Singularities in Birational Geometry

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THESIS

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To my parents, Su-Ting and Anna.

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LIST OF ABBREVIATIONS

$\mathcal{O}_X$	Structure sheaf
$\omega_X^\bullet$	Dualizing complex
K_X	Canonical divisor
L^{-1}	Dual line bundle
L^m	m-th tensor power of a line bundle
$H^i(X, \mathcal{F})$	i-th sheaf cohomology group
$h^i(X, \mathcal{F})$	Dimension of the i-th sheaf cohomology group
$\mathcal{R}^i f_* \mathcal{F}$	Higher direct image of sheaf

CHAPTER 1

INTRODUCTION

In this monograph we study singularities that occur in birational geometry. Given a variety X and a log resolution $f : Y \to X$, one way to measure the singularities of X is to compute the Jacobian of f. Since the Jacobian of f has support on $f-$exceptional divisors, we can write it as $\sum a_i E_i$ where E_i is an exceptional divisor. In this way, one gets notions of singularities based on the minimal values of these coefficients. For example, we say X is log terminal, respectively log canonical, if $\min\{a_i\} > -1$ or $\min\{a_i\} \geq -1$, respectively. (See section 2.1.1 for more detailed discussions.)

These notions are used in the minimal model program (MMP) with the aim to find a simple representative in each birational class of algebraic varieties. In dimension two, we achieve this by contracting all the -1 self intersection curves. By the Castelnuovo contraction theorem, if we start from a smooth surface then the output of MMP will also be a smooth surface.

In higher dimensions, a minimal model is defined as a variety with a nef canonical divisor. Unlike the case in dimension two, the notion of singularities is more subtle since even if we start from a smooth variety, we might end up with a singular minimal model. It turns out that log terminal is the correct notion of singularities we should allow in the most general MMP. Log canonical singularities happen naturally on the canonical model of a log pair of general type, and on the degeneration of a family of varieties.

On the other hand, one can also measure singularities by cohomological properties. For example, we say that X has rational singularities if X is normal and $R^i f_* \mathcal{O}_Y = 0$, for all $i > 0$. This implies that from the cohomological point of view X is not different from Y, since by the Leray spectral sequence we have $H^i(Y, \mathcal{O}_Y) \cong H^i(X, \mathcal{O}_X)$. More generally, for any line bundle L on X we have $H^i(Y, f^*L) \cong H^i(X, L)$. This simple observation allows us to reduce the study of cohomological properties on a singular variety to its smooth model. For example, we can show easily that the Kawamata-Viewheg vanishing theorem holds on any varieties with rational singularities. That is, for a nef and big line bundle L on X, we have $H^i(X, L^{-1}) = 0$, for all $i < \dim X$ if X has rational singularities.

It is then natural to ask if there is any relation between the notions of singularities in MMP and in the cohomological point of view. For example, it is well known that log terminal implies that X has at least rational singularities. Here we include a well known proof of this fact since it is simple and also sheds some light on the topics we are going to study in this thesis. First note that since X is log terminal, the round up of the Jacobian $\lceil K_Y - f^*K_X \rceil = \sum \lceil a_i \rceil E_i := B$ is a summation of effective exceptional divisors. Then by the Kawamata-Viehweg vanishing theorem we have $R^i f_* \mathcal{O}_Y(B) = 0$, for all $i > 0$, which implies that $\mathcal{O}_X \cong \mathbf{R} f_* \mathcal{O}_Y(B)$ in the derived category. In particular, the following sequence splits.

$$\mathcal{O}_X \to \mathbf{R} f_* \mathcal{O}_Y \to \mathbf{R} f_* \mathcal{O}_Y(B) \cong \mathcal{O}_X. \tag{1.1}$$

It follows that X has rational singularities by (1).

