p$-adic Lie group, let $\text{Aut}(G)$ be the group of topological group automorphisms of $G$, and let $\text{Inn}(G)$ be the collection of inner automorphisms. Then $\text{Aut}(G)/\text{Inn}(G)$ is finite.

**Proof.** See Theorem B.23. \hfill $\square$
Lemma 5.26. Suppose $G$ is a t.d.l.c. Polish group and $N \leq G$ is a non-elementary topologically simple $p$-adic Lie group. Then $C_G(N)N$ is a finite index closed normal subgroup of $G$.

Proof. Put $H := cl(C_G(N)N)$. Since $N$ is a non-elementary topologically simple $p$-adic Lie group, $H$ is a quasi-product of $C_G(N)$ and $N$, so via Lemma 5.24, $C_G(N)N = H$. Hence, $C_G(N)N$ is a closed normal subgroup of $G$. Additionally,

$$G/C_G(N)N \ni Aut(N)/Inn(N)$$

, and since $Aut(N)/Inn(N)$ is finite by Theorem 5.25, the lemma follows. \qed

5.4 Decomposition Theorems

Theorem 5.27. Suppose $G$ is a non-elementary t.d.l.c. Polish group which is locally of finite rank and locally pro-$\pi$ for some finite set of primes $\pi$. Then the ascending elementary series is such that $G/A_2$ is finite and $A_2/A_1 \simeq N_1 \times \cdots \times N_k$ for $0 < k < \infty$ where each $N_i$ is a non-elementary compactly generated topologically simple $p(i)$-adic Lie group of adjoint simple type.

Proof. Let $\{1\} \leq A_1 \leq A_2 \leq G$ be the ascending elementary series and put $H := A_2/A_1$. $H$ is elementary-free, locally of finite rank, and locally pro-$\pi$. Let $N_1, \ldots, N_k$ list the minimal normal subgroups of $G$ given by Lemma 5.15. Each $N_i$ is plainly non-elementary and, by Lemma 5.16, topologically simple. Applying Proposition 5.19, we conclude each $N_i$ is a non-elementary topologically simple $p(i)$-adic Lie group. Theorem 5.21 and Theorem B.12 imply each $N_i$ is of adjoint simple type and compactly generated.
Via Lemma 5.26, we have that \( C_H(N_i)N_i = H \) for each \( i \) and, thus, \( \bigcap_{i=1}^{k} C_H(N_i)N_i = H \).

It is easy to check
\[
\bigcap_{i=1}^{k} C_H(N_i)N_i = \left( \bigcap_{i=1}^{k} C_H(N_i) \right) N_1 \ldots N_k
\]

since \( N_i \trianglelefteq C_H(N_j) \) for \( i \neq j \). So \( \bigcap_{i=1}^{k} C_H(N_i) \) is a closed normal subgroup of \( H \) which meets \( N_i \) trivially for each \( i \). Since the \( N_i \) list all non-trivial minimal normal subgroups of \( H \), we conclude \( \bigcap_{i=1}^{k} C_H(N_i) = \{1\} \). Hence, \( N_1 \ldots N_k = H \), and it follows \( H \cong N_1 \times \cdots \times N_k \).

We now consider \( G/A_2 \). The conjugation action of \( G/A_1 \) on the collection of minimal normal subgroups of \( A_2/A_1 \) induces a continuous homomorphism \( \psi : G \to \text{Sym}(k) \). Plainly, \( L := \ker(\psi) \) is a finite index normal subgroup of \( G \); moreover, for each \( N_i \) we have \( N_i \trianglelefteq L \). Applying Lemma 5.26, we have that \( C_L(N_i)N_i \) is finite index in \( L \) for each \( i \). Thus,
\[
\bigcap_{i=1}^{k} C_L(N_i)N_i = \left( \bigcap_{i=1}^{k} C_L(N_i) \right) N_1 \ldots N_k
\]
is finite index in \( L \). Additionally, since \( N_1 \ldots N_k \) is normalized by \( G/A_1 \), we have \( \left( \bigcap_{i=1}^{k} C_L(N_i) \right) \trianglelefteq G/A_1 \). But now
\[
\left( \bigcap_{i=1}^{k} C_L(N_i) \right) \cong \left( \left( \bigcap_{i=1}^{k} C_L(N_i) \right) N_1 \ldots N_k \right) / (A_2/A_1) \leq G/A_2
\]

whereby \( \left( \bigcap_{i=1}^{k} C_L(N_i) \right) \) is an elementary normal subgroup of \( G/A_1 \). Since \( A_1 \) is the elementary radical of \( G \), \( \left( \bigcap_{i=1}^{k} C_L(N_i) \right) = \{1\} \). We conclude \( N_1 \ldots N_k = A_2/A_1 \) is finite index in \( G/A_1 \), so \( A_2 \) is finite index in \( G \) finishing our proof of the theorem.

We get an immediate, interesting corollary:
Corollary 5.28. Suppose $G$ is non-elementary l.c.s.c. $p$-adic Lie group. Then the ascending elementary series is such that $G/A_2$ is finite and $A_2/A_1 \cong N_1 \times \cdots \times N_k$ for $0 < k < \infty$ where each $N_i$ is a non-elementary compactly generated topologically simple $p$-adic Lie group of adjoint simple type.

Proof. Suppose $G$ is a l.c.s.c. $p$-adic Lie group. By Fact 5.18, $G$ contains a compact open subgroup which has finite rank and is pro-$p$. The corollary now follows from Theorem 5.27. $\square$

Remark 5.29. The assumptions on Theorem 5.27 may be weakened slightly to $G$ a t.d.l.c. Polish group such that $G/A_1$ is locally of finite rank and locally pro-$\pi$ for a finite set of primes $\pi$.

We now consider the general case of locally of finite rank t.d.l.c. Polish groups. We only consider elementary-free groups; there is no loss of generality since we may always reduce to such a group.

Lemma 5.30. Suppose $U$ is a second countable profinite group with finite rank. Suppose further $U$ has no non-trivial locally normal abelian subgroups and has trivial quasi-center. If $M \trianglelefteq U$ is pro-$\pi$ for some finite set of primes $\pi$ and $U/M$ is virtually solvable, then $U$ is pro-$\xi$ for some finite set of primes $\xi \supseteq \pi$.

Proof. Let $p_1, \ldots, p_n$ list $\pi$. Since $U$ has finite rank, every $p_i$-Sylow subgroup of $U$ is topologically finitely generated. Fact 5.8 gives that $U/O_{p_i}(U)$ is virtually pro-$p_i$ for each $i$. We may thus find $N_i \trianglelefteq_U U$ such that $N_i$ has a normal subgroup $K_i$ which is pro-$p_i'$ and $N_i/K_i$ is pro-$p_i$ - a so called $p_i'$-Hall subgroup - for each $1 \leq i \leq n$. 
Form $L := \bigcap_{i=1}^n N_i$. Certainly, $L$ has a normal $p'_i$-Hall subgroup, $L_{p'_i}$, for each $1 \leq i \leq n$ since $L \leq N_i$. Setting $K := \bigcap_{i=1}^n L_{p'_i}$, we see that $M \cap K = \{1\}$ as $K$ is a pro-$\pi'$ subgroup. Hence, $KLM/M \simeq K/K \cap M \simeq K$, and $K$ is virtually solvable since $LM/M$ is virtually solvable.

If $K$ is finite, then $K$ must be trivial since $U$ has trivial quasi-center. If $K$ is infinite, then we may find $W \trianglelefteq L$ such that $W \cap K$ is solvable. Since $W \cap K \leq L$, it follows $L$ has a non-trivial abelian normal subgroup contradicting our assumption on $U$. We conclude $K$ is trivial, so $L/K = L$ is pro-$\pi$ proving the lemma.

The next theorem reduces the general case to Theorem 5.27

**Theorem 5.31.** Suppose $G$ is a t.d.l.c. Polish group which is elementary-free and locally of finite rank. Then $G = \bigcup_{i \in \omega} P_i$ where $(P_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compactly generated open subgroups of $G$ such that for each $i \in \omega$ there is a finite set of primes $\pi(i)$ with $P_i/\text{Rad}_G(P_i)$ locally of finite rank and locally pro-$\pi(i)$.

**Proof.** Fix $U \in \mathcal{U}(G)$ with finite rank and let $(p_i)_{i \in \omega}$ list the primes. Define $\mathcal{H} \subseteq S(G)$ by $C \in \mathcal{H}$ if and only if for all $L \in \mathcal{U}(G)$ there is $W \leq \omega L$, $(W_i)_{i \in \omega}$ a normal basis at 1 for $W$, and $n \geq 0$ such that

$$C \leq \bigcap_{i \in \omega} \bigcap_{j=n}^\infty N_G(F(W)_{p_j} \cap W_i)$$

where $F(W)$ is the Fitting subgroup of $W$. Certainly $\mathcal{H}$ is hereditary, and it is easy to check $\mathcal{H}$ is also conjugation invariant.

Form the $\mathcal{H}$-core, $N_\mathcal{H}$, and let $N = \text{cl}(N_\mathcal{H})$.

**Claim 1.** $F(U) \leq N$. 


Proof of claim. It suffices to show $F(U)_p^k \subseteq N_\mathcal{H}$ for each $k \in \omega$. To that end, fix $k \in \omega$ and $u \in F(U)_p^k$. For $C \in \mathcal{H}$ and $L \in \mathcal{U}(G)$, there is $W \leq L$, $(W_i)_{i \in \omega}$ a normal basis at 1 for $W$ and $n \geq 0$ such that

$$C \leq \bigcap_{i \in \omega} \bigcap_{j=n}^\infty N_G(F(W)_p^j \cap W_i)$$

We may take $n > k$. Further, since $F(W) \sim_{c} F(U)$ by Lemma 5.12, there is $l \in \omega$ so that $F(W_l) = F(W) \cap W_l \leq F(U)$.

Setting $\tilde{W} := W_l$, we have that $\tilde{W} \leq L$ and $(W_{l+i})_{i \in \omega}$ is a normal basis at 1 for $\tilde{W}$. Additionally,

$$F(W)_p^j \cap \tilde{W} = F(W)_p^j \cap F(\tilde{W}) = F(\tilde{W})_p^j$$

and hence,

$$C \leq \bigcap_{l \in \omega} \bigcap_{j=n}^\infty N_G(F(\tilde{W})_p^j \cap W_{l+i})$$

Since $F(\tilde{W})_p^j \leq F(U)_p^j$, we have that $u$ centralizes $F(\tilde{W})_p^j$ for $j \geq n$. Thus,

$$cl(\langle u, C \rangle) \leq \bigcap_{i \in \omega} \bigcap_{j=n}^\infty N_G(F(\tilde{W})_p^j \cap W_{l+i})$$

and we have the claim. □

By Claim 1 and Fact 5.9, $G/N$ is locally solvable and, via Theorem 3.6, elementary. $G$ elementary free implies $N = G$ and $N_\mathcal{H}$ is dense in $G$. Let $(n_i)_{i \in \omega}$ list a countable dense subset of $N_\mathcal{H}$ and put $P_i := \langle U, n_0, \ldots, n_i \rangle$. So $(P_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of open subgroups of $G$ which exhaust $G$. By passing to a subsequence, we may assume each $P_i$ is non-elementary.
By construction, \( P_i \in \mathcal{H} \); so there is \( W \leq_o U \) and \( n \geq 0 \) such that \( P_i \leq \bigcap_{j=n}^{\infty} N_G\left( F(W)_{p_j} \right) \). We thus have \( F(W)_{p_j} \leq \text{Rad}_E(P_i) \) for each \( j \geq n \). Letting \( \pi : P_i \to P_i/\text{Rad}_E(P_i) \) be the usual projection, \( \pi(F(W)) \) is pro-\({p_0, \ldots, p_{n-1}}\). Now \( \pi(W) \) is a profinite group with finite rank, \( \pi(F(W)) \leq \pi(W) \) is pro-\({p_0, \ldots, p_{n-1}}\), and \( \pi(W)/\pi(F(W)) \) is virtually solvable. Further, \( \pi(W) \) has trivial quasi-center and, via Corollary 4.18, no non-trivial locally normal abelian subgroups. In view of Lemma 5.30, we conclude \( P_i/\text{Rad}_E(P_i) \) is locally pro-\( \xi \) for some finite set of primes \( \xi \) as required.

Combining Theorem 5.27 and Theorem 5.31, we obtain a complete decomposition result for locally of finite rank t.d.l.c. Polish groups:

**Corollary 5.32.** Let \( G \) be a non-trivial t.d.l.c. Polish group which is elementary-free and locally of finite rank. Then \( G = \bigcup_{i \in \omega} P_i \) where \( (P_i)_{i \in \omega} \) is an \( \leq \)-increasing sequence of compactly generated open subgroups and for each \( P_i \) the ascending elementary series \( A_1(i) \leq A_2(i) \leq P_i \) is such that \( P_i/A_2(i) \) is finite and \( A_2(i)/A_1(i) \simeq N_1(i) \times \cdots \times N_{k(i)}(i) \) for \( 0 < k(i) < \infty \) where each \( N_j(i) \) is a non-elementary compactly generated topologically simple \( p(i,j) \)-adic Lie group of adjoint simple type.

### 5.5 The Class \( \mathcal{E} \mathcal{P} \)

Let \( \mathcal{P} \) be the collection of all l.c.s.c. \( p \)-adic Lie groups for all primes \( p \); certainly this class is closed under subgroups and quotients. Form \( \mathcal{E} \mathcal{P} \), the elementary closure of \( \mathcal{P} \), as defined in Definition 2.17.
**Theorem 5.33.** The class $\mathcal{P}$ is elementarily robust. In particular, $\mathcal{E}\mathcal{P}$ enjoys the same closure properties as $\mathcal{E}$.

**Proof.** We need to show $\mathcal{P}$ satisfies (a), and (b) of Theorem 2.18.

For (a), suppose $H$ is a t.d.l.c. Polish group and $\psi : H \to G$ is a continuous injective homomorphism with $G \in \mathcal{P}$. Let $U \in \mathcal{U}(G)$ be a pro-$p$ compact open subgroup of finite rank given by Fact 5.18. Since $\psi$ is continuous, there is $W \in \mathcal{U}(H)$ such that $W \simeq \psi(W) \leq U$. We conclude that $W$ has finite rank and is pro-$p$ and, via Fact 5.18, $H \in \mathcal{E}\mathcal{P}$.

For (b), suppose $G$ is a t.d.l.c. Polish group, $H, L \lhd G$, $[H, L] = \{1\}$, and $H \in \mathcal{P}$. We may assume $H$ is non-elementary since else we are done via Lemma 2.11. Let $A_1 \leq A_2 \leq H$ be the ascending elementary series for $H$. Since $A_1$ is elementary, $cl(A_1 L)/L$ is elementary via Lemma 2.11.

Via Theorem 5.27, $A_2/A_1 \simeq N_1 \times \cdots \times N_k$ with $N_i$ topologically simple $p$-adic Lie groups. Letting $\psi : H/A_1 \to cl(HL)/cl(A_1 L)$ be the obvious map, we have that $\psi(N_i)$ must be non-trivial since else elementary. Lemma 5.23 gives that the $\psi(N_i)$ are closed subgroups of $cl(HL)/cl(A_1 L)$. Since the $\psi(N_i)$ centralize each other, $cl(\psi(N_1) \cdots \psi(N_k))$ is a subgroup of $cl(HL)/cl(A_1 L)$ which is a quasi-product with quasi-factors $\psi(N_1) \cdots \psi(N_k)$. Lemma 5.24 implies that $\psi(N_1) \cdots \psi(N_k)$ is already closed.

So $\psi(A_2/A_1)$ is a closed subgroup of $cl(HL)/cl(A_1 L)$. Furthermore, since $A_2/A_1$ is finite index in $H/A_1$, we have that $\psi(H/A_1)$ is closed, whereby $\psi(H/A_1) = cl(HL)/cl(A_1 L)$. Thus, $cl(HL)/cl(A_1 L)$ is a quotient of a $p$-adic Lie group and, therefore, a $p$-adic Lie group. We conclude that $cl(HL)/L$ is elementary-by-$p$-adic, so $cl(HL)/L \in \mathcal{E}\mathcal{P}$ as required. \qed
We remark that all locally of finite rank groups lie in $\mathcal{E} \mathcal{P}$; however, it is easy to see these groups do not exhaust the class.

5.6 **Examples**

We first compute the decomposition given by Theorem 5.27 in an example. Let $S_3$ be the symmetric group on three elements and let $S_3 \acts (\text{PSL}_3(\mathbb{Q}_p))^3 \times \text{PSL}_4(\mathbb{Q}_7)$ by permuting the first three coordinates. We form $G := \mathbb{Z} \times (\text{PSL}_3(\mathbb{Q}_p))^3 \times \text{PSL}_4(\mathbb{Q}_7) \rtimes S_3$. It is easy to check that the ascending elementary series for $G$ is $\{1\} \leq \mathbb{Z} \leq \mathbb{Z} \times (\text{PSL}_3(\mathbb{Q}_p))^3 \times \text{PSL}_4(\mathbb{Q}_7) \leq G$.

An example for Corollary 5.32 takes more work. Recall

**Fact 5.34.** (1) $GL_n(\mathbb{Z}_p)$ is a compact open subgroup of $GL_n(\mathbb{Q}_p)$. ([36, LG 4.3.2] )

(2) For any prime $p$, there is $U \leq_o GL_n(\mathbb{Z}_p)$ with rank $n^2$. ([17, 5.1])

(3) For a finite nilpotent group $F$, let $m(F)$ be the cardinality of a minimal set of generators.

Then $m(F) \leq \sup\{m(F_p) \mid p \text{ prime}\} + 1$ where $F_p$ is the Sylow $p$-subgroup of $F$. (This is an exercise or see [19] for a much more general, deep result.)

In view of Fact 5.34, $\text{PSL}_3(\mathbb{Q}_p)$ has a compact open subgroup of rank 9 for any $p$. Let $\{p_i\}_{i \in \omega}$ list the primes and for each $i$ fix $U_i \leq U(\text{PSL}_3(\mathbb{Q}_{p_i}))$ with rank 9. Define $G := \bigoplus_{i \in \omega} (\text{PSL}_3(\mathbb{Q}_{p_i}), U_i)$. Certainly, $W := \prod_{i \in \omega} U_i$ is a compact open subgroup of $G$.