The above argument fails if we do not assume X has log terminal singularities. For in log terminal cases, we have

$$B = K_Y - f^*K_X + \text{boundary}$$

where the boundary is a combination of simple normal crossing divisors with coefficients strictly less then one, and we are able to apply the Kawamata-Viehweg vanishing theorem. Without the assumption of being log terminal, the boundary will contain some reduced divisors, so that the Kawamata-Viehweg vanishing theorem does not hold. In other words, we can think that the existence of reduced parts in the boundary is the reason why X fails to have rational singularities.

In this monograph we study the case of isolated log canonical points, and ask how far it is from being rational. Instead of using the definition, we approach this question by using the following equivalent formulation of rational singularities.

- X has rational singularities if and only if X is Cohen-Macaulay and $f_*\omega_Y = \omega_X$.

So given an isolated log canonical point $p \in X$, we ask how far is X from being Cohen-Macaulay? We begin by recalling the definition of Cohen-Macaulay. There are two ways to understand codimension of a subvariety p in X. The first is from the more geometric view point: we calculate the length of the increasing chain of varieties containing p, which we call $\dim\mathcal{O}_{X_p}$. The second is from the more algebraic view point. Roughly speaking, we calculate the length of equations needed to cut out p, which is $\text{depth}_p\mathcal{O}_X$. We say X is Cohen-Macaulay at p if the geometric codimension and algebraic codimension coincide. In our case, we take p

as a closed point, so X is Cohen-Macaulay at p if $\dim X = \mathrm{depth}_p \mathcal{O}_X$. To investigate how far $p \in X$ is from being rational, we calculate $\mathrm{depth}_p \mathcal{O}_X$. We have the following theorem.

Theorem 1.0.1. *(=corollary 2.3.3) Given $p \in X$, suppose p is an isolated log canonical center. Let E be the union of exceptional divisors with coefficient -1 in the Jacobian $K_{Y/X} = \sum a_i \cdot E_i$. Then for any integer $3 \leq t \leq n$, we have $depth_p \mathcal{O}_X \geq t$ if and only if $H^{i-1}(E, \mathcal{O}_E) = 0, \forall 1 < i < t$. (Note that by assumption X is normal, so we know $depth_p \mathcal{O}_X$ is at least two.) Moreover, $f_* \omega_Y \cong \omega_X$ if and only if $H^{n-1}(E, \mathcal{O}_E) = 0$.*

For example, by this theorem we will see in section 2.3.3 that a cone over a K3 surface is Cohen-Macaulay but not rational. A cone over an Abelian variety is Cohen-Maculay if and only if the dimension of the Abelian variety is one.

A crucial ingredient to the proof of this theorem is a vanishing theorem due to Kovács (2), which says that in this case $R^i f_* \mathcal{O}_Y(-E) = 0, \forall i > 0$. With this theorem, the conclusion of Theorem 1.0.1 follows from a standard application of the Grothendieck duality theorem. In Chapter 2, we will prove a vanishing theorem for dlt pairs, a special case of log canonical singularity. We then deduce Kovács's theorem from this vanishing theorem.

In the second part of this monograph, we prove a vanishing theorem of log canonical varieties. We begin by asking the following question:

- Given an ideal sheaf $\mathcal{I}$ and a line bundle L on a log canonical variety X, then how positive L has to be to guarantee $H^i(X, \mathcal{O}_X(K_X) \otimes L \otimes \mathcal{I}) = 0, \forall i > 0$?

One possible answer is by the Nadel vanishing theorem. It says that if $\mathcal{J}(\mathcal{I})$ is the multiplier ideal associated to $\mathcal{I}$, and $L = A \otimes M$ such that A is nef and big and $M \otimes \mathcal{I}$ is globally generated, then we have $H^i(X, \mathcal{O}_X(K_X) \otimes L \otimes \mathcal{J}(\mathcal{I})) = 0, \forall i > 0$. But in general $\mathcal{I} \subsetneq \mathcal{J}(\mathcal{I})$, so it is not exactly the answer to our question.

Another answer is given by de Fernex and Ein (3). They consider a locally complete intersection variety X and a subvariety V of codimension e. They conclude that if as a pair (X, eV) is log canonical, then $H^i(X, \mathcal{O}_X(K_X) \otimes L^{\otimes k} \otimes A \otimes \mathcal{I}_Z) = 0, \forall i > 0$. Here k is an integer depending on the degree of the generators of $\mathcal{I}_Z$, and A is a nef and big line bundle. Their result generalizes a result by Bertram, Ein and Lazarsfeld (4) in which they assume the variety X is smooth.

The result of de Fernex and Ein is essentially an application of Nadel vanishing theorem. They assume X is a locally complete intersection, then they apply inversion of adjunction to approximate $\mathcal{I}_Z$ by some multiplier ideal sheaf. In the second part of this thesis, we generalize their result from a different view point. More precisely, we prove the following,

Theorem 1.0.2. *(=Theorem3.2.1) Suppose the pair (X, eZ) is log canonical. Then for any nef line bundles A and M, such that $A \otimes \mathcal{O}(-K_X)$ is ample and $M \otimes \mathcal{I}_Z^{\otimes e}$ is globally generated, we have*

$$H^i(X, A \otimes M \otimes \mathcal{I}_Z) = 0 \ \textit{for}\ i > 0.$$

In this theorem, we do not assume X is a local complete intersection, and in the proof we do not use the Nadel vanishing theorem. Instead, we use a torsion freeness theorem due to

Ambro and Fujino, which is a generalization of Kollár's torsion freeness theorem to log canonical singularities cases.

With a tiny revision of the proof, we have the following more general theorem. The same statement for pair $(\mathbb{P}^n, eZ)$ was proved in (5) by a method of generic linkage.

Corollary 1.0.3. *The conclusion of Theorem 1.0.2 is still true if* (X, eZ) *is log canonical except at finitely many points.*

In the third part of this monograph, we consider non $\mathbb{Q}-$Gorenstein singularities. In the MMP, we consider not only a single variety but a pair (X, Δ) (See section 2.1.1). Here Δ is a linear combination of Weil divisors with rational coefficients between zero and one. To study the singularities of (X, Δ), we need to assume that $(K_X + \Delta)$ is $\mathbb{Q}-$Gorenstein so that its pull back to any smooth model over X is of pure codimension one.

In a paper of de Fernex and Hacon (6), they propose a new method to study singularities in birational geometry. Instead of using the notion of pairs, they pull back the canonical divisor by an asymptotic approach. Using this new approach, they also define log terminal and log canonical singularities. In this thesis, we show that this notion of singularities satisfies the following transversality theorem.

Theorem 1.0.4. *Let* X *be a homogeneous variety with a group variety* G *acting on it. Let* Y, Z *be subvarieties of* X *such that* Z *is smooth and* Y *is log canonical (resp. log terminal). For any* $g \in G$*, denote* Y^g *the transition of* Y *by* g*. Then there exists a non empty open* $U \subset G$ *such that* $\forall g \in U$*,* $Y^g \times_X Z$ *is log canonical (resp. log terminal).*

On the other hand, to study singularities first we need to take a log resolution. This is usually a difficult task when we consider concrete examples. To this aim, we show the following theorem and apply it to the study of singularities of generic determinantal varieties.

Theorem 1.0.5. *Let $f : Y \to X$ be a small resolution of a normal variety X, such that Y is smooth and $-K_Y$ is relatively nef. Then X is log terminal (in the sense of de Fernex and Hacon). Moreover, there is a boundary Δ such that (X, Δ) is canonical (in the classical sense).*

Here by small we mean that the codimension of the exceptional locus is at least two.

CHAPTER 2

DEPTH OF ISOLATED LOG CANONICAL CENTERS

In this chapter, we study isolated log canonical (lc) centers. By definition, lc centers are the image of exceptional divisors with discrepancy equal to negative one, which is called lc places. We say that a lc center $p \in X$ is isolated if outside of p, X has only log terminal singularities. Our result Theorem 2.3.3 shows that the singularity property of a lc center may be determined by lc places along, even there may be some other exceptional divisors mapped to p.