**Claim 1.** $W$ has rank at most 10.

**Proof.** Let $N \leq W$. Since $W$ is pro-nilpotent, we have that $N$ is pro-nilpotent. So $N = \prod_{i \in \omega} N_{p_i}$ with $N_{p_i}$ the $p_i$-Sylow subgroup of $N$. By the uniqueness of Sylow subgroups in a pro-nilpotent group, $N_{p_i} \leq U_{p_i}$ and, thereby, is topologically 9-generated.
Let \((K_i)_{i \in \omega}\) be a normal basis at 1 for \(N\). So \(N = \varprojlim(N/K_i)\). For each \(i\), we have \(N/K_i \simeq \prod_{j=1}^{m} N_{p_j}/(N_{p_j} \cap K_i)\) for some \(m\) depending on \(i\). Via Fact 5.34, we conclude \(N/K_i\) is 10 generated since \(N_{p_j}/(N_{p_j} \cap K_i)\) is nine generated for each \(j\). It now follows \(N\) is topologically 10-generated.

\(G\) is thus locally of finite rank and, just as in Proposition 2.35, elementary-free. Putting

\[H_n := \prod_{i=0}^{n} PSL_3(\mathbb{Q}_{p_i}) \times \prod_{i \geq n+1} U_i\]

we have that \((H_n)_{n \in \omega}\) is an \(\subseteq\)-increasing sequence of open compactly generated subgroups of \(G\) with \(G = \bigcup_{n \in \omega} H_n\). It is easy to check the ascending elementary series for \(H_n\) is \(\{1\} \leq \prod_{i \geq n+1} U_i \leq H_n\). We note that this example shows Corollary 5.32 is sharp.
CHAPTER 6

THE SETS $P_n(G)$

**Definition 6.1.** Let $G$ be a t.d.l.c. Polish group. An element $g \in G$ is **periodic** if $\text{cl}((g))$ is compact. An $n$-tuple $(g_1, \ldots, g_n) \in G^\times n$ is **periodic** if $\text{cl}((g_1, \ldots, g_n))$ is compact.

Periodic elements and tuples appear to play an important role in the structure theory of non-discrete t.d.l.c. Polish groups. Indeed, while discrete groups may be, indeed often are assumed to be, torsion free, van Dantzig’s theorem shows there is an unavoidable abundance of periodic elements and tuples in the non-discrete setting. Moreover, as the previous chapters and the many results in the literature indicate, these non-trivial compact open subgroups and the elements thereof have a deep relationship with the global structure.

In this chapter, we study the sets $P_n(G) := \{ \overline{g} \in G^\times n \mid \overline{g} \text{ is periodic} \}$ for $G$ a t.d.l.c. Polish group. We specifically consider the following question:

**Question 6.2** (Rosendal). For a t.d.l.c. Polish group $G$ and $n \geq 1$, is $P_n(G)$ closed?

In the mid 90s, Willis proved $P_1(G)$ is closed for any t.d.l.c. group $G$ by way of his theory of tidy subgroups [41]. We here give an elementary proof of his result. We go on to contribute to the study of Question 6.2 via the following theorems:

**Theorem 6.3.** If $G$ is a elementary group, then $P_n(G)$ is closed for all $n$.

**Theorem 6.4.** Let $\mathcal{P}$ be the class of all $p$-adic Lie groups for all primes $p$. If $G$ is in $\mathcal{E}\mathcal{P}$, the elementary closure of $\mathcal{P}$, then $P_n(G)$ is closed for all $n$. 93
These theorems, while certainly far from answering Question 6.2 completely, demonstrate that there are no “easy” counterexamples to be found.

6.1 Locally Elliptic Groups

The following proposition is more or less implicit in Platonov’s paper [31]; we give a proof for completeness.

Proposition 6.5. Let $G$ be a t.d.l.c. Polish group. The following are equivalent:

1. For all $X \subseteq G$ finite, $cl(\langle X \rangle)$ is compact; i.e. $G$ is locally elliptic.
2. $P_n(G)$ is dense in $G \times \mathbb{N}$ for all $n$.
3. $G = \bigcup_{i \in \omega} U_i$ where $(U_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compact open subgroups; i.e. $G$ is compactly approximable.
4. For all $X \subseteq G$ compact, $cl(\langle X \rangle)$ is compact.

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1) are immediate.

For (2) $\Rightarrow$ (3), let $(g_i)_{i \in \omega}$ be a countable dense subset of $P_1(G)$ and note $(g_i)_{i \in \omega}$ is dense in $G$. We proceed by induction to produce compact open subgroups $U_0 \subseteq U_1 \subseteq \ldots$ such that $g_0, \ldots, g_n \in U_n$ for each $n \in \omega$. For the base case, we may find $U_0 \in \mathcal{U}(G)$ containing $g_0$ since $g_0 \in P_1(G)$.

Suppose we have constructed $U_n$. Since $g_{n+1} \in P_1(G)$, we may find $V$ a compact open subgroup containing $g_{n+1}$. Since $VU_n$ is compact and $U_n$ open, we may find $A \subseteq V$ finite, symmetric, and containing 1 such that $VU_n = AU_n$. On the other hand, $U_n A$ is compact, so there is a finite, symmetric $B$ with $A \subseteq B \subseteq U_n AU_n$ and $U_n A \subseteq BU_n$. Thus, $U_n BU_n = BU_n$. 


Since $P_{|B|}(G)$ is dense, there is symmetric $C \in P_{|B|}(G)$ containing 1 such that $CU_n = BU_n$ and, therefore, $U_nCU_n = CU_n$. Induction on $k$ shows $(CU_n)^k = C^kU_n$ for all $k \geq 1$. We conclude

$$VU_n \subseteq \langle BU_n \rangle = \langle CU_n \rangle = \bigcup_{k \geq 1} (U_nCU_n)^k = \bigcup_{k \geq 1} C^kU_n = \langle C \rangle U_n =: U_{n+1}$$

$U_{n+1}$ is a product of a relatively compact group and a compact open group and, thereby, is compact and open. We have thus produced $U_{n+1}$ that is a compact open super group of $U_n$ containing $g_{n+1}$. This finishes the induction, and (3) follows.

We collect a few additional facts:

**Theorem 6.6** (Platonov [31]). Let $G$ be a t.d.l.c. group. If $H \leq G$ is locally elliptic and $G/H$ is locally elliptic, then $G$ is locally elliptic.

**Corollary 6.7** (Platonov [31]). Let $G$ be a t.d.l.c. group. Then $\text{Rad}_{LE}(G/\text{Rad}_{LE}(G)) = \{1\}$.

**Theorem 6.8** (Usakov). Let $G$ be a t.d.l.c. group and let $B \subseteq P_1(G)$ be a conjugation invariant set with compact closure. Then $\langle B \rangle$ is normal with compact closure. In particular, if $g \in P_1(G)$ is such that $g^G$ is relatively compact, then $cl(\langle g^G \rangle)$ is compact and normal.

A nice proof of Theorem 6.8 may be found in S.P. Wang’s paper [39]. Theorem 6.8 and the uniqueness of the locally elliptic radical imply a useful fact: For $G$ a t.d.l.c. Polish group, if $g \in P_1(G)$ is such that $g^G$ is relatively compact, then $g \in \text{Rad}_{LE}(G)$.

We now prove a slight adaptation of Theorem 6.6.
Lemma 6.9. If $G$ is a t.d.l.c. Polish group and $L \leq_{cc} G$ is central in $G$, then $\text{Rad}_{LE}(G)$ is open.

Proof. Fix $U \in \mathcal{U}(G)$. Plainly, $UL \leq G$ is open and $G/UL$ is finite. Since $L$ is central, we have that $U \leq UL$. The normal core of $UL$, $W$, is then normal in $G$ and has a compact open normal subgroup. We conclude that $\text{Rad}_{LE}(W)$ is open and, since $\text{Rad}_{LE}(W)$ characteristic, $\text{Rad}_{LE}(G)$ is open. \qed

Proposition 6.10. Let $G$ be a t.d.l.c. Polish group. If $H \leq G$ is compactly generated with an open locally elliptic radical and $G/H$ is compactly generated with an open locally elliptic radical, then $G$ is compactly generated with an open locally elliptic radical.

Proof. Let us first check $G$ is compactly generated: Say $X$, $Y$ are compact generating sets for $H$ and $G/H$, respectively, and take $U \in \mathcal{U}(G)$. Since $Y$ is compact, we may find $g_0, \ldots, g_n$ in $G$ such that $\psi(g_0 U), \ldots, \psi(g_n U)$ is an open cover for $Y$. We claim $X \cup g_0 U \cup \cdots \cup g_n U$ is a generating set for $G$. Indeed, take $g \in G$. Certainly, $\psi(g) = g_i u_1 \cdots g_n u_k H$ for some $g_i u_j \in g_i U$; so $g = g_i u_1 \cdots g_n u_k h$ for some $h \in H$. Since $X$ generates $H$, it is now clear $X \cup g_0 U \cup \cdots \cup g_n U$ generates $G$.

We now consider the non-trivial claim. Since $\text{Rad}_{LE}(H)$ is characteristic in $H$, $\text{Rad}_{LE}(H) \leq G$. We may thus form $\tilde{H} = H/\text{Rad}_{LE}(H)$, $\tilde{G} = G/\text{Rad}_{LE}(H)$, and let $\pi : \tilde{G} \to G/H$ be the usual projection. These give

$$1 \to \tilde{H} \to \tilde{G} \to G/H \to 1$$

a short exact sequence of topological groups.
Note that $\tilde{H}$ is discrete and finitely generated. It follows that $C_{\tilde{G}}(\tilde{H})$ is open and normal in $\tilde{G}$. On the other hand, $\pi^{-1}(Rad_{LE}(G/H))$ is also open and normal in $\tilde{G}$. So $J := C_{\tilde{G}}(\tilde{H}) \cap \pi^{-1}(Rad_{LE}(G/H)) \leq_o \tilde{G}$.

**Claim 1.** $Rad_{LE}(J)$ is open.

**Proof of claim.** Since $J$ is open, we have $K := J\tilde{H}/\tilde{H} \cong J/J \cap \tilde{H}$. This gives a short exact sequence $1 \to J \cap \tilde{H} \to J \to K \to 1$ of topological groups. Observe that $Z := J \cap \tilde{H}$ is central and $K$ is locally elliptic.

Since $K$ is locally elliptic, $K = \bigcup_{i \in \omega} U_i$ such that $(U_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence of compact open subgroups of $K$. Putting $C_i := \pi^{-1}(U_i)$, we have $J = \bigcup_{i \in \omega} C_i$ with $(C_i)_{i \in \omega}$ an $\subseteq$-increasing sequence of open subgroups such that $|C_{i+1} : C_i| < \infty$ and $C_i/Z$ compact. Since $Z$ is central, $Rad_{LE}(C_i)$ is open for each $i$ by Lemma 6.9.

Now $Rad_{LE}(C_i)Z/Z \cong Rad_{LE}(C_i)/Rad_{LE}(C_i) \cap Z$. Since $Z \leq \tilde{H}$,

$$(Rad_{LE}(C_i) \cap Z) \leq Rad_{LE}(Z) \leq Rad_{LE}(\tilde{H}) = \{1\}$$

where the last equality follows by Corollary 6.7. Hence, $Rad_{LE}(C_i)$ is compact.

For $r \in Rad_{LE}(C_i)$, we thus have that $r^{C_i}$ is relatively compact and, so, $r^{C_{i+1}}$ is also relatively compact since $C_i$ has finite index in $C_{i+1}$. Since $r \in P_1(C_{i+1})$, Theorem 6.8 implies $r \in Rad_{LE}(C_{i+1})$. We conclude $(Rad_{LE}(C_i))_{i \in \omega}$ is $\subseteq$-increasing. Hence, $R := \bigcup_{i \in \omega} Rad_{LE}(C_i)$ is an open normal locally elliptic subgroup of $J$, and $Rad_{LE}(J)$ is open. \qed
By Claim 1, $\text{Rad}_{\mathcal{LE}}(\tilde{G})$ is open. Letting $\pi : G \to \tilde{G}$ be the usual projection, $\pi^{-1}(\text{Rad}_{\mathcal{LE}}(\tilde{G}))$ is open and normal in $G$. Further,

$$1 \to \text{Rad}_{\mathcal{LE}}(H) \to \pi^{-1}(\text{Rad}_{\mathcal{LE}}(\tilde{G})) \to \text{Rad}_{\mathcal{LE}}(\tilde{G}) \to 1$$

is a short exact sequence of topological groups. Theorem 6.6 implies $\pi^{-1}(\text{Rad}_{\mathcal{LE}}(\tilde{G}))$ is locally elliptic and we conclude $\text{Rad}_{\mathcal{LE}}(G)$ is open.

We remark that the compact generation hypothesis cannot be omitted.

We close this section by noting a couple of facts about the locally elliptic radical and the sets $P_n(G)$:

**Lemma 6.11.** If $G$ is a t.d.l.c. Polish group, then $P_n(G)\text{Rad}_{\mathcal{LE}}(G)^{\times n} = P_n(G)$.

*Proof.* Since the reverse inclusion obviously holds, take $\mathbf{g} \in P_n(G)$ and $\mathbf{r} \in \text{Rad}_{\mathcal{LE}}(G)^{\times n}$ and consider $X := \{r_1, \ldots, r_n\}^{cl(\langle \mathbf{g} \rangle)}$. We see $X \subseteq \text{Rad}_{\mathcal{LE}}(G)$ is compact since a finite union of compact sets and, thus, $cl(\langle X \rangle)$ is compact via Proposition 6.5. We conclude $cl(\langle \mathbf{g} \rangle)cl(\langle X \rangle)$ is a compact group and $cl(\langle \mathbf{g}\mathbf{r} \rangle) \leq cl(\langle \mathbf{g} \rangle)cl(\langle X \rangle)$. So $\mathbf{g}\mathbf{r} \in P_n(G)$ as required.

**Lemma 6.12.** If $G$ is a t.d.l.c. Polish group with an open locally elliptic radical, then $P_n(G)$ is closed for all $n$.

*Proof.* Certainly, $P_n(G)\text{Rad}_{\mathcal{LE}}(G)^{\times n}$ is closed since a collection of left cosets of the open subgroup $\text{Rad}_{\mathcal{LE}}(G)^{\times n} \leq G^{\times n}$. Lemma 6.11 gives that $P_n(G) = P_n(G)\text{Rad}_{\mathcal{LE}}(G)^{\times n}$, and therefore, $P_n(G)$ is closed.
6.2 $P_1(G)$

We here give an elementary proof of Willis’ result that $P_1(G)$ is closed. The original argument uses the highly non-trivial Theorem 1.4. Our argument uses only a small portion of the theory developed in [40].

Let $G$ be a t.d.l.c. group, fix $g \in G$, and take $U \in \mathcal{U}(G)$. Define $U_k := \bigcap_{i=0}^{k} g^i U g^{-i}$ for $k \geq 0$. Certainly, $gU_{k+1}g^{-1}/U_{k+2} \hookrightarrow gU_kg^{-1}/U_{k+1}$ for any $k \geq 0$. Furthermore, since $gU_kg^{-1}/U_{k+1}$ is finite, the injection must be a bijection for large enough $k$. Let us fix $N$ such that $gU_{k+1}g^{-1}/U_{k+2} \hookrightarrow gU_kg^{-1}/U_{k+1}$ for all $k \geq N$.

It is easy to see $gU_kg^{-1}/U_{k+1} \hookrightarrow gU_Ng^{-1}/U_{N+1}$ for all $k \geq N$. So

$$gU_Ng^{-1} = gU_kg^{-1}U_{N+1} \subseteq gU_kg^{-1}U_N$$

for all $k \geq N$. We conclude that $U_N \subseteq \bigcap_{k \geq N} (U_k g^{-1} U_N g)$.

Putting $U_+ := \bigcap_{k \in \omega} U_k$, we have $U_+ g^{-1} U_N g \subseteq \bigcap_{k \geq N} (U_k g^{-1} U_N g)$. Conversely, take $x \in \bigcap_{k \geq N} (U_k g^{-1} U_N g)$. For each $k \geq N$, we have $x = u_k w_k$ where $u_k \in U_k$ and $w_k \in g^{-1} U_N g$. Since $g^{-1} U_N g$ is compact, we may pass to a subsequence $(w_{k_i})_{i \in \omega}$ which converges to some $w \in g^{-1} U_N g$; $(u_{k_i})_{i \in \omega}$ must also converge to some $u$. As $(U_k)_{k \geq 0}$ forms an $\subseteq$-decreasing sequence of sets, $u \in U_k$ for all $k \geq 0$. We conclude $x \in U_+ g^{-1} U_N g$. It now follows that $U_N g^{-1} U_N = U_+ g^{-1} U_N$. Setting $V := U_N$, we have $V g^{-1} V = V_+ g^{-1} V$.

**Claim 1.** Let $g$ and $V$ be as above. Then $(V g^{-1} V)^n = V g^{-n} V$ for all $n \geq 1$. 

Proof. We proceed by induction on \( n \). Since the base is obvious, suppose the claim holds up to \( n \). Applying the induction hypothesis, \((Vg^{-1}V)^{n+1} = Vg^{-n}Vg^{-1}V\). We see, by our work above, that

\[
Vg^{-n}Vg^{-1}V = Vg^{-n}Vg^{-1}V \subseteq Vg^{-n}g^nVg^{-n}g^{-1}V = Vg^{-n-1}V
\]

Hence, \((Vg^{-1}V)^{n+1} = Vg^{-n-1}V\) as desired. \(\square\)

**Theorem 6.13** (Willis [41]). If \( G \) is a t.d.l.c. group, then \( P_1(G) \) is closed.

Proof. Suppose \((g_i)_{i \in \omega} \subset P_1(G)\) is such that \( g_i \to g \). By Claim 1, we may find \( V \in \mathcal{U}(G) \) be such that \((Vg^{-1}V)^n = Vg^{-n}V\) for \( n \geq 1 \). Find \( i \) such that \( g_i^{-1} \in Vg^{-1}V \); say \( g_i^{-1} = ug^{-1}v \). Since \( g_i^{-1} \) is periodic, there is \( n \geq 0 \) such that \((g_i^{-1})^n \in V \). We thus have that \( u'g^{-n}v' = (ug^{-1}v)^n \in V \) where \( u',v' \in V \). So \( g^{-n} \in V \), and it follows that \( g \in P_1(G) \). \(\square\)

### 6.3 Elementary Groups

We now show any elementary group \( G \) has \( P_n(G) \) closed for every \( n \). The troublesome step in the obvious induction argument is the group extension case. We therefore begin by developing some machinery.

**Lemma 6.14.** Let \( G \) be a t.d.l.c. Polish group and suppose \( N \trianglelefteq G \) is such that \( P_n(G/N) \) is closed. If \( \overline{g} \in \text{cl}(P_n(G)) \), then there is a compactly generated \( W \leq_o G \) containing \( \overline{g} \) such that \( N \cap W \) is cocompact in \( W \).
Proof. Let $\mathbf{g}_i \to \mathbf{g}$ witness $\mathbf{g} \in \operatorname{cl}(P_n(G))$ and let $\pi : G \to G/N$ be the usual projection. Since $\pi$ is continuous, $\pi(\mathbf{g}_i) \in P_n(G/N)$, and therefore $\pi(\mathbf{g}) \in P_n(G/N)$. We may thus find $O \in \mathcal{U}(G/N)$ such that $\pi(\mathbf{g}) \in O$.

Finding $V \in \mathcal{U}(G)$ such that $VN \leq_o \pi^{-1}(O)$, we have that $W := \langle V, \mathbf{g} \rangle \leq_o \pi^{-1}(O)$, and $WN/N \leq_o O$. Thus, $WN/N$ is compact, and since $WN/N \simeq W/W \cap N$, the lemma follows. □

Let $G$ be t.d.l.c. Polish group, fix $\mathbf{g} \in cl(P_k(G))$, and let $\Omega \leq_o G$ be compactly generated and containing $\mathbf{g}$. We now inductively define a sequence of pairs of subgroups $(R_i, \Omega_i)_{i \in \omega}$ such that $\mathbf{g} \in \Omega_i$, $\Omega_i$ is open and compactly generated, $\Omega_{i+1} \leq \Omega_i$, and $R_i \cap \Omega_i \leq_{cc} \Omega_i$.

For the base case, we set $R_0 := \operatorname{Res}(\Omega)$, so $\Omega/R_0$ is residually discrete and, thus, a SIN group via Theorem 1.12. Since SIN groups have an open locally elliptic radical, Lemma 6.12 implies $P_k(\Omega/R_0)$ is closed. We may thus find $\Omega_0 \leq_o \Omega$ compactly generated such that $\mathbf{g} \in \Omega_0$ and $R_0 \cap \Omega_0 \leq_{cc} \Omega_0$ via Lemma 6.14.

Suppose we have built $(R_n, \Omega_n)$ and set $R_{n+1} := \operatorname{Res}(R_n \cap \Omega_n)$. So $R_{n+1} \leq R_n \cap \Omega_n \leq_{cc} \Omega_n$, and since $R_{n+1}$ is characteristic in $R_n \cap \Omega_n$, we have that $R_{n+1} \leq \Omega_n$. This gives a short exact sequence of topological groups:

$$1 \to (R_n \cap \Omega_n)/R_{n+1} \to \Omega_n/R_{n+1} \to \Omega_n/(R_n \cap \Omega_n) \to 1$$

By the induction hypothesis, $\Omega_n/(R_n \cap \Omega_n)$ is compact and $\Omega_n$ compactly generated. So $R_n \cap \Omega_n$ is compactly generated. Since $R_n \cap \Omega_n/R_{n+1}$ is a SIN group, Proposition 6.10 implies
\( \Omega/R_{n+1} \) has an open locally elliptic radical. Applying Lemma 6.12, we conclude \( P_k(\Omega_n/R_{n+1}) \) is closed.

Lemma 6.14 now gives \( \Omega_{n+1} \leq_o \Omega_n \) compactly generated and containing \( \overline{g} \) such that \( R_{n+1} \cap \Omega_{n+1} \leq_{cc} \Omega_{n+1} \). This finishes our inductive construction.

**Lemma 6.15.** Suppose \( G \) is an elementary group and \( \overline{g} \in \text{cl}(P_k(G)) \). Let \( M \leq G, \Omega \leq_o G \) be compactly generated with \( \overline{g} \in \Omega \), and form \( (\Omega_n, R_n)_{n \in \omega} \) for \( \Omega \) and \( \overline{g} \). If \( R_n \leq M \) for some \( n \) and \( L \trianglelefteq M \), then there is \( m \geq n \) with \( R_m \leq L \).

**Proof.** We induct on the rank of \( M/L \) simultaneously for all \( M \leq G \) with \( R_n \leq M \) for some \( n \).

Suppose \( \text{rk}(M/L) = 0 \), so \( M/L \) is discrete or compact. In either case,

\[
R_{n+1} = \text{Res}(R_n \cap \Omega_n) \leq \text{Res}(M) \leq L
\]

We conclude \( R_{n+1} \leq L \).

For the induction step, suppose \( M \leq G, R_n \leq M \) for some \( n \), and \( L \trianglelefteq M \) with \( \text{rk}(M/L) = \alpha + 1 \). It is easy to see \( (M \cap \Omega_n) / (L \cap \Omega_n) \leq_o M/L \) and, thus,

\[
\text{rk}((M \cap \Omega_n) / (L \cap \Omega_n)) \leq \alpha + 1
\]

Since \( M \cap \Omega_n \leq_{cc} \Omega_n \), we have that \( (M \cap \Omega_n) / (L \cap \Omega_n) \) is compactly generated. There is thus \( N \trianglelefteq (M \cap \Omega_n) / (L \cap \Omega_n) \) such that

\[
\text{rk} \left( ((M \cap \Omega_n) / (L \cap \Omega_n)) / N \right) \leq \alpha
\]
and \( rk(N) \leq \alpha \).

Let \( \tilde{N} \trianglelefteq (M \cap \Omega_n) \) be the pre-image of \( N \) under the usual projection. Since

\[
\text{rk}\left( (M \cap \Omega_n) / \tilde{N} \right) \leq \alpha
\]

and \( R_{n+1} \leq (M \cap \Omega_n) \), we may find \( k \geq n + 1 \) such that \( R_k \leq \tilde{N} \) via the induction hypothesis.

On the other hand, \( \text{rk}(\tilde{N} / (L \cap \Omega_n)) \leq \alpha \) and \( R_k \leq \tilde{N} \). The induction hypothesis therefore implies there is \( m \geq k \) such that \( R_m \leq L \cap \Omega_n \leq L \). This finishes the induction. \( \square \)

**Theorem 6.16.** If \( G \) is a elementary group, then \( P_n(G) \) is closed for all \( n \).

**Proof.** We induct on the rank of \( G \). Since the base case, \( rk(G) = 0 \), is trivial, suppose the theorem holds up to rank \( \alpha \). Say \( rk(G) = \alpha + 1 \) and let \( \overline{g} \in cl(P_n(G)) \).

If \( G = \bigcup_{i \in \omega} H_i \) where \((H_i)_{i \in \omega}\) is an \( \subseteq \)-increasing sequence of open subgroups with \( rk(H_i) \leq \alpha \), then \( \overline{g} \in H_i \) for some sufficiently large \( i \). The induction hypothesis gives \( \overline{g} \in P_n(H_i) \subseteq P_n(G) \), and the theorem follows.

Suppose \( G \) is a group extension of lower rank groups; let \( N \trianglelefteq G \) be such that

\[
\text{rk}(G/N), rk(N) \leq \alpha
\]

By the induction hypothesis and Lemma 6.14, we may find \( \Omega \leq_o G \) compactly generated with \( \overline{g} \in \Omega \) and \( N \cap \Omega \leq_{cc} \Omega \). Letting \((R_i, \Omega_i)_{i \in \omega}\) be as above for \( \overline{g} \) and \( \Omega \), we have that \( R_0 \leq N \).
Since \( \{1\} \subseteq N \), Lemma 6.15 implies there is \( m \geq 0 \) such that \( R_m \subseteq \{1\} \). We conclude \( \Omega_m \) is compact and, therefore, \( g \in P_n(G) \) finishing the induction.

\[\square\]

6.4 \( \mathcal{E} \mathcal{P} \) Groups

Let \( \mathcal{P} \) be the collection of all l.c.s.c. \( p \)-adic Lie groups for all primes \( p \). Form \( \mathcal{E} \mathcal{P} \), the elementary closure of \( \mathcal{P} \), as defined in Definition 2.17. We equip \( \mathcal{E} \mathcal{P} \) with a rank as follows:

For \( G \in \mathcal{E} \mathcal{P} \), put

- \( G \in \mathcal{E} \mathcal{P}_0 \) if and only if \( G \) is profinite, discrete, or \( p \)-adic Lie for some prime \( p \).
- For \( \alpha + 1 \), put \( G \in \mathcal{E} \mathcal{P}_\alpha \) if and only if there exists \( N \subseteq G \) such that \( N \in \mathcal{E} \mathcal{P}_\alpha \) and \( G/N \in \mathcal{E} \mathcal{P}_\alpha \). Put \( G \in \mathcal{E} \mathcal{P}_l^\alpha \) if and only if \( G = \bigcup_{i \in \omega} H_i \) where \( (H_i)_{i \in \omega} \) is an \( \subseteq \)-increasing sequence of open subgroups of \( G \) with \( H_i \in \mathcal{E} \mathcal{P}_\alpha \) for each \( i \in \omega \). Define \( \mathcal{E} \mathcal{P}_{\alpha+1} := \mathcal{E} \mathcal{P}_\alpha \cup \mathcal{E} \mathcal{P}_l^\alpha \).
- For \( \lambda \) a limit ordinal, \( \mathcal{E} \mathcal{P}_\lambda := \bigcup_{\alpha < \lambda} \mathcal{E} \mathcal{P}_\alpha \).

We define \( rk_{\mathcal{E} \mathcal{P}}(G) := \min \{ \alpha \mid G \in \mathcal{E} \mathcal{P}_\alpha \} \). We call \( rk_{\mathcal{E} \mathcal{P}} \) the \( \mathcal{E} \mathcal{P} \)-rank.

Observation 6.17. Let \( G \in \mathcal{E} \mathcal{P} \). Then

1. \( rk_{\mathcal{E} \mathcal{P}}(G) \) is a successor ordinal.

2. If \( G \) is compactly generated, then the rank of \( G \) is given by a group extension.

3. If \( O \leq_o G \), then \( O \in \mathcal{E} \mathcal{P} \) and \( rk_{\mathcal{E} \mathcal{P}}(O) \leq rk_{\mathcal{E} \mathcal{P}}(G) \).

The observation follows by induction on \( rk_{\mathcal{E} \mathcal{P}} \).

We now show the sets \( P_n(G) \) are closed in any \( G \in \mathcal{E} \mathcal{P} \). To that end, we first consider \( p \)-adic Lie groups. Recall
Theorem 6.18 (Parreau [29]). Every compactly generated, periodic subgroup of $GL_n(\mathbb{Q}_p)$ is relatively compact.

A $p$-adic Lie group has a canonical representation: For $G$ a $p$-adic Lie group there is a continuous homomorphism $Ad : G \to GL_n(\mathbb{Q}_p)$ called the adjoint representation of $G$.

Fact 6.19 (Glöckner, Willis [18]). Let $G$ be a $p$-adic Lie group with $x \in G$ and $s : G \to \mathbb{Z}$ be the scale function. Then $s(x) = 1 = s(x^{-1})$ if and only if $Ad(x)$ is a periodic element.

Theorem 6.20 (Glöckner, Willis [18]). A compactly generated uniscalar $p$-adic Lie group is a SIN group.

Proposition 6.21. If $G$ is a l.c.s.c. $p$-adic Lie group, then the sets $P_n(G)$ are closed for every $n$.

Proof. Suppose $\overline{g} \in cl(P_n(G))$ is witnessed by $\overline{g}_i \to g$. Plainly, $Ad(\overline{g}_i) \in P_n(GL_k(\mathbb{Q}_p))$ and $Ad(\overline{g}_i) \to Ad(\overline{g})$. Further, since $P_1(GL_k(\mathbb{Q}_p))$ is closed, we have that $cl(\langle Ad(\overline{g}) \rangle)$ is a periodic group. Theorem 6.18 thus implies $cl(\langle Ad(\overline{g}) \rangle)$ is compact.

Find $U \in U(GL_k(\mathbb{Q}_p))$ such that $Ad(\overline{g}) \in U$; without loss of generality, $Ad(\overline{g}_i) \in U$ for all $i$. So $H := Ad^{-1}(U)$ is a clopen subgroup of $G$ which contains $(\overline{g}_i)_{i<\omega}$ and $\overline{g}$. Furthermore, $H/Ker(Ad \upharpoonright_H)$ is elementary via Theorem 2.16. On the other hand, $Ker(Ad \upharpoonright_H)$ is uniscalar by Fact 6.19. Let $(P_i)_{i<\omega}$ be an $\subseteq$-increasing exhaustion of $Ker(Ad \upharpoonright_H)$ by compactly generated relatively open subgroups. Since each $P_i$ is again uniscalar, Theorem 6.20 implies each $P_i$ is SIN. We conclude $Ker(Ad \upharpoonright_H)$ is elementary and, therefore, $H$ is elementary.

In view of Theorem 6.16, we conclude $\overline{g} \in cl(P_n(H)) = P_n(H)$ and, therefore, $\overline{g} \in P_n(G)$. $\square$
For our next lemma, we need a fact about pro-$p$ groups of finite rank.

**Fact 6.22 ([17]).** If $U$ is a finite rank pro-$p$ group, then $\text{Aut}(U)$ is virtually a finite rank pro-$p$ group under the congruence subgroup topology.

**Lemma 6.23.** Suppose $G$ is a t.d.l.c. Polish group and $H \trianglelefteq G$ is a non-discrete $p$-adic Lie group with $G/H$ elementary. Then $G/\text{Rad}_e(G)$ is a $p$-adic Lie group.

**Proof.** Fix $U \in \mathcal{U}(G)$. Certainly, $U \cap H \leq U$ is infinite, and Lemma 6.23 implies that $U/C_U(U \cap H)$ is virtually pro-$p$ and of finite rank. Since $U \cap H \in \mathcal{U}(H)$ and $H \trianglelefteq G$, we also have that $\text{Comm}_G(U \cap H) = G$.

Define $\mathcal{H} \subseteq S(G)$ by $C \in \mathcal{H}$ if and only if for all $V \in \mathcal{U}(G)$ there is $N_W \leq W \leq_a V$ and $(W_i)_{i \in \omega}$ a normal base at 1 for $W$ such that

1. $N_W \sim_c U \cap H$

2. $C \leq \bigcap_{i \in \omega} N_G(N_W \cap W_i)$.

$\mathcal{H}$ is plainly hereditary, and since $U \cap H$ is commensurated, it follows as in Lemma 4.13 that $\mathcal{H}$ is conjugation invariant. We may thus form the $\mathcal{H}$-core, $\bar{N}_\mathcal{H}$.

**Claim 1.** $C_U(U \cap H) \trianglelefteq N_\mathcal{H}$.

**Proof of claim.** Take $g \in C_U(U \cap H)$ and $C \in \mathcal{H}$. Fix $V \in \mathcal{U}(G)$ and find $W \leq_a V$ as given by the definition of $C \in \mathcal{H}$. So there is $(W_i)_{i \in \omega}$ a normal basis at 1 for $W$ and $N_W \leq W$ with $N_W \sim_c U \cap H$ and $C \leq \bigcap_{i \in \omega} N_G(N_W \cap W_i)$. 

Since $N_W \sim_c U \cap H$, there is $i \in \omega$ such that $N_W \cap W_i \leq U \cap H$. Fix such an $i$ and put $N_{W_i} := N_W \cap W_i$. We now have that $N_{W_i} \leq W_i$, $N_{W_i} \sim_c U \cap H$, and $(W_j)_{j \geq i}$ is a normal basis at 1 for $W_i$. We further have that $N_{W_i} \leq U \cap H$ and, therefore, $cl((C, g)) \leq \bigcap_{j \geq i} N_G(N_{W_i} \cap W_j)$. It now follows $cl((C, g)) \in \mathcal{H}$ and $g \in N_H$.

Claim 2. $N := cl(N_H)$ is elementary

Proof of claim. Let $(n_i)_{i \in \omega}$ be a countable dense subset of $N_H$ and put $P_i := \langle U, n_0, \ldots, n_i \rangle$. We see $(P_i)_{i \in \omega}$ is an $\subseteq$-increasing sequence and, since $\{n_0, \ldots, n_i\} \subseteq N_H$ and $U \in \mathcal{H}$, $P_i \in \mathcal{H}$.

Find $W \leq_o U$, $N_W \leq W$, and $(W_i)_{i \in \omega}$ a normal basis at 1 as given by $P_i \in \mathcal{H}$. We now see that $N_W \leq P_i$ and, since $N_W \sim_c H \cap U$, $\pi(H \cap P_i)$ is discrete in $P_i/N_W$ where $\pi$ is the usual projection.

We now consider $(P_i/N_W)/\pi(H \cap P_i) \simeq P_i/H \cap P_i$. Since $P_i$ is open, $P_iH/H$ is a clopen subgroup of $G/H$, so $P_iH/H \simeq P_i/(H \cap P_i)$ is elementary. We conclude that $P_i/N_W$ is discrete-by-elementary and, thus, elementary. Seeing as $N_W$ is compact, $P_i$ is elementary. Since $N \leq \bigcup_{i \in \omega} P_i$, Theorem 2.16 gives that $N$ is elementary. □

Form $G/Rad_{\rho}(G)$. By Claim 2, $N \leq Rad_{\rho}(G)$, so $\pi(U)$, where $\pi$ is the usual projection, is a quotient of $U/C_U(U \cap H)$. We conclude $\pi(U)$ is a virtually pro-$p$ and virtually of finite rank compact open subgroup of $G/Rad_{\rho}(G)$. Fact 5.18 implies $G/Rad_{\rho}(G)$ is a $p$-adic Lie group as desired. □

Lemma 6.24. If $G$ is a compactly generated t.d.l.c. Polish group which is residually a $p$-adic Lie group for some fixed prime $p$, then $G$ is elementary-by-$p$-adic Lie.
**Proof.** Let $\mathcal{F}$ be the collection of normal subgroups of $G$ whose quotient is a $p$-adic Lie group. Observe that $\mathcal{F}$ is a filtering family. Indeed, take $M, N \in \mathcal{F}$. The obvious map $\xi : M/M \cap N \to G/N$ is continuous. Fix $U \in U(G/N)$ which is pro-$p$ and finite rank. Since $\xi$ is continuous, there is $V \in U(M/M \cap N)$ such that $\xi(V) \leq U$. Since $V \simeq \xi(V)$, we conclude that $V$ is pro-$p$ and has finite rank. By Fact 5.18, $M/M \cap N$ is a $p$-adic Lie group. It is now clear $G/M \cap N$ is a $p$-adic Lie group, and therefore, $M \cap N \in \mathcal{F}$. Hence, $\mathcal{F}$ is a filtering family.