We remark that in general isolated lc centers are not isolated lc singularities. The reason is that outside of the center there might still be singular, just better than log canonical. So when we study isolated lc centers, we are not allowed to take good resolution. That is, a log resolution which is an isomorphism outside the center. Note that the method of taking good resolution is required in the papers of studying isolated lc singularities. ((7) and (8).)

A crucial ingredient of the proof of Theorem 2.3.3 is a vanishing theorem for dlt pairs. This vanishing theorem is proved by a trick using Grothendieck duality theorem. Besides Theorem 2.3.3 we also give some other applications of this trick. Most of the results in this chapter have been published by the author in (9).

2.1 Preliminaries

2.1.1 Singularity of Pairs

In this section we recall some notions of singularity of pairs. The main reference is (10) and (11). Through out the whole monograph we work over $\mathbb{C}$.

We call (X, Δ) a log pair if X is a normal variety and Δ is an effective linear combination of Weil divisors with rational coefficients between 0 and 1. We also assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Since we are working over $\mathbb{C}$, by Hironaka's theorem we can take a resolution $f : Y \to X$ such that the exceptional locus and the strict transform $f_*^{-1}\Delta$ of Δ are simple normal crossing divisors.

Fix a canonical divisor K_Y on Y such that $f_*K_Y = K_X$. Let m be an integer such that $m \cdot (K_X + \Delta)$ is Cartier. Then we have

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta)) \cong f^*\mathcal{O}_X(m(K_X + \Delta))(\sum_i m \cdot a(E_i, X, \Delta)E_i).$$

Use numerical equivalence, divide this equation by m and write

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum_{E_i \text{ exceptional}} a(E_i, X, \Delta)E_i.$$

Then $a(E_i, X, \Delta)$ is called the discrepancy of E_i respect to (X, Δ).

In the minimal model program, we define the following notion,

Definition 2.1.1. *The discrepancy of* (X, Δ) *is given by*

$$discrep(X, \Delta) := inf\{a(E, X, \Delta) : E \text{ is an exceptional divisor over } X\}.$$

Now we give some classes of singularities used most often in the minimal model program. We say (X, Δ) is

- *terminal* if $\mathrm{discrep}(X, \Delta) > 0$, this is the singularities occurring on minimal models if we start from a smooth variety. It is known that the singular locus of terminal singularities has codimension at least three, So the minimal models of smooth surfaces are smooth.
- *canonical* if $\mathrm{discrep}(X, \Delta) \geq 0$. In the case of $\Delta = 0$, it is the singularities occurring on canonical models of varieties of general type. Under the surface case, it is also called Du Val singularities and we have complete classification of them by the combinatorial information of exceptional divisors on the minimal resolutions. (A-D-E type singularities.)
- *Kawamata log terminal* if $\mathrm{discrep}(X, \Delta) > -1$, and $\lfloor \Delta \rfloor = 0$. It is the singularities suitable for (relative) Kawamata-Viehweg vanishing theorem.
- *log canonical* if $\mathrm{discrep}(X, \Delta) \geq -1$. The singularities occur in the degeneration of log pairs of general type. As discussed in the introduction, in general the relative Kawamata-Viehweg vanishing theorem does not apply to this kind of singularities.

Another very useful notion of singularity is divisorial log terminal (dlt) defined in the following way

Definition 2.1.2. *Let* (X, Δ) *be a log pair and assume that* $K_X + \Delta$ *is* $\mathbb{Q}-$*Cartier. We say that* (X, Δ) *is dlt if there is a closed subset* $Z \subset X$ *such that*

(i). $X \setminus Z$ *is smooth and* $\Delta_{X \setminus Z}$ *is a simple normal crossing divisor.*

(ii). If $f : Y \to X$ *is birational and* $E \subset Y$ *is an irreducible divisor such that* $\text{center}_X E \subset Z$ *and* $a(E, X, \Delta) > -1$.