By Corollary 1.15, there is some $N \in \mathcal{F}$ which is compact by discrete - i.e. elementary. $N$ now witnesses $G$ elementary-by-$p$-adic Lie. \hfill \Box

We remark that, via Theorem 5.33, $\mathcal{E} \mathcal{P}$ is elementarily robust. We may thus take subgroups and quotients of members of $\mathcal{E} \mathcal{P}$ and remain in $\mathcal{E} \mathcal{P}$.

**Lemma 6.25.** Suppose $G \in \mathcal{E} \mathcal{P}$ group. Suppose further $\overline{g} \in cl(P_k(G))$, $\Omega \leq_o G$ with $\Omega$ compactly generated, $\overline{g} \in \Omega$, and there is $M \leq \Omega$ such that $Res(\Omega) \leq M$. Then for all $L \leq M$ there is a compactly generated $\Omega_L \leq_o \Omega$ and $R_L \leq_{cc} \Omega_L$ such that $\overline{g} \in \Omega_L$ and $R_L \leq L$.

**Proof.** We induct on the $\mathcal{E} \mathcal{P}$-rank of $M/L$ over all subgroups $M$ and compactly generated open $\Omega$ containing $\overline{g}$ with $Res(\Omega) \leq M$. For the base case $M/L$ is either profinite, discrete, or a $p$-adic Lie group. Suppose $M/L$ is profinite or discrete and put $R_0 := Res(\Omega)$. So $R_1 := Res(R_0) \leq L$, and $\Omega/R_1$ is elementary. By Lemma 6.14 and Theorem 6.16, there is $\Omega_L \leq_o \Omega$ such that $\overline{g} \in \Omega_L$ and $R_L := R_1 \cap \Omega_L \leq_{cc} \Omega_L$. Since $R_L \leq L$, we have the profinite and discrete cases.

Suppose $M/L$ is $p$-adic Lie. As $\Omega/R_0$ is elementary with $R_0$ as above, we may pass to a compactly generated $\Omega_1 \leq_o \Omega$ such that $R_0 \cap \Omega_1 \leq_{cc} \Omega_1$ and $\overline{g} \in \Omega_1$. Put $R_1 := R_0 \cap \Omega_1$ and
note $R_1$ is compactly generated. Fix $D := \{h_i \mid i \in \omega\}$ a countable dense subgroup of $\Omega_1$ and let $N_i := \bigcap_{l=0}^{n} h_i(R_1 \cap L)h_i^{-1}$. The $N_i$ form a filtering family of closed normal subgroups of $R_1$. Furthermore,

**Claim 1.** $R_1/N_i$ is a $p$-adic Lie group for all $i \in \omega$.

*Proof of claim.* We induct on $i$. For the base case, $i = 0$, $R_1/h_0(R_1 \cap L)h_0^{-1}$ is isomorphic to $R_1/(R_1 \cap L)$, so it suffices to consider the latter group. Let $V \in U(M/L)$ be pro-$p$ with finite rank and let $\pi : R_1/(R_1 \cap L) \to M/L$ be the obvious map. Since $\pi$ is continuous, there is $U \in U(R_1/(R_1 \cap L))$ such that $\pi(U) \subseteq V$. Since $U$ is isomorphic to $\pi(U)$, $U$ is pro-$p$ with finite rank. Fact 5.18 implies $R_1/(R_1 \cap L)$ is a $p$-adic Lie group.

Suppose the result holds up to $k$. It is easy to see we have a continuous homomorphism $N_k/N_{k+1} \to R_1/h_{k+1}(R_1 \cap L)h_{k+1}^{-1}$. So, just as in the base case, $N_k/N_{k+1}$ is a $p$-adic Lie group. Since $R_1/N_k$ is a $p$-adic Lie group by induction, we conclude $G/N_{k+1}$ is also a $p$-adic Lie group.

Putting $A = \bigcap_{i \in \omega} N_i$, we have that $R_1/A$ is elementary-by-$p$-adic Lie via Claim 1 and Lemma 6.24. Thus, $R_1/B$ is a $p$-adic Lie group for $B := \pi^{-1}(Rad_\leq(R_1/A))$. We further have that $B \subseteq \Omega_1$ since $A \subseteq \Omega_1$ and the elementary radical is characteristic.

We now have that $\Omega_1/B$ has a cocompact $p$-adic Lie group, $R_1/B$. By Lemma 6.23, we have that

$$(\Omega_1/B)/Rad_\leq(\Omega_1/B)$$
is a $p$-adic Lie group. It now follows there is a series of normal subgroups $A \leq P \leq \Omega_1$ with $P/A$ elementary and $\Omega_1/P$ a $p$-adic Lie group.

Applying Proposition 6.21 and Lemma 6.14, there is a compactly generated $\Omega_2 \leq_o \Omega_1$ with $\mathfrak{g} \in \Omega_2$ and $P \cap \Omega_2 \leq_{cc} \Omega_2$. It is easy to see $\Omega_2/(A \cap \Omega_2)$ is elementary. Applying Theorem 6.16 and Lemma 6.14, there is a compactly generated $\Omega_3 \leq_o \Omega_2$ with $\mathfrak{g} \in \Omega_3$ such that $A \cap \Omega_3 \leq_{cc} \Omega_3$. Setting $\Omega_L := \Omega_3$, we have $R_L := (A \cap \Omega_L) \leq L$ and $R_L \leq_{cc} \Omega_L \leq_o \Omega$. This finishes the base case.

Suppose the lemma holds up to $rk_{\mathcal{P}}(M/L) \leq \alpha$ and suppose $L \leq M$ is such that $rk_{\mathcal{P}}(M/L) = \alpha + 1$ and $R_0 := \text{Res}(\Omega) \leq M$. Since $\Omega/R_0$ is elementary. There is a compactly generated $\Omega_1 \leq_o \Omega$ such that $\mathfrak{g} \in \Omega_1$ and $(R_0 \cap \Omega_1) \leq_{cc} \Omega_1$. Set $M_1 := (M \cap \Omega_1) \leq_{cc} \Omega_1$ and $L_1 := (L \cap \Omega_1)$. Thus, $M_1$ is compactly generated, $M_1/L_1$ has $\mathcal{P}$-rank at most $\alpha + 1$, and $M_1/L_1$ is a group extension of lower rank groups. We may thus find $L_1 \leq K \leq M_1$ which witnesses the rank of $M_1/L_1$. Furthermore, since $R_1 := \text{Res}(\Omega_1) \leq M_1$, the induction hypothesis implies there is a compactly generated $\Omega_K \leq_o \Omega_1$ and $R_K \leq_{cc} \Omega_K$ such that $\mathfrak{g} \in \Omega_K$ and $R_K \leq K$.

On the other hand, since

$$\frac{(K \cap \Omega_K)}{(L_1 \cap \Omega_K)} \leq_o \frac{K}{L_1}$$

we have,

$$rk_{\mathcal{P}}((K \cap \Omega_K)/(L_1 \cap \Omega_K)) \leq \alpha$$
As $\text{Res}(\Omega_K) \leq (K \cap \Omega_K)$, the induction hypothesis may be applied again to find $\Omega_L \leq_{_0} \Omega_K$ compactly generated containing $\overline{g}$ and $R_L \leq_{cc} \Omega_L$ such that $R_L \leq (L \cap \Omega_L) \leq L$. Since $\Omega_L \leq_{_0} \Omega$, this finishes the induction and we have the lemma. \hfill \Box

**Theorem 6.26.** If $G \in \mathcal{E}\mathcal{P}$, then $P_n(G)$ is closed for all $n$.

**Proof.** We proceed by induction on the $\mathcal{E}\mathcal{P}$-rank of $G$. The base case follows from Proposition 6.21.

Suppose the theorem holds for $\mathcal{E}\mathcal{P}$-groups of rank at most $\alpha$ and suppose $\text{rk}_{\mathcal{E}\mathcal{P}}(G) = \alpha + 1$.

If $G$ is an increasing union of lower rank subgroups, then the theorem follows easily by induction.

Suppose $G$ is a group extension. Say $H \leq G$ is such that $\text{rk}_{\mathcal{E}\mathcal{P}}(H), \text{rk}_{\mathcal{E}\mathcal{P}}(G/H) \leq \alpha$. Take $\overline{g} \in \text{cl}(P_n(G))$. By the induction hypothesis and Lemma 6.14, we may find a compactly generated $\Omega \leq_{_0} G$ such that $(H \cap \Omega) \leq_{cc} \Omega$ and $\overline{g} \in \Omega$. Since $\text{Res}(\Omega) \leq (H \cap \Omega)$ and $\{1\} \leq (H \cap \Omega)$, Lemma 6.25 implies there is $\Omega_1 \leq_{_0} \Omega$ containing $\overline{g}$ such that $\{1\}$ is cocompact in $\Omega_1$. We conclude $\overline{g} \in P_n(G)$, and the theorem is proved. \hfill \Box
CHAPTER 7

CONJUGACY CLASS CONDITIONS

For our final chapter, we consider the structure of t.d.l.c. Polish groups with topological conditions on a conjugacy class. The motivation for the results herein is to answer a question of Kechris and Rosendal:

**Question 7.1** (Kechris, Rosendal). Can a non-trivial t.d.l.c. Polish group admit a comeagre conjugacy class?

To answer the question, we first study t.d.l.c. Polish groups with a non-meager conjugacy class:

**Theorem 7.2.** Suppose $U$ is a profinite Polish group and $g \in U$. Then the following are equivalent:

1. $gU$ is non-meagre.
2. $gU$ is open.
3. $\mu(gU) > 0$ where $\mu$ is the normalized Haar measure on $U$.
4. There is an integer $M > 0$ such that $|C_{U/V}(gV)| \leq M$ for every open normal subgroup $V \trianglelefteq U$.

**Corollary 7.3.** Suppose $G$ is a t.d.l.c. Polish group and $g \in G$ is such that $g$ is periodic. Then the following are equivalent:
(1) $g^G$ is open.

(2) $g^G$ is non-meagre.

(3) $g^G$ is non-null.

We next consider t.d.l.c. Polish groups with a dense conjugacy class.

**Theorem 7.4.** If $G$ is a non-trivial t.d.l.c. Polish group and $g^G$ is dense, then $g^G$ is meagre and Haar null.

We thus answer the motivating question.

**Corollary 7.5.** A non-trivial t.d.l.c. Polish group does not admit a comeagre conjugacy class.

### 7.1 Non-meagre Conjugacy Classes

The key is to initially consider profinite groups with a non-null conjugacy class.

**Lemma 7.6.** Let $U$ be a profinite group with normalized Haar measure $\mu$. If $h \in U$ is such that $\mu(h^U) > 0$, then

(1) For all $N \trianglelefteq_o U$, $|C_{U/N}(hN)| \leq \frac{1}{\mu(h^U)}$.

(2) $|C_U(h)| \leq \frac{1}{\mu(h^U)}$; in particular, $h$ is torsion.

**Proof.** Let $h \in U$ have a non-null conjugacy class, take $N \trianglelefteq_o U$, and let $\pi : G \to G/N$ be the usual projection. The pushforward of the normalized Haar measure $\mu$ on $G$ to $G/N$, $\pi^*\mu$, is normalized counting measure on $G/N$. That is to say $\pi^*\mu(C) = |C|\mu(N)$ for any $C \subseteq U/N$. Now $\pi(h^U) \subseteq (hN)^{U/N}$, so

$$0 < \mu(h^U) \leq \pi^*\mu((hN)^{U/N}) = |(hN)^{U/N}|\mu(N)$$
On the other hand, since $1 = \mu(U) = |U/N|\mu(N)$, we have

$$|(hN)^{U/N}|\mu(N) = |U/N : C_{U/N}(hN)|\mu(N) = \frac{|U/N|\mu(N)}{|C_{U/N}(hN)|} = \frac{1}{|C_{U/N}(hN)|}$$

So $|C_{U/N}(hN)| \leq \frac{1}{\mu(hU)}$, and it follows $|C_U(h)| \leq \frac{1}{\mu(hU)}$.

\[\square\]

**Theorem 7.7.** Suppose $U$ is a profinite Polish group and $g \in U$. The following are equivalent:

(1) $g^U$ is non-meagre.

(2) $g^U$ is open.

(3) $\mu(g^U) > 0$ where $\mu$ is the normalized Haar measure on $U$.

(4) There is $M > 0$ such that $|C_{U/N}(gN)| \leq M$ for all $N \trianglelefteq_U U$.

**Proof.** For (1) $\Rightarrow$ (2), since $g^U$ is non-meagre, it is somewhere dense. $g^U$ is also closed since the continuous image of a compact set and, therefore, contains an non-empty open set. Say $O \subseteq g^U$. Thus, $g^U = \bigcup_{u \in U} uOu^{-1}$, and $g^G$ is open.

(2) $\Rightarrow$ (3) is immediate from properties of the Haar measure.

For (3) $\Rightarrow$ (4), let $g \in U$ be as hypothesized and take $N \trianglelefteq_U U$. By Lemma 7.6, $|C_{U/N}(gN)| \leq \frac{1}{\mu(gU^*)}$. We obtain (4) by fixing $M \geq \frac{1}{\mu(gU^*)}$.

For (4) $\Rightarrow$ (1), suppose $g \in U$ satisfies (4), fix $(N_i)_{i \in \omega}$ a normal basis at 1 for $U$, and take $M > 0$ which witnesses (4) for $g$. For each $i$, let $A_i \subseteq g^U$ be a minimal set of coset representatives for $(gN_i)^{U/N_i}$ in $U/N_i$. We see that $g^U \subseteq A_iN_i := \bigcup_{a \in A_i} aN_i$ and $\bigcap_{i \in \omega} A_iN_i = g^U$. Furthermore, since $(A_iN_i)_{i \in \omega}$ is an $\subseteq$-decreasing sequence, $\mu(A_iN_i) \to \mu(g^U)$ by continuity from above for $\mu$. 
On the other hand, \(|C_{U/N_i}(gN_i)| \leq M\) for any \(i \in \omega\), and therefore, \(|C_{U/N_i}(gN_i)|\) only takes on finitely many values as \(i\) varies. By passing to a subsequence, we may assume \(|C_{U/N_i}(gN_i)| = k \leq M\) for all \(i\). So

\[
\mu(A_iN_i) = \frac{1}{|C_{U/N_i}(gN_i)|} = \frac{1}{k}
\]

for each \(i\).

It now follows \(\mu(A_iN_i) = \mu(gU)\) for all \(i\). Fixing an \(i\) and setting \(E := (A_iN_i) \setminus gU\), we have that \(E\) is open and \(\mu(E) = 0\). As the only Haar null open set is \(\emptyset\), \(E = \emptyset\) and \(A_iN_i = gU\). Hence, \(gU\) is open and, a fortiori, non-meagre.

We now extend this result to all t.d.l.c. Polish groups.

**Lemma 7.8.** Let \(G\) be a t.d.l.c. Polish group. If \(g^G\) is non-null, then \(g^U\) is non-null for any compact open subgroup \(U\) of \(G\). If \(g\) is also periodic, then \(g\) is torsion and \(g^U\) is open for any compact open subgroup \(U\).

**Proof.** Fix \(U \in \mathcal{U}(G)\). Since \(G\) is second countable, there is a countable set \((h_i)_{i \in \omega}\) such that \(G = \bigcup_{i \in \omega} h_iU\). So \(g^G = \bigcup_{i \in \omega} g^{h_iU}\), and for some \(i \in \omega\), \(\mu(g^{h_iU}) > 0\). Fix such an \(i\); now

\[
0 < \mu(g^{h_iU}) = \mu(h_ig^Uh_i^{-1}) = \mu(g^U)\Delta(h_i^{-1})
\]

where \(\Delta\) is the modular function. Since the modular function is strictly positive, we conclude \(\mu(g^U) > 0\) as desired.
If \( g \) is also periodic, we may find \( W \) a compact open subgroup containing \( g \). By the uniqueness of the Haar measure, \( g^W \) is non-null in \( W \) with respect to the normalized Haar measure on \( W \). Lemma 7.6 implies \( g \) is torsion. Theorem 7.7 gives that \( g^W \) is open.

If \( U \) is an arbitrary compact open subgroup, then, since \( W \) is compact and \( U \) open, \( g^W \subseteq g^{w_1U} \cup \cdots \cup g^{w_nU} \) for some \( w_1, \ldots, w_n \in W \). By the Baire category theorem, \( g^{w_iU} = w_ig^Uw_i^{-1} \) is non-meagre for some \( 1 \leq i \leq n \). Certainly, \( g^U \) is non-meagre, and since \( g^U \) is closed, there is \( O \subseteq g^U \) a non-empty open set. We conclude \( g^U = \bigcup_{u \in U} uOu^{-1} \), and \( g^U \) is open.

**Corollary 7.9.** Suppose \( G \) is a t.d.l.c. Polish group and \( g \in G \) is periodic. The following are equivalent:

1. \( g^G \) is open.
2. \( g^G \) is non-meagre.
3. \( g^G \) is non-null.

**Proof.** (1) \( \Rightarrow \) (2) follows by the Baire category theorem.

For (2) \( \Rightarrow \) (3), since \( G \) is \( K_{\sigma} \), we may write \( G = \bigcup_{i \in \omega} K_i \) with the \( K_i \) compact. So \( g^G = \bigcup_{i \in \omega} g^{K_i} \), and the Baire category theorem implies \( g^{K_i} \) is non-meagre for some \( i \). For such an \( i \), we have that \( g^{K_i} \) is closed and, thereby, has non-empty interior. It follows \( \mu(g^G) > 0 \), and we have (3).

(3) \( \Rightarrow \) (1) is immediate from Lemma 7.8.

We remark that it is not clear non-discrete examples of the groups discussed in this section exist. We here present an example due to C. Rosendal of a non-discrete profinite Polish group.
with a non-null conjugacy class. The author wishes to express his thanks to Rosendal for allowing this example to be included.

Let $D_3$ be the dihedral group of the triangle. Recall $D_3 = \langle s, r \mid r^3 = 1, s^2 = 1, \text{ and } sr = r^{-1}s \rangle$ and every element of $D_3$ is of the form $r^i$ or $sr^i$ for $i = 0, 1, 2$. Form $D_3^N$ and define $G \trianglelefteq D_3^N$ to be the collection of $\alpha$ such that all coordinates of $\alpha$ are of the form $sr^i$ or all coordinates are of the form $r^i$. Certainly, $G$ is a closed subset of $D_3^N$ and is closed under inverses. It is easy to check $G$ is also closed under multiplication.

Let $H \triangleleft G$ be the collection of $\alpha \in G$ for which all coordinates are of the form $r^i$. We see $H$ is closed and index two in $G$, whereby $H$ is also open. We now consider the element $\beta \in G$ which is constantly equal to $s$.

**Claim 1.** Every $\gamma \in G \setminus H$ is conjugate to $\beta$ by an element of $H$.

**Proof.** Let $\gamma \in G \setminus H$ and say $\gamma(i) = sr^{j(i)}$ where $j(i) \in \{0, 1, 2\}$. Define $\eta \in H$ by

$$
\eta(i) = \begin{cases} 
1 & \text{if } j(i) = 0 \\
1 & \text{if } j(i) = 1 \\
r & \text{if } j(i) = 2
\end{cases}
$$

One checks $\eta \gamma \eta^{-1} = \beta$.

By the claim, $G \setminus H$ is the conjugacy class of $\beta$, so $\beta^G$ is open and non-null.

### 7.2 Dense Conjugacy Classes

We now consider t.d.l.c. Polish groups with a dense conjugacy class. We begin by recalling a few facts from profinite group theory.
Fact 7.10 ([35, Lemma 2.8.7]). If $U$ is pro-$p$, then $\Phi(U) = U^p[U,U]$ where $[U,U]$ is the closure of the commutator subgroup and $U^p$ is the closed subgroup generated by all $p$ powers.

Theorem 7.11 (Zel’manov [45]). Every torsion pro-$p$ group is locally finite.

Theorem 7.12 (Wilson [43]). Let $U$ be a compact Hausdorff torsion group. Then $U$ has a finite series $\{1\} = U_0 \leq U_1 \leq \ldots \leq U_n = U$ of closed characteristic subgroups in which each factor $U_i/U_{i-1}$ is either (1) pro-$p$ for some prime $p$ or (2) isomorphic to a Cartesian product of isomorphic finite simple groups.

Lemma 7.13. A torsion pro-$p$ group with an open conjugacy class is finite

Proof. Suppose $U$ is a torsion, pro-$p$ group with an open conjugacy class $h^U$. Since $h^{-1}h^U \subseteq [U,U]$, we see that $[U,U]$ is open and, via Fact 7.10, $\Phi(U)$ is open.

Letting $u_1, \ldots, u_k$ be coset representatives for $\Phi(U)$ in $U$, we have $U = cl(\langle u_1, \ldots, u_n \rangle)\Phi(U)$. Thus, $U = cl(\langle u_1, \ldots, u_n \rangle)$, and Theorem 7.11 implies $U$ is finite. □

Theorem 7.14. If $G$ is a non-trivial t.d.l.c. Polish group and $g^G$ is dense, then $g^G$ is meagre and null.

Proof. Suppose toward a contradiction that $G$ is a non-trivial t.d.l.c. Polish group and $g^G$ is dense and either non-meagre or non-null. Since $g^G$ is dense, $g \in P_1(G)$. Corollary 7.9 now implies $g^G$ is open, and by Lemma 7.8, $g$ is torsion. Say $|g| = n$ and observe that $G$ must have exponent $n$.

We now consider $U \in U(G)$. Since $U$ is torsion, there is a series of characteristic subgroups $\{1\} = U_0 \leq U_1 \leq \ldots \leq U_n = U$ given by Theorem 7.12. Let $k < n$ be greatest such that $U_k$
is not open in $U$. Since $U_{k+1}$ is open, $g^G$ meets $U_{k+1}$; without loss of generality, $g \in U_{k+1}$. Lemma 7.8 implies $g^{U_{k+1}}$ is open. Thus, $U_{k+1}/U_k$ has an open conjugacy class.

By Lemma 7.13, $U_{k+1}/U_k$ cannot be pro-$p$ since else $U_k$ is finite index and, therefore, open. So $U_{k+1}/U_k$ must be isomorphic to a Cartesian product of isomorphic finite simple groups. Say $U_{k+1}/U_k \cong \prod_{i \in I} S_i =: S$ and let $(s_i)_{i \in I} =: s \in S$ have an open conjugacy class. Lemma 7.6 implies $s$ has a finite centralizer. However, $C_S(s) = \prod_{i \in I} C_{S_i}(s_i)$, and each $C_{S_i}(s_i)$ contains at least two elements. We conclude $S$ must be a finite product and $U_k$ is open. This contradicts the choice of $k$. \hfill \square

**Corollary 7.15.** If $G$ is a non-trivial t.d.l.c. Polish group, then $G$ does not admit a comeagre or co-null conjugacy class.

We conclude by noting the hypotheses of Theorem 7.14 are not vacuous. Indeed, Akin, Glasner, and Weiss have an example of such a group [2]. We include a sketch of their example for completeness.

Let $J = \{ J_i \mid i \in \omega \}$ be a sequence of non-empty finite subsets of $\mathbb{N}$ which partition $\mathbb{N}$ and have strictly increasing cardinality. Let $J^k := \bigcup_{i=0}^k J_i$ and $K_n$ be the collection of permutations in $\text{Sym}(\mathbb{N})$ which setwise stabilize each of $J^n, J_{n+1}, J_{n+2}$, ..., i.e. $K_n$ is the collection of permutations of $\mathbb{N}$ which preserve the partition beyond the $n$-th part. It is easy to see $K_n$ is compact as a subset of $\text{Sym}(\mathbb{N})$ and $K_n \preceq_o K_{n+1}$.

Put $G := \bigcup_{i \in \omega} K_i$ and give $G$ the inductive topology: $A \subseteq G$ is open if and only if $A \cap K_i$ is open in $K_i$ for all $i$. Under this topology, $G$ is a t.d.l.c. Polish group. Further, the sets
\( G(\pi) := \{ g \in K_n \mid g \mid_{J^n} = \pi \} \) where \( n \in \mathbb{N} \) and \( \pi \in \text{Sym}(J^n) \) vary form a basis for the topology on \( G \).

**Claim 1.** \( G \) has a dense conjugacy class.

**Proof.** It is enough to show \( G \) is topologically transitive. I.e. for all basic open sets \( G(\pi), G(\xi) \) there is \( k \in G \) such that \( kG(\pi)k^{-1} \cap G(\xi) \neq \emptyset \).

Without loss of generality, we may assume \( \pi, \xi \in \text{Sym}(J^n) \) for some \( n \). Choose \( k > n \) such that \( |J_k| > |J^n| \); this is possible since the \( J_i \) are strictly increasing in cardinality. Fix an injective map \( \beta : J^n \to J_k \) and extend \( \beta \) to an element of \( G \), \( b \), by

\[
  b(i) = \begin{cases} 
    \beta(i) & i \in J^n \\
    \beta^{-1}(i) & i \in \beta(J^n) \\
    \text{id} & \text{else}
  \end{cases}
\]

It is easy to check that we may find \( a \in G(\pi) \) such that \( ab \mid_{J^n} = b\xi \mid_{J^n} \). We thus have \( b^{-1}ab \in b^{-1}G(\pi)b \cap G(\xi) \) as desired. \( \square \)

**Remark 7.16.** It is worth nothing the above example has ample dense elements. That is to say the diagonal action by conjugation of \( G \) on the \( n \)-th Cartesian power of \( G \) has a dense orbit for every \( n \geq 1 \).
CHAPTER 8

CONCLUSION

This thesis first contributes to the topological theory of t.d.l.c. Polish groups by isolating the class of elementary groups. The results herein suggest elementary groups play an important role in the theory. Indeed, the ascending and descending elementary series show that a t.d.l.c. Polish group decomposes into elementary and elementary-free groups. Moreover, elementary-free groups appear to have many nice properties as evidenced by the results of chapters 3, 4, and, in particular, 5.

Many new questions concerning elementary groups arise as a result of this work. For example, it is unknown if there are elementary groups of arbitrary high rank below $\omega_1$. Alternatively, is there a bound on the elementary rank of dimension $d$ elementary $p$-adic Lie groups? Perhaps most important to the topological theory of t.d.l.c. Polish groups,

**Question 8.1.** Can every t.d.l.c. Polish group be decomposed into topologically simple groups and elementary groups via group extension and countable increasing union?

The results of Chapter 5 demonstrate, surprisingly, a positive answer to Question 8.1 in certain interesting classes.

Chapter 6 contributes to the theory of t.d.l.c. Polish groups by demonstrating the sets $P_n(G)$ are closed in a large family of t.d.l.c. Polish groups $G$. This work suggests a positive answer to Rosendal’ question:
**Question 8.2** (Rosendal). For a t.d.l.c. Polish group $G$ and $n \geq 1$ is $P_n(G)$ closed?

To answer Question 8.2 in general, it seems new techniques must be developed. It is thus sensible to explore Question 8.2 in interesting, non-elementary examples. For example,

**Question 8.3** (Caprace). Are the sets $P_n(AAut(T_d))$ closed for $AAut(T_d)$ Neretin’s group of spheromorphisms?

Our work on the sets $P_n(G)$ also suggests generalizations:

**Question 8.4** (Caprace). For $G$ a t.d.l.c. Polish group and $n \geq 1$ is the set

$$Am_n(G) := \{ \overline{g} \in G^{\times n} \mid cl((g_1, \ldots, g_n)) \text{ is amenable} \}$$

closed?

We remark that Question 8.4 is known to have a positive answer for elementary groups by recent results of the author.

**Question 8.5** (Willis). For $G$ a t.d.l.c. Polish group and $n \geq 1$ is the set

$$T_n(G) := \{ \overline{g} \in G^{\times n} \mid \text{the elements of } cl((g_1, \ldots, g_n)) \text{ have a common tidy subgroup} \}$$

closed? What about for $G$ elementary?

The final chapter gives a complete answer to the following question:
**Question 8.6** (Kechris, Rosendal). Can a non-trivial t.d.l.c. Polish group admit a comeagre conjugacy class?

This question is motivated by the existence, indeed the abundance, of non-locally compact Polish groups with a comeagre conjugacy class. Our work in Chapter 7 suggests considering if other “strange” properties arising in non-locally compact Polish groups can or cannot arise in the locally compact setting. For example, $Aut(\mathcal{R})$ is a non-locally compact Polish group which is topologically simple, topologically finitely generated, and generically periodic. On the other hand, the following is open:

**Question 8.7** (Willis). Does there exist a non-discrete compactly generated t.d.l.c. Polish group which is topologically simple and periodic? Does there exist a non-elementary t.d.l.c Polish group which is topologically simple and periodic?

In a related direction, we have

**Question 8.8** (Caprace). Is there a compactly generated t.d.l.c. Polish group with a dense conjugacy class?

The topological theory of t.d.l.c. Polish groups is a fruitful and interesting area of study. This thesis contributes to this theory. Indeed, this work gives new tools, demonstrates useful theorems, and suggests new and interesting questions.
APPENDICES
Appendix A

TECHNICAL GROUP CONSTRUCTIONS

A.1 Basic Group Theory

For a group $D$ and a countable set $N$,

$$D^{<N} := \{ f : M \to D \mid \text{for all but finitely many } m \in M \, f(m) = 1 \}$$

**Proposition A.1.** Let $F$ be a non-abelian discrete simple group and $D$ a countable discrete group with an action on a countable set $\Omega$. Then

(1) If $H \trianglelefteq F^\Omega$ is a closed normal shift invariant subgroup of $F^\Omega$, then there is a $D$-invariant $X \subseteq \Omega$ such that $H = F^X \times \{1\}_\Omega \setminus X$.

(2) If $H \trianglelefteq F^{<\Omega}$ is a normal shift invariant subgroup of $F^{<\Omega}$, then there a $D$-invariant $X \subseteq \Omega$ such that $H = F^{<X} \times \{1\}_\Omega \setminus X$

**Proof.** Since the arguments are identical, we only prove (1). Let $H$ be a closed normal shift invariant subgroup of $F^\Omega$. Since $F$ is simple and $H$ normal, we have that the projection $\pi_\omega(H) \in \{1, F\}$ for every $\omega \in \Omega$. Let $X \subseteq \Omega$ be the collection of elements $\omega \in \Omega$ such that $\pi_\omega(H) = F$. We claim $X$ satisfies the proposition.

In view of shift invariance, $\pi_\delta(H) = \pi_\omega(H)$ for every $\delta \in orb(\omega, D)$, so $X$ is $D$-invariant as required. Fixing $\delta \in X$, there is $\beta \in H$ such that $\beta(\delta) = s \neq 1$. Certainly, we may find $h \in F$
Appendix A (Continued)

such that $h$ does not centralize $s$. Let $\alpha_{\delta,h} \in F_{\Omega}$ have image 1 except on $\delta$ where it takes value $h$. We now have

$$\alpha_{\delta,h} \beta \alpha_{\delta,h}^{-1} \beta^{-1} = \alpha_{\delta,hsh^{-1}s^{-1}}$$

Certainly, $\alpha_{\delta,hsh^{-1}s^{-1}} \in H$ since $H$ is normal, and further, $hsh^{-1}s \neq 1$. By conjugating $\alpha_{\delta,hsh^{-1}s^{-1}}$ with $\alpha_{\delta,k}$ while varying $k$, we have $\alpha_{\delta,f} \in H$ for all $f \in (hsh^{-1}s^{-1})^F$. Hence, $(\alpha_{\delta,f} \mid f \in (hsh^{-1}s^{-1})^F) \leq H$, and since $F$ is simple,

$$\pi_\delta((\alpha_{\delta,f} \mid f \in (hsh^{-1}s^{-1})^F)) = F$$

We conclude $\alpha_{\delta,f} \in H$ for all $f \in F$.

It now follows that $F^{<X} \subseteq H$, and since $H$ is closed, we have proved the lemma.

When considering the action of an element $d$ of a group $D$ on a point $\omega \in \Omega$, we write $d.D\omega$.

The subscript “$D$” is omitted when clear from context.

**Proposition A.2.** Suppose $H$ and $G$ are groups and $K \rtimes L \leq H \rtimes G$ such that $K \leq H$, $L \leq G$, and $L \lhd H/K$ trivially. Then the action of $G$ on $H$ lifts to an action of $G/L$ on $H/K$ by group automorphisms and $(H \rtimes G)/(K \rtimes L) \simeq H/K \rtimes G/L$.

**Proof.** First let us see the action of $G$ on $H$ lifts to an action of $G/L$ on $H/K$; let $G \lhd H$ be denoted by $\cdot_G$. Define $gL.hK := (g.G h)K$. Let us see this is well defined. Fix $l \in L$ and $k \in K$. So

Appendix A (Continued)

Since \( L \lhd H/K \) trivially, we have that \((gl)_* hK = g_* hK\) and the action is well defined.

Define \( \psi : H \rtimes G \to H/K \rtimes G/L \) in the obvious manner. It is now easy to check this map is a surjective homomorphism with kernel \( K \rtimes L \).

\[ \text{Proposition A.3.} \quad \text{Suppose we have groups } K, H, \text{ and } G \text{ and homomorphisms } \psi : G \to \text{Aut}(H) \]
and \( \phi : H \rtimes \psi G \to \text{Aut}(K) \). Then

1. The map \( \chi_g(k, h) := (\phi_g(k), \psi_g(h)) \) is a group automorphism of \( K \rtimes \phi \downarrow_H H \).
2. The map \( g \mapsto \chi_g \) is a homomorphism \( \chi : G \to \text{Aut}(K \rtimes \phi \downarrow_H H) \).
3. The identity map induces a group isomorphism \( K \rtimes \phi (H \rtimes \psi G) \cong (K \rtimes \phi \downarrow_H H) \rtimes \chi G \).

\[ \text{Proof.} \text{ The proof is an easy exercise.} \]

\[ \text{Definition A.4.} \quad \text{Suppose that } F \text{ is a finite group and } D \text{ is a group with an action on a countable set } \Omega. \text{ The wreath product of } F \text{ by } D \text{ over } \Omega \text{ is defined to be } F^\Omega \rtimes D \text{ where } D \lhd F^\Omega \text{ by shift; i.e. } d.\alpha(l) := \alpha(d^{-1}.l). \text{ The wreath product is denoted } F \wr \Omega D. \text{ We put } F \wr \Omega D := F^\Omega \rtimes D \text{ when } D \text{ is a countable discrete group and } D \lhd D \text{ by left multiplication.} \]

Observe that \( F \wr \Omega D \) is a t.d.l.c. Polish group; more precisely, \( F \wr \Omega D \) is a elementary group.

\[ \text{Lemma A.5.} \quad \text{Suppose } G \text{ is a t.d.l.c. Polish group with a continuous action on a countable set } \Omega \text{ such that } G \text{ acts on } \text{orb}(\omega, G) \text{ faithfully for each } \omega \in \Omega. \text{ If } F \text{ is a non-abelian finite simple group, then any closed normal subgroup of } F \wr \Omega G \text{ is either contained in } F^\Omega \text{ or contains } F^\Omega. \]

\[ \text{Proof.} \text{ Suppose } N \leq F \wr \Omega G \text{ and } N \text{ is not contained in } F^\Omega. \text{ Consider } H := N \cap F^\Omega. \text{ If } H = F^\Omega, \text{ then we are done. Else, by Proposition A.1, there is } \omega \in \Omega \text{ such that } F_{\text{orb}(\omega, G)} \cap N = \{1\}. \text{ Plainly, } F_{\text{orb}(\omega, G)} \leq F \wr \Omega G, \text{ so } F_{\text{orb}(\omega, G)} \text{ and } N \text{ centralize each other.} \]
Appendix A (Continued)

Take \((\alpha, d) \in N\) such that \(d \neq 1\). For \((\beta, 1) \in F_{\text{orb}(\omega, G)}\), we have

\[(\beta, 1)(\alpha, d)(\beta, 1)^{-1} = (\beta \cdot \alpha \cdot (d, \beta^{-1}), d)\]

Since \(G \curvearrowright \text{orb}(\omega, G)\) faithfully, we may find \(\beta \in F_{\text{orb}(\omega, G)}\) and \(\delta \in \text{orb}(\omega, G)\) such that \(\beta = 1\) except on \(\delta\) and \(d^{-1}\delta \neq \delta\). For such a \(\beta\), we have \((\beta \cdot \alpha \cdot (d, \beta^{-1}))(\delta) \neq \alpha(\delta)\). This contradicts the assumption \(F_{\text{orb}(\omega, G)}\) centralizes \(N\).