Here $\text{center}_X E \subset Z$ means the image $f(E)$ on X.

The notion of dlt singularity is very useful because of the following theorem due to Szabó (12).

Theorem 2.1.3. *(Theorem 2.44 in (10)) Given a log pair* (X, Δ)*,the following are equivalent:*

(i). (X, Δ) *dlt.*

(ii). There exists a log resolution $f : Y \to X$ *of* (X, Δ) *such that* $a(E_i, X, \Delta) > -1$ *for every exceptional divisors* $E_i \subset Y$.

Roughly speaking, this theorem implies that the log canonical locus are coming from the boundary Δ. In particular, a dlt pair (X, Δ) is klt if and only if $\lfloor \Delta \rfloor = 0$. In the following, we call a resolution in (ii) as Szabó resolution.

Given a birational morphism $f : Y \to X$ such that the exceptional locus are of pure codimension one. Write $K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum a_i E_i$ and $\Delta_Y := f_*^{-1}\Delta + \sum_{a_i \le -1} E_i$. We say that (Y, Δ_Y) is a minimal dlt model over (X, Δ) if (Y, Δ_Y) is a dlt pair,$K_Y + \Delta_Y$ is relative nef over X and the discrepancy of every f exceptional divisors is at most negative one. So in particular if (X, Δ) is log canonical, we have

$$K_Y + \Delta_Y \equiv f^*(K_X + \Delta)$$

By running relative minimal model program over a log pair, we have the following important theorem due to Hacon,

Theorem 2.1.4. *(Theorem 3.1 in (13)) Let (X, Δ) be a log pair such that $K_X + \Delta$ is $\mathbb{Q}-$Cartier. Then (X, Δ) admits a $\mathbb{Q}-$factorial minimal dlt model.*

2.1.2 Duality Theorems

In this section we recall duality theorems which will be used in this thesis. First we recall Grothendieck duality theorem (III.11.1, VII.3.4 in (14)). Let $f : Y \to X$ be a proper morphism between finite dimensional Noetherian schemes. Suppose that both X and Y admit dualizing complexes, for example when they are quasi-projective varieties. Then for any $\mathcal{F}^\bullet \in D^-_{qcoh}(Y)$, we have

$$Rf_* R\mathcal{H}om_Y(\mathcal{F}^\bullet, \omega_Y^\bullet) \cong R\mathcal{H}om_X(Rf_*\mathcal{F}^\bullet, \omega_X^\bullet)$$

Here $\omega_X^\bullet$ is the dualizing complex. Let n be the dimension of X and assume that X is normal, then $h^{-n}(\omega_X^\bullet) := \omega_X = \mathcal{O}_X(K_X)$, the extension of regular n-forms on smooth locus. In this thesis we only consider normal varieties, so we will use ω_X and K_X interchangeably. If X is Cohen-Macaulay, then $h^i(\omega_X^\bullet) = 0$, if $i \neq -n$, and $h^{-n}(\omega_X^\bullet) = \omega_X$. Or equivalently, $\omega_X^\bullet = \omega_X[n]$.

Now we recall local duality (V.6.2 in (14)). Suppose that (R, p) is a local ring. An injective hull I of the residue field $k = R/p$ is a an injective R module I such that for any non-zero submodule $N \subset I$ we have $N \cap k \neq 0$. (See (15) Proposition 3.2.2. for more discussion.) Matlis duality says that the functor $Hom(\cdot, I)$ is a faithful exact functor on the category of Noetherian R modules.

Theorem 2.1.5. *(Local duality) Let (R, p) be a local ring and $\mathcal{F}^\bullet \in D^+_{coh}(R)$. Then*

$$\mathbf{R}\Gamma_p(\mathcal{F}^\bullet) \to \mathbf{R}Hom(\mathbf{R}Hom(\mathcal{F}^\bullet, \omega_R^\bullet), I)$$

is an isomorphism.