A.2 Extending Wreath Products

We now develop a general technique for extending wreath products; the delicate point is we wish to stay in the category of locally compact groups. Let \(F\) be a finite group, \(D\) be a discrete group, \(G\) some t.d.l.c. Polish group, and \(\Omega\) a countable set. Suppose further,

(i) \(D \curvearrowright \Omega\) by permutations.

(ii) \(G \curvearrowright \Omega\) continuously by permutations and \(G \curvearrowright D\) continuously via group automorphisms.

(iii) For all \(g \in G\), \(d \in D\) and \(\omega \in \Omega\), we have \(g \cdot_G (d \cdot_D \omega) = (g \cdot_G d) \cdot_D (g \cdot_G \omega)\). We say the \(G\) actions distribute over the \(D\) action on \(\Omega\).

Since \(D \curvearrowright \Omega\), we may form \(F \wr_\Omega D\); furthermore,

**Proposition A.6.** \(G\) acts continuously by continuous group automorphisms on \(F \wr_\Omega D\) via \(g.(\alpha, d) := (g \cdot_G \alpha, g \cdot_G d)\) where \(g \cdot_G d\) is the action of \(G\) on \(D\) and \(g \cdot_G \alpha\) is the shift action induced by the action of \(G\) on \(\Omega\).
Appendix A (Continued)

Proof. It is easy to check the definition above indeed gives a continuous action by homeomorphisms. It thus remains to show this action is by group automorphisms.

We only check inverses as multiplication is similar: Fix \( g \in G \) and \((\alpha, d) \in F \wr_\Omega D \). Now,

\[
g.((\alpha, d)^{-1}) = g.(d^{-1}.D\alpha^{-1}, d^{-1}) = (g_G(d^{-1}.D\alpha^{-1}), g_Gd^{-1})
\]

Since the \( G \) actions distribute over the \( D \) action, we have

\[
(g_G(d^{-1}.D\alpha^{-1}), g_Gd^{-1}) = ((g_Gd^{-1}).D(g_G\alpha^{-1}), g_Gd^{-1}) = (g_G\alpha, g_Gd)^{-1}
\]

The action of \( G \) on \( F \rtimes_\Omega D \) thus respects inverses.

Remark A.7. Upon building a wreath product \( F \rtimes_\Omega D \) along with a t.d.l.c. Polish group \( G \lhd \Omega \) and \( G \lhd D \) as above, we may use Proposition A.6 to produce \((F \rtimes_\Omega D) \rtimes G\), a new t.d.l.c. Polish group.

We note a special case of particular interest in this work; the proof is obvious and, therefore, omitted.

Proposition A.8. Suppose \( F \) is a non-abelian finite simple group, \( D \) is a discrete group, and \( \Omega' \) is a countable set. Suppose \( G \) is a t.d.l.c. Polish group with a continuous action on \( \Omega' \) and \( \Omega \subseteq \Omega' \) is a \( G \)-invariant subset. Form \( H := F \rtimes_{D^{<\Omega}} D^{<\Omega} \) where \( D^{<\Omega} \lhd D^{<\Omega'} \) by left multiplication. Then \( G \) has a continuous action by group automorphisms on \( H \) via
Appendix A (Continued)

\[ g.(\alpha, f) = (g.\alpha, g.f) \] where \( g.f \) is the usual shift and \( g.\alpha \) is the shift given by the shift action of \( G \) on \( D^{<\Omega'} \).

\( H \) as above has a canonical action on two countable sets. Put

\[ C := \{(f, s) \mid f \in D^{<\Omega'} \text{ and } s \in F\} \]

Define \( H \curvearrowright C \) by \((\alpha, g).(f, s) = (gf, \alpha(gf)s)\). This action is continuous and faithful. We also consider the restriction of the action to \( B := \{(f, s) \mid f \in D^{<\Omega} \text{ s} \in F\} \). It is easy to see the action on \( B \) is transitive; however, this action has a non-trivial kernel.

**Remark A.9.** The set \( C \) may be seen as the collection of cosets of the open subgroups \( \Sigma_{f \mapsto 1} = \{\alpha \in F^{D^{<\Omega'}} \mid \alpha(f) = 1\} \) in \( F^{D^{<\Omega'}} \). The action is, essentially, the action of \( H \) on the collection cosets by left multiplication; we call this action the **coset action**.

It is important to understand how the coset action lifts when we apply Proposition A.8 to extend \( H \). Indeed, forming \( H \rtimes G \), the coset action lifts to

\[
\left( (\alpha, g), k \right).(f, s) = \left( g \cdot (k.f), \alpha(g \cdot (k.f))s \right)
\]

**Lemma A.10.** The definition above gives a continuous group action of \( H \rtimes G \) on \( C \).
Appendix A (Continued)

Proof. Certainly, \(((1,1),1).(f,s) = (f,s)\). It remains to show compatibility:

\[
\left((\beta, h), k\right) \cdot \left((\alpha, h'), k'\right).(f,s) = \left((\beta, h), k\right) \cdot (h' \cdot (k'.f), \alpha(h' \cdot (k'.f))s)
\]

which equals

\[
\left(h \cdot k.\cdot (kk').f, \beta(h \cdot k.h' \cdot (kk').f)\alpha(h' \cdot k'.f)s\right) \quad (A.1)
\]

On the other hand,

\[
\left((\beta, h), k\right) \left((\alpha, h'), k'\right) = \left((\beta, h) (k.\alpha, k.h'), kk'\right) = \left((\beta \cdot h.(k.\alpha), h \cdot k.h'), kk'\right)
\]

Now consider

\[
\left((\beta \cdot h.(k.\alpha), h \cdot k.h'), kk'\right).(f,s) \quad (A.2)
\]

The first coordinates of (A.1) and (A.2) agree as required. Let us evaluate the second coordinate of (A.2):

\[
\beta \cdot h.(k.\alpha)(h \cdot k.h' \cdot (kk').f) = \beta(h \cdot k.h' \cdot (kk').f) \cdot k.\alpha(k.h' \cdot (kk').f) = \beta(h \cdot k.h' \cdot (kk').f) \cdot \alpha(h' \cdot k'.f)
\]

The second coordinates of (A.1) and (A.2) thus agree, and our definition satisfies the compatibility requirement to be a group action. \(\square\)

For \(X \subseteq \Omega\) and \(F\) a finite group, we define \(K_F(X) := \{\alpha \in F^\Omega \mid \alpha \mid_X = 1\}\). We suppress the subscript \(F\) when clear from context.
Lemma A.11. For the construction above, we have

(1) \( \ker(H \ltimes G \blacktriangleright C) \leq \ker(G \blacktriangleright \Omega') \),

(2) \( \ker(H \ltimes G \blacktriangleright B) = K_{D^{<\Omega}}(D^{<\Omega}) \rtimes \ker(G \blacktriangleright \Omega) \), and

(3) If \( g \in H \ltimes G \) centralizes \( H \), then \( g \) is an element of \( \ker(G \blacktriangleright \Omega') \).

Proof. For (1), suppose \( ((\alpha, g), k) \in \ker(H \ltimes G \blacktriangleright C) \). For any \( (f, s) \in C \), we have

\[
(f, s) = ((\alpha, g), k)(f, s) = (g \cdot (k.f), \alpha(g \cdot (k.f))s)
\]

So \( g \cdot k.f = f \) and \( k.f = g^{-1} \cdot f \). Further, for any \( h \in D^{<\Omega'} \) we have \( h \cdot f \in D^{<\Omega'} \), whereby

\[
k.(h \cdot f) = g^{-1} \cdot h \cdot f \iff k.h \cdot g^{-1} \cdot f = g^{-1} \cdot h \cdot f \iff k.h = g^{-1} \cdot h \cdot g
\]

The last equivalent must hold for any \( h \in D^{<\Omega'} \). It follows \( k \blacktriangleright \Omega' \) trivially. It is now easy to see \( g = 1 \) and \( \alpha = 1 \). We have thus demonstrated (1).

For (2), suppose for any \( (f, s) \in B \) we have

\[
(f, s) = ((\alpha, g), k)(f, s) = (g \cdot (k.f), \alpha(g \cdot (k.f))s)
\]

So \( g \cdot (k.f) = f \). Similar to (1), \( k \) must act trivially on \( \Omega \) and \( g = 1 \). It must also be the case that \( \alpha(h) = 1 \) for any \( h \in D^{<\Omega} \). Hence,

\[
\ker(H \ltimes G \blacktriangleright B) = K(D^{<\Omega}) \rtimes \ker(G \blacktriangleright \Omega)
\]
Appendix A (Continued)

For (3) suppose \(((\alpha, f), k) \in H \rtimes G\) centralizes \(H\). For any \(\beta \in F^{<\Omega'}\), we have

\[
(\alpha, f), k = ((\beta, 1), 1) (\alpha, f), k ((\beta^{-1}, 1), 1) = ((\beta \cdot \alpha \cdot f \cdot (k \cdot \beta), f), k)
\]

Thus, \(\beta(h) \cdot \alpha(h) \cdot \beta(k^{-1}(f^{-1} \cdot h)) = \alpha(h)\), and it follows \(\beta(f \cdot h) \cdot \alpha(f \cdot h) \cdot \beta(k^{-1} \cdot h) = \alpha(f \cdot h)\).

Letting \(h = id\) and \(\beta(x) \neq 1\) exactly for \(x \neq id\), we see \(f = id\).

If \(k\) acts non-trivially on \(\Omega'\), then we may find \(h \in D^{<\Omega'}\) such that \(k^{-1}h \neq h\). Here we consider \(\beta\) such that \(\beta(h) \neq 1\) and 1 else. Such choices imply \(\beta(h) \cdot \alpha(h) \cdot \beta(k^{-1} \cdot h) \neq \alpha(h)\).

Hence, \(k\) acts trivially on \(\Omega'\).

Since \(F\) is a non-abelian finite simple group, it is easy to see \(\alpha = id\). We Conclude \(((\alpha, f), k) = ((1, 1), k)\) for \(k \in \ker(G \ltimes \Omega')\) demonstrating (3).

Combining the work above, we have the following theorem:

**Theorem A.12.** Let \(F\) be a non-abelian finite simple group, \(D\) a discrete group, \(G\) a t.d.l.c. Polish group and \(\Omega'\) a countable set. Suppose \(G\) acts on \(\Omega'\) continuously and \(\Omega \subseteq \Omega'\) is a \(G\)-invariant subset. Then

1. Form \(H := F \wr D^{<\Omega} D^{<\Omega}\) where \(D^{\Omega} \ltimes D^{\Omega'}\) via left multiplication. Then \(G \ltimes H\) by \(g.(\alpha, f) = (g.\alpha, g.f)\) where \(g.f\) is the usual shift and \(g.\alpha\) is the shift induced by the action of \(G\) on \(D^{<\Omega'}\) by shift. Further, this is a continuous action by topological group automorphisms.
Appendix A (Continued)

(2) Put $C := \{(f, s) \mid f \in D^{<\Omega} \text{ and } s \in F\}$ and $B := \{(f, s) \mid f \in D^{<\Omega} \text{ and } s \in F\}$. Then $H$ acts on $C$ and $B$ continuously and on $B$ transitively via the coset action: $(\alpha, g). (f, s) = (gf, \alpha(gf)s)$.

(3) The coset action lifts to a continuous action of $H \rtimes G$ on $C$ via

$((\alpha, g), k). (f, s) = (g \cdot (k.f), \alpha(g \cdot (k.f))s)$

Further, $B$ is an invariant set under this action on which $G$ acts transitivity, and

(a) $\ker(H \rtimes G \lhd C) \leq \ker(G \lhd \Omega)$,

(b) $\ker(H \rtimes G \lhd B) = K(D^{<\Omega}) \rtimes \ker(G \lhd \Omega)$, and

(c) If $g \in H \rtimes G$ centralizes $H$, then $g$ is an element of $\ker(G \lhd \Omega)$.

A.3 Examples of Elementary Groups

Let $A_5$ be the alternating group on five letters; recall $A_5$ is a non-abelian finite simple group. Let $S$ denote the infinite finitely generated simple group built by Higman [21]. Define

(i) $L_1 := A_5 \wr S$

(ii) $C_1 := \{(f, s) \mid f \in S \text{ and } s \in A_5\}$ where $L_1 \lhd C_1$ via the coset action.

Suppose we have built the pairs $(L_i, C_i)$ up to stage $n$. Put

(i) $L_{n+1} := A_5 \wr S^{<C_n}$

(ii) $C_{n+1} := \{(f, s) \mid f \in S^{<C_n} \text{ and } s \in A_5\}$ where $L_{n+1} \lhd C_{n+1}$ via the coset action.
Observation A.13. The coset action of $L_n$ on $C_n$ is continuous, transitive, and faithful.

Using Theorem A.12, we inductively define a sequence of groups: For the base case, set $M_1 := L_1$. By construction, $M_1$ acts continuously, transitively, and faithfully on $C_1$. Suppose we have built $M_n$ such that $M_n$ acts continuously, transitively, and faithfully on $C_n$. Applying Theorem A.12, $M_n$ acts on $L_{n+1}$ by group automorphisms. We may thus form $M_{n+1} := L_{n+1} \rtimes M_n$. By Theorem A.12, we have that $M_{n+1}$ acts continuously on $C_{n+1}$; it is obvious the action is transitive. Further, $\ker(M_{n+1} \rtimes C_{n+1}) = \ker(M_n \rtimes C_n) = \{1\}$ showing the action is also faithful.

Observation A.14. $M_n$ is compactly generated for all $n$.

Lemma A.15. Any closed normal subgroup of $M_n$ is of the form $L_n \rtimes \cdots \rtimes L_{i+1} \rtimes J$ where $i \geq 0$ and $J \in \{A_5^{S^{<C_i}}, 1\}$.

Proof. We induct on $n$. For the base case, say $H \unlhd M_1 = L_1$. By Proposition A.1 and since $S$ acts transitively on itself by left multiplication, $H$ is either $\{1\}$, $A_5^S$, or properly extends $A_5^S$. In the first two cases, we are done. For the last, $M_1/H$ is a non-trivial normal subgroup of $S$ and, therefore, equal to $S$. So $H = L_1$. This finishes the base case.

Suppose the result holds for $M_n$ and suppose $H \unlhd M_{n+1}$. Consider $H \cap L_{n+1}$. By Proposition A.1 and since $S^{<C_n}$ acts transitively on itself, $H \cap L_{n+1}$ is either $\{1\}$, $A_5^{S^{<C_n}}$, or properly extends $A_5^{S^{<C_n}}$.

For the last case, consider the quotient $M_{n+1}/(A_5^{S^{<C_n}}) = S^{<C_n} \rtimes M_n$. Letting $\pi$ be the usual projection, we have that $\pi(H) \cap S^{<C_n} \leq S^{<C_n}$ is non-trivial. Thus, $\pi(H) \cap S^{<C_n}$ must
contain $S^<C_n$ by Proposition A.1, so $H$ extends $L_{n+1}$. Letting $\xi : M_{n+1} \to M_n$ be the obvious projection, we have $\xi(H) \leq L_n$. The induction hypothesis implies $\xi(H) = L_n \times \ldots \times L_{i+1} \times J$.

We conclude $H = L_{n+1} \times L_n \times \ldots \times L_{i+1} \times J$ and the induction step holds in this case.

Suppose $H \cap L_{n+1} = \{1\}$. So $H$ centralizes $L_{n+1}$. By Theorem A.12, we have that $H \leq \ker(M_n \rtimes C_n) = \{1\}$ and, plainly, the induction hypothesis holds.

Lastly, suppose $H \cap L_{n+1} = A_{5}^{S^<C_n}$ and consider the quotient $M_{n+1}/(H \cap L_{n+1}) = S^<C_n \rtimes M_n$. Now $\pi(H)$, with $\pi$ the usual projection, centralizes $S^<C_n$. Take $(g,k) \in \pi(H)$. For $f \in S^<C_n$, we have $(g,k)(f,1)(k^{-1}g^{-1},k^{-1}) = (g(k.f)g^{-1},1) = (f,1)$. Since we are free to choose $f \in S^<C_n$, $k$ must act trivially on $C_n$. The simplicity of $S$ implies $g$ must be the identity. Hence, $\pi(H) \leq \ker(M_n \rtimes C_n) = \{1\}$, and we conclude $H = A_{5}^{S^<C_n}$.

The induction is now finished, and we conclude the proposition.

The $M_n$ are plainly elementary. Furthermore,

\textbf{Lemma A.16.} $rk(M_n) \leq rk(M_{n+1})$ for all $n \geq 0$.

\textit{Proof.} Fix $n \geq 0$. By Proposition 2.7, $rk\left(M_{n+1}/(A_{5}^{S^<B_n})\right) \leq rk(M_{n+1})$. Since $M_n \leq_o M_{n+1}/(A_{5}^{S^<B_n})$, Proposition 2.6 implies $rk(M_n) \leq rk\left(M_{n+1}/(A_{5}^{S^<B_n})\right)$ and the lemma follows.

We now wish to argue $rk(M_{4n}) \geq n$ for every $n$; however, to do so, we must consider a slightly more general construction. The problem with the construction above is that it is not clear how to apply an induction hypothesis to a normal subgroup. E.g. the normal subgroups do not need to be compactly generated. We thus consider a more general construction.
Appendix A (Continued)

Suppose we have a countable set $N$ and $A_5, S$ as above. Identify $S$ as a subgroup of $S^{<N}$.

We define a sequence of triples as follows:

(i) $L_1(N) := A_5 \wr_{S^{<N}} S$

(ii) $C_1(N) := \{(f, s) \mid f \in S^{<N} \text{ and } s \in A_5\}$ where $L_1(N) \curvearrowright C_1(N)$ by the coset action

(iii) $B_1 := \{(f, s) \mid f \in S \text{ and } s \in A_5\}$. Observe $B_1 \subseteq C_1(N)$ and is invariant under the coset action.

Suppose we have built the triples up to stage $n$. Put

(i) $L_{n+1}(N) := A_5 \wr_{S^{<C_n(N)}} S^{<B_n(N)}$ where we identify $S^{<B_n(N)}$ with the obvious subgroup of $S^{<C_n(N)}$. We may do this since $B_n(N) \subseteq C_n(N)$ by induction.

(ii) $C_{n+1}(N) := \{(f, s) \mid f \in S^{<C_n(N)} \text{ and } s \in A_5\}$ and $L_{n+1}(N) \curvearrowright C_{n+1}(N)$ by the coset action.

(iii) $B_{n+1} := \{(f, s) \mid f \in S^{<B_n(N)} \text{ and } s \in A_5\}$. Observe $B_{n+1} \subseteq C_{n+1}(N)$ and is invariant under the coset action.

We now inductively define a sequence of groups just as in our first construction. Put $M_1(N) := L_1(N)$. By construction, $M_1(N)$ acts continuously on $C_1(N)$, setwise stabilizes $B_1 \subseteq C_1(N)$, and the action is transitive on $B_1$. Additionally, $k(1) := \ker(M_1(N) \curvearrowleft B_1) = K(S)$ via Theorem A.12.

Suppose we have built $M_n(N)$ such that $M_n(N)$ acts continuously on $C_n(N)$, setwise stabilizes $B_n \subseteq C_n(N)$, and the action is transitive on $B_n$. Applying Theorem A.12, $M_n(N)$ acts on
Appendix A (Continued)

$L_{n+1}(N)$ by group automorphisms. We may now form $M_{n+1}(N) := L_{n+1}(N) \rtimes M_n(N)$. By Theorem A.12, $M_{n+1}(N)$ acts continuously on $C_{n+1}(N)$, setwise stabilizes $B_{n+1} \subseteq C_{n+1}(N)$, and the action is transitive on $B_{n+1}$. Further, $k(n+1) := \ker(M_{n+1}(N) \actson B_{n+1}) = K(S_{<B_n}) \ltimes k(n)$.

Observation A.17.

(1) $B_n = C_n$ with $C_n$ as defined in the construction of $L_n$.

(2) $M_n(N)$ is compactly generated for all $n$.

(3) $k(n)$ is compact for all $n$.

(4) If a group $J \acts N$ and fixes a point, then we may run the above construction to produce $M_n(N) \rtimes J$ such that the coset action of $M_n(N)$ on $B_n$ extends to $M_n(N) \rtimes J \actson B_n$.

Proposition A.18. For each $n \geq 1$ there is a topological group isomorphism

$$\psi_n : M_n(N)/k(n) \to M_n$$

such that for all $x \in B_n = C_n$ and $g \in M_n(N)$, $g.x = \psi_n(gk(n)).x$

Proof. We argue by induction on $n$. For $n = 1$, we have $\left( A_5^{S_{<N}}/K(S) \right) \rtimes S = M_1(N)/k(1)$. Let $\xi : A_5^{S_{<N}} \to A^S$ be the projection homomorphism onto $A_5^S$ and define $\psi_1 : M_1(N)/k(1) \to M_1$ by $(\alpha, g)k(1) \mapsto (\xi(\alpha), g)$. It is easy to see that this map is an isomorphism of topological groups since $\xi$ is $S$-equivariant.

We now consider the action of $M_1(N)$ on $B_1$. Fix $(f, s) \in B_1$. So

$$(\alpha, g)(f, s) = (g \cdot f, \alpha(g \cdot f) \cdot s)$$
Appendix A (Continued)

Since \( g \cdot f \in S \), we see \( \alpha(g \cdot f) \) only depends on the value of \( \alpha \) restricted to \( S \). I.e. \( \alpha(g \cdot f) = \xi(\alpha)(g \cdot f) \). Hence,

\[
(g \cdot f, \alpha(g \cdot f) \cdot s) = (g \cdot f, \xi(\alpha)(g \cdot f) \cdot s) = \psi_1((\alpha, g)k(1)).(f, s)
\]

Suppose the proposition holds up to \( n \). Recall \( M_{n+1}(N) = L_{n+1}(N) \rtimes M_n(N) \) and \( k(n+1) = K(S^{<B_n}) \rtimes k(n) \). Since \( k(n) \) acts trivially on \( L_{n+1}(N)/K(S^{<B_n}) \), we may take

\[
M_{n+1}(N)/k(n+1) = \left( L_{n+1}(N)/K(S^{<B_n}) \right) \rtimes M_n(N)/k(n)
\]

Let \( \xi : A_0^{S^{<C\alpha}(N)} \rightarrow A_0^{S^{<B_n}} \) be the projection onto \( A_0^{S^{<B_n}} \). Observe that \( \xi \) is equivariant under any shift action that stabilizes \( S^{<B_n} \).

We now define \( \psi_{n+1} : M_{n+1}(N)/k(n+1) \rightarrow M_{n+1} \) by

\[
\psi_{n+1}\left(\left( (\alpha, g)K(S^{<B_n}), kk(n) \right) \right) := \left( (\xi(\alpha), g), \psi_n(kk(n)) \right)
\]

It is easy to see that \( \psi_{n+1} \) is a homeomorphism. We now check \( \psi_{n+1} \) is a homomorphism.

Take \( x := ((\alpha, g), s)k(n+1) \) and \( y := ((\beta, h), l)k(n+1) \) in \( M_{n+1}(N)/k(n+1) \). So

\[
x \cdot y = \left( (\alpha \cdot g \cdot (s, \beta), g \cdot s \cdot h)K(S^{<B_n}), slk(n) \right)
\]
Thus
\[
\psi_{n+1}(x \cdot y) = \left( (\xi(\alpha) \cdot g, \xi(s, \beta), g \cdot s, h), \psi_n(sl_k(n)) \right)
\]

By induction, the action of \( s \) on \( B_n \) is the same as the action of \( \psi_n(sk(n)) \) on \( B_n \). We thus have
\[
\psi_{n+1}(x \cdot y) = \left( (\xi(\alpha) \cdot g, \xi(\psi_n(sk(n)), \beta), g \cdot \psi_n(sk(n)).h, \psi_n(sl_k(n))) \right)
\]

\psi_{n+1} is thus an isomorphism of topological groups.

We lastly must check that for all \( g \in M_{n+1}(N) \) and \((f, s) \in B_n, g.(f, s) = \psi(gk(n+1)).(f, s). \)

This is follows similarly to showing \( \psi_{n+1} \) is a homomorphism.

We now utilize both of the above constructions to show the desired theorem:

**Theorem A.19.** For all \( n \geq 0 \), \( rk(M_{4^n}) \geq n \).

**Proof.** We induct on \( n \). For the base case, \( n = 0 \), we have \( M_1 = A_5 \rtimes S_5 \). Certainly, \( M_1 \) is neither compact nor discrete and, therefore, has rank at least one.

Suppose the theorem holds up to \( n \). Consider \( M_{4^n+1} \). By Observation A.14, we have that \( M_{4^n+1} \) is compactly generated and, thus, the rank is witnessed by a group extension. Let \( H \leq M_{4^n+1} \) witness the rank. Via Lemma A.15, \( H = L_{4^n+1} \rtimes \ldots \rtimes L_{i+1} \rtimes J \) with \( J \in \{ A_5^{S_{C_i}} 1 \} \).

This gives two cases: \( i + 1 \leq 4^n + 1 \) and \( i + 1 > 4^n + 1 \).
Appendix A (Continued)

Suppose \( i + 1 \leq 4^n + 1 \), fix \( x \in C_i \), and identify \( S \) with the subgroup of \( S^{<C_i} \) of functions which have support \( x \). Since \( J \) acts on \( C_i \) continuously, the stabilizer of \( x \) in \( J \) is open; call this stabilizer \( J' \). Applying Theorem A.12, we have

\[
L_1(C_i) \rtimes J' = A_5 \wr S^{<C_i} S \rtimes J' \leq_o L_{i+1} \rtimes J
\]

A simple induction argument gives \( M_{4^n+1-i}(C_i) \rtimes J' \leq_o H \). Another induction shows \( k(l) \leq M_l(C_i) \rtimes J' \) for each \( l \leq 4^{n+1} - i \).

**Claim 1.** \( J' \) acts trivially on \( M_l(C_i)/k(l) \) for all \( l \leq 4^{n+1} - i \).

**Proof of claim.** By an easy induction argument, we see that the coset action of \( M_l(C_i) \) on \( B_l \) and \( C_l(C_i) \) lifts to an action of \( M_l(C_i) \rtimes J' \) on \( B_l \) and \( C_l(C_i) \) for each \( l \leq 4^{n+1} - i \). Further, \( J' \) is in the kernel of the action on \( B_l \).

We now induct on \( l \) up to \( 4^{n+1} - i \) for the claim. For the base case, \( J' \rtimes M_1(C_i)/k(1) \) via

\[
j(\alpha K(S), g) = (j.\alpha K(S), j.g) = (j.\alpha K(S), g)
\]

Since \( j \) fixes \( S \), we have that \( \alpha^{-1} j.\alpha \in K(S) \). Thus, \( j.\alpha K(S) = \alpha K(S) \), and \( J' \) acts trivially on \( M_1(C_i)/k(1) \).

Suppose \( J' \) acts trivially on \( M_l(C_i)/k(l) \) and consider \( M_{l+1}(C_i)/k(l+1) \). Since

\[
M_{l+1}(C_i)/k(l+1) \rtimes J' = L_{l+1}(C_i)/K(S^{<B_l}) \rtimes M_l(C_i)/k(l) \rtimes J'
\]
we may consider $J'$ as acting diagonally on

\[ L_{l+1}(C_i)/K(S^{<B_l}) \rtimes M_l(C_i)/k(l) \]

by the associativity of semi-direct products. The action on the first coordinate is the shift given by the extension of the coset action of $M_l(C_i)$ on $B_l$ and $C_l(C_i)$ to an action of $M_l(C_i) \rtimes J'$ on $B_l$ and $C_l(C_i)$. The action on the second coordinate is just the action of $J'$ on $M_l(C_i)$. But, as in the base case, $J'$ acts trivially on $L_{l+1}(C_i)/K(S^{<B_l})$ since $J'$ acts trivially on $S^{<B_l}$. Since $J'$ acts trivially on $M_l(C_i)/k(l)$ by induction, we conclude $J'$ acts trivially on $M_{l+1}(C_i)/k(l + 1)$.

This finishes the induction and we conclude the claim. \qed

Via the claim and Proposition A.18, we have

\[ M_{4^{n+1}-i}(C_i)/k(4^{n+1} - i) \rtimes J' \cong M_{4^{n+1}-i} \times J' \]

Since $4^{n+1} - i \geq 4^n$, the induction hypothesis and Lemma A.16 imply $rk(M_{4^{n+1}-i}) \geq n$. Proposition 2.7 now gives

\[ n \leq rk(M_{4^{n+1}-i}) = rk(M_{4^{n+1}-i} \times J') \leq rk(M_{4^{n+1}-i} \times J') \]

Via an application of Proposition 2.7 and Proposition 2.6, we conclude

\[ n \leq rk(M_{4^{n+1}-i}(C_i) \rtimes J') \leq rk(H) \]
Appendix A (Continued)

We thus have that \( rk(H) \) is at least \( n \) and, therefore, \( rk(M_{4n+1}) \geq n + 1 \).

For the second case, suppose \( i + 1 > 4^n + 1 \). If \( J = \{1\} \), then consider \( M_i = L_1 \times \cdots \times L_1 \).

By the choice of \( i \), we have that \( M_4 \leq M_i \). Lemma A.16 along with the induction hypothesis imply \( rk(M_i) \geq n \), so \( rk(M_{4n+1}/H) \geq n \) and \( rk(M_{4n+1}) \geq n + 1 \).

If \( J = A_5^{<C_{i-1}} \), then consider \( M_{i-1} = L_{i-1} \times \cdots \times L_1 \). It is easy to see \( M_{i-1} \leq_o M_{4n+1}/H \), whereby

\[
rk(M_{i-1}) \leq rk\left(\frac{M_{4n+1}}{H}\right)
\]

By the choice of \( i \), we have that \( M_4 \leq M_{i-1} \). Lemma A.16 along with the induction hypothesis imply \( rk(M_{i-1}) \geq n \). Hence, \( rk(M_{4n+1}/H) \geq n \) and \( rk(M_{4n+1}) \geq n + 1 \) finishing the induction.

\( \Box \)
Appendix B

ALGEBRAIC GROUPS

All fields are taken to be characteristic zero.

B.1 Generalities on Algebraic Groups

Definition B.1. An algebraic group is an algebraic variety $G$ with group operations given by $\cdot$ and $\ast^{-1}$ such that $\cdot : G \times G \to G$ and $\ast^{-1} : G \to G$ are morphisms of algebraic varieties.

An algebraic group $G$ is connected if it is connected in the Zariski topology. A subgroup $H \leq G$ is algebraic if $H$ is an algebraic subvariety of $G$. When an algebraic group is a variety defined over a field $k$, we say $G$ is a $k$-algebraic group. More generally, the prefix “$k$-” is applied to indicate an algebraic object is defined over the field $k$.

Fact B.2 ([28, 0.13]). Let $G$ be a $k$-algebraic group and $G^0$ the connected component containing the identity. Then

(1) $G^0$ is a connected $k$-algebraic subgroup of $G$.

(2) $G/G^0$ is finite.

The radical of a $k$-algebraic group $G$ is the maximal connected algebraic solvable normal subgroup of $G$. $G$ is said to semisimple if its radical is trivial. There are a number of additional notions of simplicity in the category of algebraic groups: An algebraic group $G$ is called absolutely simple if $\{1\}$ is the only proper algebraic normal subgroup of $G$. $G$ is called almost
Appendix B (Continued)

absolutely simple if all such subgroups are finite. $G$ is called $k$-simple $\{1\}$ is the only proper $k$-algebraic normal subgroup of $G$. $G$ is almost $k$-simple if all such subgroups are finite.

A surjective group homomorphism with finite kernel is called an isogeny. If $\phi$ is an isogeny and $\ker(\phi)$ is central, we say $\phi$ is a central isogeny. We remark that every isogeny of a connected algebraic group is central.

**Definition B.3.** An algebraic group $G$ is called simply connected if $G$ is connected, semisimple, and if every central isogeny $\phi : H \to G$ for $H$ a connected algebraic group is an algebraic group isomorphism.

**Definition B.4.** An algebraic group $G$ is called adjoint if $G$ is connected, semisimple, and if every central isogeny $\phi : G \to H$ for $H$ a connected algebraic group is an algebraic group isomorphism.

Under the assumption $\text{char}(k) = 0$, a connected semisimple $k$-algebraic group $G$ is adjoint if and only if $Z(G) = \{1\}$.

**Fact B.5** ([28, 1.4.11]). Let $G$ be a connected semisimple $k$-algebraic group. Then there is a sequence $\tilde{G} \overset{\tilde{\rho}}{\to} G \overset{\rho}{\to} \overline{G}$ where $\tilde{G}$ is a simply connected $k$-algebraic group, $\overline{G}$ is an adjoint $k$-algebraic group, and $\tilde{\rho}, \rho$ are central $k$-isogenies. We call $\tilde{G}$ the simply connected covering of $G$ and $\overline{G}$ the adjoint group of $G$.

If $G$ is an algebraic group, then the set of $k$-rational points is denoted $G(k)$. $G(k)$ is naturally a group under the group operations of $G$. When $k$ is a local field, i.e. a commutative non-discrete locally compact topological field, the set of $k$-rational points of a $k$-algebraic group
Appendix B (Continued)

$G$ inherits a topology from the local field $k$. Under this topology, $G(k)$ is a locally compact, $\sigma$-compact, and metrizable topological group.

**Definition B.6.** For $G$ a connected algebraic group, $G(k)^+$ is the normal subgroup of $G(k)$ generated by all unipotent elements.

We remark that Definition B.6 is equivalent with the more general definition usually given as we assume $\text{char}(k) = 0$.

We require the notions of $k$-isotropic and $k$-anisotropic algebraic groups. For the precise definition of these terms, see [28, .025]; one may, for our purposes, take the following proposition as a definition where $k$-anisotropic is understood to be the negation of $k$-isotropic.

**Proposition B.7** ([28, 1.5.3]). Let $G$ a semisimple algebraic group. Then $G(k)^+ = \{1\}$ if and only if $G$ is $k$-anisotropic.

If $G$ is a $k$-algebraic group and is $k$-isotropic or $k$-anisotropic, we drop the prefix “$k$-” and say $G$ is isotropic or anisotropic, respectively.

We collect a few more facts which will be used in the following sections.

**Proposition B.8** ([28, 1.5.5]). If $f : G \to G'$ is a central $k$-isogeny of semisimple $k$-algebraic groups, then $f(G(k)^+) = G'(k)^+$.

**Theorem B.9** ([28, 1.5.6]). Suppose $\text{char}(k) = 0$ and $G$ is an almost $k$-simple algebraic group. Then any subgroup of $G(k)$ normalized by $G(k)^+$ is either contained in $Z(G)$ or contains $G(k)^+$.

**Theorem B.10** ([28, 2.3.1]). Suppose $k$ is a local field of characteristic zero and $G$ is a semisimple $k$-algebraic group. Then
Appendix B (Continued)

(1) If $G$ is simply connected, isotropic, and almost $k$-simple, then $G(k)^+ = G(k)$.

(2) If $G$ is isotropic and almost $k$-simple, then $G(k)^+$ is a finite index open normal subgroup of $G(k)$.

**Proposition B.11** ([28, 2.3.4]). Let $\beta : H \to G$ be a central $k$-isogeny of connected semisimple $k$-groups with $k$ a local field and $\text{char}(k) = 0$. Then the restriction $\beta : H(k) \to G(k)$ is continuous in the locally compact group topology induced by $k$, $\beta(H(k)) \leq_o G(k)$, and $H(k)/\ker(\beta) \simeq \beta(H(k))$ as topological groups.

**Theorem B.12** ([28, 2.3.5]). For $k$ a local field and $G$ a semisimple $k$-algebraic group, $G(k)$ and $G(k)^+$ are compactly generated.

**B.2 Simple $p$-adic Lie Groups**

For a l.c.s.c. $p$-adic Lie group $G$, let $L(G)$ denote the Lie algebra. Recall there is a canonical representation $\text{Ad} : G \to \text{Aut}(L(G))$ and $\text{Aut}(L(G))$ is a closed subgroup of $\text{GL}_n(\mathbb{Q}_p)$ for some $n$. This induces a morphism of Lie algebras $\text{ad} : L(G) \to L(\text{Aut}(L(G))) \subseteq \text{gl}_n(\mathbb{Q}_p)$. We remark that for $p$-adic Lie groups we may not assume $\text{Ad}(G)$ is closed in $\text{GL}_n(\mathbb{Q}_p)$.

**Lemma B.13.** If $G$ is a non-elementary topologically simple l.c.s.c. $p$-adic Lie group, then $G$ is isomorphic to a closed subgroup of $\text{GL}_n(\mathbb{Q}_p)$ and $L(G)$ is simple.

**Proof.** Let $\mathfrak{r}$ be the solvable radical of $L(G)$. Suppose for contradiction $\mathfrak{r}$ is non-trivial. Via the Levi-Malcev theorem, see [7, I.6.8 Theorem 5], $\mathfrak{r}$ has a supplement. There is thus $O \leq_o G$ and a non-trivial $N \leq O$ such that $L(N) = \mathfrak{r}$; cf. [7, III.7.1 Proposition 2].
Appendix B (Continued)

Since \( r \) is solvable, there is \( W \leq_o N \) which is solvable. We may thus find \( U \in U(G) \) with \( U \leq O \) and \( U \cap N \leq W \). The penultimate term in the derived series of \( N \cap U \) is thus a non-trivial locally normal abelian group of \( G \). This contradicts Corollary 4.18. We conclude \( r \) is trivial and, therefore, \( L(G) \) is semisimple. Say that \( L(G) = \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_n \) with the \( \mathfrak{s}_i \) simple Lie algebras.

Observe that \( Ad \) is injective. Indeed, else \( G = \ker(Ad) \). Applying Fact 6.19, every element of \( G \) is uniscalar. Write \( G = \bigcup_{i \in \omega} O_i \) with \( (O_i)_{i \in \omega} \) an \( \subseteq \)-increasing sequence of compactly generated open subgroups of \( G \). Each \( O_i \) is also uniscalar and, via Theorem 6.20, elementary. This contradicts that \( G \) is non-elementary.

Fix \( U \in U(G) \) and note that \( L(U) = L(G) \). Certainly, \( Ad(U) \) is compact since \( U \) is compact. In view of [7, III.3.8 Proposition 28], we further have that \( U \simeq Ad(U) \) as Lie groups and \( L(U) = L(Ad(U)) \). On the other hand, since \( L(G) \) is semisimple, [36, LA 6.7 Corollary 2] and [7, III.3.11 Corollary 5] imply \( L(G) = L(Aut(L(G))) \). Thus, \( Ad(U) \) is open in \( Aut(L(G)) \) via [7, III.7.1 Theorem 2], and it follows that \( G \simeq Ad(G) \). Seeing as \( Aut(L(G)) \) is a closed subgroup of \( GL_n(\mathbb{Q}_p) \), we conclude that \( G \) is isomorphic to a closed subgroup of \( GL_n(\mathbb{Q}_p) \).

For the simplicity of \( L(G) \), let \( W \leq Aut(L(G)) \) be the collection of \( g \in Aut(L(G)) \) such that \( g(\mathfrak{s}_i) = \mathfrak{s}_i \) for each \( 1 \leq i \leq n \). Certainly, \( W \) is finite index in \( Aut(L(G)) \), so \( W \cap Ad(G) = Ad(G) \).

Identifying \( W \) with \( Aut(\mathfrak{s}_1) \times \cdots \times Aut(\mathfrak{s}_n) \), we consider \( Ad(G) \cap Aut(\mathfrak{s}_i) \). If \( Ad(G) \cap Aut(\mathfrak{s}_i) = \)
Appendix B (Continued)

{1} for each $i$, then $Aut(s_i)$ is discrete for each $i$ which is absurd. Thus, $Ad(G) \leq Aut(s_i)$ for some $i$, and since $Ad(G)$ is open, we conclude that

$$L(G) = L(Ad(G)) = L(Aut(s_i)) = s_i$$

and $L(G)$ is simple.

\[ \square \]

**Remark B.14.** The above uses and proves a result of Caprace, Reid, and Willis in forthcoming work [11]: An $[A]$-semisimple $p$-adic Lie group has a semisimple Lie algebra.

We now show non-elementary topologically simple $p$-adic Lie groups have a close relationship with $\mathbb{Q}_p$-algebraic groups. This requires additional results from the literature:

**Definition B.15 ([15])**. A $p$-adic Lie group $G$ is said to be of simple type if $G \simeq S(\mathbb{Q}_p)^+$ where $S$ is an almost $\mathbb{Q}_p$-simple isotropic $\mathbb{Q}_p$-algebraic group. We say $G$ is of adjoint simple type if $S$ is also adjoint.

We remark that adjoint almost $k$-simple $k$-algebraic groups are $k$-simple when $\text{char}(k) = 0$.

**Fact B.16 ([15] Proposition 6.3)**. Let $G$ be a non-compact closed subgroup of $GL_m(\mathbb{Q}_p)$ such that every proper, closed, characteristic subgroup is compact. Then one of the following cases holds:

1. The closed subgroup generated by all solvable normal subgroups of $G$ is compact open, central in $G$.
2. $G$ is isomorphic to $\mathbb{Q}_p^n$ for some $n > 0$. 

Appendix B (Continued)

(3) $G \simeq S^{xk}/Z$ where $S$ is $p$-adic of simple type, $k \geq 1$, and $Z$ is a central subgroup of $S^{xk}$ which is invariant under a transitive group of permutations of $\{1, \ldots, k\}$.

**Theorem B.17.** If $G$ is a non-elementary topologically simple $p$-adic Lie group, then $G \simeq H(\mathbb{Q}_p)^+$ with $H$ an adjoint $\mathbb{Q}_p$-simple isotropic $\mathbb{Q}_p$-algebraic group. i.e. $G$ is of adjoint simple type.

**Proof.** Via Lemma B.13, we may identify $G$ with a closed subgroup of $GL_m(\mathbb{Q}_p)$ and apply Fact B.16; certainly, cases (1) and (2) are impossible since $G$ is non-elementary. So $G \simeq S^{xk}/Z$ for some $k \geq 1$. Suppose $k > 1$ and let $\psi : S^{xk} \to G$ be the obvious homomorphism. Writing $S^{xk} = S_1 \times \cdots \times S_k$, we have $\psi(S_i) \leq G$; hence, $\text{cl}(\psi(S_i)) = G$ for some $i$. We may assume $i = 1$.

Take $i \neq 1$ and observe $\psi(S_i)$ must commute with $\psi(S_1)$. It follows $G$ commutes with $\psi(S_i)$. We thus have that $\psi(S_i)$ is central in $G$ and, therefore, equals $\{1\}$, so $S_i \leq Z$ for all $i \neq 1$. Certainly, $S_i$ is abelian, and since $Z$ is invariant under a transitive group of permutations of $\{1, \ldots, k\}$, $S^{xk}$ is abelian. This is absurd as $G$ is non-elementary. We conclude $G \simeq S/Z$ where $S = T(\mathbb{Q}_p)^+$ for $T$ some almost $\mathbb{Q}_p$-simple isotropic $\mathbb{Q}_p$-algebraic group.

Consider $\bar{p} : T \to \overline{T}$ as given by Fact B.5. Since $T$ is almost $\mathbb{Q}_p$-simple and isotropic, $\overline{T}$ is an adjoint $\mathbb{Q}_p$-simple isotropic $\mathbb{Q}_p$-algebraic group; further, $\bar{p} : T(\mathbb{Q}_p)^+ \to \overline{T}(\mathbb{Q}_p)^+$ surjectively. Via Theorem B.9, $\overline{T}(\mathbb{Q}_p)^+$ has trivial center, whereby $\ker(\bar{p} \upharpoonright_{T(\mathbb{Q}_p)^+}) = Z$. It now follows $G \simeq T(\mathbb{Q}_p)^+$, and we conclude that $G$ is of adjoint simple type. \qed
Appendix B (Continued)

In the next section, \( \mathbb{Q}_p \)-simplicity is not sufficient for our purposes. There is, fortunately, a way to account for this:

**Fact B.18** ([28, 1.7]). Let \( k \) be a local field. If \( S \) is an adjoint \( k \)-simple isotropic \( k \)-algebraic group, then there is \( k' \) a finite separable extension of \( k \) and an adjoint absolutely simple isotropic \( k' \)-algebraic group \( S' \) such that \( S'(k')^+ \simeq S(k)^+ \) as topological groups.

The next corollary follows immediately from the above fact and Theorem B.17.

**Corollary B.19.** If \( G \) is a non-elementary topologically simple \( p \)-adic Lie group, then \( G \cong H(k)^+ \) as topological groups with \( H \) an adjoint absolutely simple isotropic \( k \)-algebraic group for \( k \) a finite extension of \( \mathbb{Q}_p \).

**B.3 Abstract Homomorphisms**

For \( k \) a local field and \( S \) an algebraic group, we consider \( Aut(S(k)^+) \) to be the group of topological group automorphisms of \( S(k)^+ \), \( Inn(S(k)^+) \) to be the collection of inner automorphisms, \( Aut(S) \) to be the group of algebraic automorphisms of \( S \), and \( Aut(k) \) to be the group of field automorphisms of \( k \) - these are not assumed to be continuous.

**Theorem B.20** ([28, 1.8.2]). Suppose \( k \) is a finite extension of \( \mathbb{Q}_p \) and \( S \) is a \( k \)-algebraic group which is adjoint, absolutely simple, and isotropic. If \( \alpha \in Aut(S(k)^+) \), then there is a unique \( \phi \in Aut(k) \) and unique \( k \)-isomorphism \( \beta : \phi S \to S \) such that \( \alpha = \beta \circ \phi^0 \mid_{S(k)^+} \)

A precise definition of \( \phi S \) and \( \phi^0 \) may be found in [28, 1.7].

Suppose \( k \) is a finite extension of \( \mathbb{Q}_p \) and \( \phi \in Aut(k) \). Via [5, 2.3], \( \phi \) is continuous and, therefore, is the identity on \( \mathbb{Q}_p \). We conclude that \( Aut(k) = Aut(k/\mathbb{Q}_p) \) and is finite. Suppose
that $S$ is an adjoint absolutely simple isotropic $k$-algebraic group. For each $\alpha \in \text{Aut}(S(k)^+)$, let $\phi_\alpha \in \text{Aut}(k)$ and $\beta_\alpha : \phi_\alpha S \to S$ be as given by Theorem B.20. For $\phi \in \text{Aut}(k)$, put

$$\Sigma_\phi := \{ \alpha \in \text{Aut}(S(k)^+) \mid \phi_\alpha = \phi \}$$

Observe that $\text{Inn}(S(k)^+) \leq \Sigma_1$, cf. Theorem B.22, and $\text{Aut}(S(k)^+) = \bigcup_{\phi \in \text{Aut}(k)} \Sigma_\phi$.

**Claim 1.** $\Sigma_1$ is a finite index subgroup of $\text{Aut}(S(k)^+)$. 

**Proof.** Let us first check that $\Sigma_1$ is a subgroup. Take $\alpha_1, \alpha_2 \in \text{Aut}(S(k)^+)$ and let $\beta_1$ and $\beta_2$ be the respective $k$-isomorphisms given by Theorem B.20. Certainly, $\beta_1 \circ \beta_2 : G \to G$ is again a $k$-isomorphism; further,

$$\beta_1 \circ \beta_2 \mid_{S(k)^+} = \alpha_1 \circ \alpha_2$$

By the uniqueness of $\beta_{\alpha_1 \alpha_2}$ and $\phi_{\alpha_1 \alpha_2}$, we conclude that $\beta_{\alpha_1 \alpha_2} = \beta_1 \circ \beta_2$ and $\phi_{\alpha_1 \alpha_2} = 1$, whereby $\Sigma_1$ is closed under composition. That $\Sigma_1$ is closed under taking inverses follows similarly.

To see $\Sigma_1$ has finite index, it suffices to show for each $\phi \in \text{Aut}(k)$, $\Sigma_\phi$ is contained in a right coset of $\Sigma_1$. To this end, fix $\phi \in \text{Aut}(k)$, take $\alpha_1$ and $\alpha_2$ in $\Sigma_\phi$, and let $\beta_1$ and $\beta_2$ be the respective $k$-isomorphisms $\phi S \to S$ given by Theorem B.20. Thus, $\beta_1 \circ \beta_2^{-1}$ is a $k$-automorphism of $S$, and for $x \in S(k)^+$,

$$\beta_1(\beta_2^{-1}(x)) = \beta_1(\phi(\alpha_2^{-1}(x))) = \alpha_1 \circ \alpha_2^{-1}(x)$$
By the uniqueness of $\beta_{\alpha_1\alpha_2^{-1}}$ and $\phi_{\alpha_1\alpha_2^{-1}}$, we conclude $\beta_{\alpha_1\alpha_2^{-1}} = \beta_1 \circ \beta_2^{-1}$ and $\phi_{\alpha_1\alpha_2^{-1}} = 1$. Hence, $\alpha_1 \circ \alpha_2^{-1} \in \Sigma_1$, so $\Sigma_\phi$ is contained in a right coset of $\Sigma_1$. \hfill \Box

By the proof of Claim 1, the map $\Phi : \Sigma_1 \to Aut(S)$ defined by $\alpha \mapsto \beta_\alpha$ is an injective group homomorphism where $Aut(S)$ is the group of algebraic automorphisms of $S$. To prove the main theorem of this section, we require two additional facts concerning $Aut(S)$.

**Theorem B.21** ([38, 5.7.2]). Let $G$ be a semisimple $k$-algebraic group. Then there exists a $k$-algebraic group $Aut(G)$ such that for any field extension $l$ of $k$ we have that $Aut(G)(l)$ is the group of $l$-automorphisms of $G$.

**Theorem B.22** ([38, 5.7.2]). If $G$ is an adjoint semisimple $k$-algebraic group, then $G$ is $k$-isomorphic to $Aut(G)^0$ where $Aut(G)^0$ is the connected component of 1 and the isomorphism is given by the self action of $G$ by conjugation.

**Theorem B.23.** Let $G$ be a non-elementary topologically simple $l.c.s.c.$ $p$-adic Lie group, let $Aut(G)$ be the group of topological group automorphisms of $G$, and let $Inn(G)$ be the collection of inner automorphisms. Then $Aut(G)/Inn(G)$ is finite.

**Proof.** In view of Corollary B.19, we may take $G = S(k)^+$ where $k$ is a finite extension of $\mathbb{Q}_p$ and $S$ is an adjoint absolutely simple isotropic $k$-algebraic group.

Let $\xi : S \to Aut(S)^0$ send $s \in S$ to the algebraic automorphism given by conjugation with $s$. Observe that, by Theorem B.22, $\xi$ is a $k$-isomorphism of $k$-algebraic groups. So $\xi(S(k)) = Aut(S)^0(k)$; cf. [28, 2.1.2]. On the other hand, let $\Phi : \Sigma_1 \to Aut(S)$ be as described above and identify $S(k)^+ \simeq Inn(S(k)^+) \leq Aut(S(k)^+)$. For $x \in S(k)^+$, we see that $\xi(x) \mid_{S(k)^+}$
Appendix B (Continued)

Φ(x) ↾_{S(k)^+}. Since ξ(x) is a k-automorphism of S, the uniqueness of Φ(x) implies ξ(x) = Φ(x).

It follows that Φ^{-1}(ξ(S(k)^+)) = Inn(S(k)^+).

Now observe that Φ^{-1}(Aut(S)^0) is finite index in Σ_1 and

$$\Phi : \Phi^{-1}(Aut(S)^0) \to Aut(S)^0(k) = ξ(S(k))$$

Applying Theorem B.10, ξ(S(k)^+) is finite index in ξ(S(k)), so Φ^{-1}(ξ(S(k)^+)) = Inn(S(k)^+)

is finite index in Φ^{-1}(Aut(S)^0). We conclude Inn(S(k)^+) is finite index in Σ_1. In view of

Claim 1, Inn(S(k)^+) is indeed finite index in Aut(S(k)^+) proving the theorem. □


VITA

Education

Adviser: Christian Rosendal


Invited Talks

04/2014 *Elementary totally disconnected locally compact Polish groups and applications*, American Mathematical Society Spring Western Sectional Meeting, University of New Mexico, Albuquerque.

02/2014 *Elementary totally disconnected locally compact Polish groups and applications*, Visitor Seminar, Fields Institute, Toronto, Canada.

11/2013 *Constructible totally disconnected locally compact Polish groups and applications*, Logic Seminar, University of Illinois, Urbana-Champaign.

10/2013 *Conjugacy classes in locally compact subgroups of $S_\infty$*, Workshop on Homogeneous Structures, Hausdorff Research Institute for Mathematics, Bonn, Germany.

10/2013 *A decomposition theorem for $p$-adic Lie groups*, Séminaires liés à la théorie des groupes à l’UCL, Université catholique de Louvain, Louvain-la-Neuve, Belgium.

09/2013 *Conjugacy class conditions in totally disconnected locally compact Polish groups*, ESI 2013 Set Theory Programme: Forcing, Large Cardinals, and Descriptive Set Theory, Erwin Schrödinger International Institute for Mathematical Physics, Vienna, Austria.

03/2013 *Conjugacy classes in totally disconnected locally compact Polish groups*, American Mathematical Society Spring Southeastern Sectional Meeting, University of Mississippi, Oxford.

02/2013 *Conjugacy classes in totally disconnected locally compact Polish groups*, Logic Seminar, University of Illinois, Urbana-Champaign.

Publications


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**Teaching Experience**

  Led recitation sections for large lecture courses.

- **06/2010–08/2010** Summer Enrichment Workshop Lecturer, *University of Illinois, Chicago.*
  Lectured for *Math 075: Beginning Algebra.*

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**Committees**

  Organized the Descriptive Set Theory Seminar. Descriptive set theory students presented papers and their research.

- **08/2013–05/2014** Graduate Student Colloquium, *University of Illinois, Chicago.*
  Co-organized the Graduate Student Colloquium. Advanced graduate students in mathematics presented their research results in colloquium style talks.

  Co-organized the Louise Hay Logic Seminar. Logic students presented papers and their research.

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**Community Service**

  Volunteered to judge high school students’ mathematics research posters.

  Gave a colloquium talk for undergraduate students titled *Descriptive set theory and Polish groups.*

- **09/2013** Association for Women in Mathematics Panel, *University of Illinois, Chicago.*
  Served as a panel member for the Association for Women in Mathematics panel *Surviving Graduate School.*