In particular, if we take i-th cohomology on both hand sides, we have

$$\mathrm{H}^i_p(\mathcal{F}^\bullet) \cong Hom(\mathrm{H}^{-i}(\mathbf{R}Hom(\mathcal{F}^\bullet, \omega_R^\bullet)), I)$$

The $-i$ comes from switching the cohomology functor $\mathrm{H}^i(\cdot)$ and $Hom(\cdot, I)$.

2.2 A vanishing theorem for dlt pairs

In this section, we prove the main theorem in this chapter. This theorem will then be used to deduced a vanishing theorem due to Kovács. Then we apply it to study isolated log canonical centers.

Theorem 2.2.1. *Let (X, Δ_X) be a dlt pair and let $f : Y \to X$ be a Szabó resolution. Then we can write*

$$K_Y + \Delta_Y = f^*(K_X + \Delta_X) + A - B,$$

where A, B are effective exceptional divisors, $\lfloor B \rfloor = 0$ and Δ_Y is the strict transform of Δ_X. Then for any reduced subset $\Delta' \subseteq \Delta_Y$, we have

$$R^i f_* \mathcal{O}_Y(-\Delta') = 0$$

for every $i > 0$.

Proof. Write

$$K_Y - f^*(K_X + \Delta_X) + \Delta_Y = A - B,$$

Then

$$\lceil A \rceil = K_Y - f^*(K_X + \Delta_X) + \Delta_Y + B + \lceil A \rceil - A,$$

which is f-exceptional and effective. Consider the following diagram of complexes,

$$\begin{array}{ccc} f_* \mathcal{O}_Y(-\Delta') & \xrightarrow{\alpha} & \mathbf{R} f_* \mathcal{O}_Y(-\Delta') \\ & \searrow^{\beta} & \downarrow \gamma \\ & & \mathbf{R} f_* \mathcal{O}_Y(\lceil A \rceil - \Delta') \end{array}$$

Note that

$$\lceil A \rceil - \Delta' = K_Y - f^*(K_X + \Delta_X) + \text{strict transform} + \delta,$$

where δ is some effective simple normal crossing divisors such that $\lfloor \delta \rfloor = 0$. So by Reid-Fukuda type vanishing $R^i f_* \mathcal{O}_Y(\lceil A \rceil - \Delta') = 0$ for $i > 0$. On the other hand, Since $\lceil A \rceil$ is exceptional and Δ' is strict transform, so $f_* \mathcal{O}_Y(\lceil A \rceil - \Delta') = f_* \mathcal{O}_Y(-\Delta')$. (Lemma 12 in (16)). So β is a quasi isomorphism.

Dualize this diagram we have

$$\begin{array}{ccc} \mathbf{R}Hom(f_*\mathcal{O}_Y(-\Delta'), \omega_X^\bullet) & \xleftarrow{\alpha^*} & \mathbf{R}Hom(Rf_*\mathcal{O}_Y(-\Delta'), \omega_X^\bullet) \\ & \nwarrow^{\beta^*} & \uparrow \gamma^* \\ & & \mathbf{R}Hom(Rf_*\mathcal{O}_Y(\lceil A \rceil - \Delta'), \omega_X^\bullet) \end{array}$$

Apply Grothendieck duality we get the following composition,

$$\mathbf{R}f_*\omega_Y^\bullet(\Delta' - \lceil A \rceil) \xrightarrow{\gamma^*} \mathbf{R}f_*\omega_Y^\bullet(\Delta') \xrightarrow{\alpha^*} \mathbf{R}\mathcal{H}om(f_*\mathcal{O}_Y(-\Delta'), \omega_X^\bullet), \quad \beta^* = \alpha^* \circ \gamma^*$$

By Lemma 2.2.2, the complex $\mathcal{R}f_*\omega_Y^\bullet(\Delta')$ has vanishing higher cohomology. Note that β^* is a quasi isomorphism, so it is in fact a composition of sheaf morphisms as following,

$$f_*\omega_Y(\Delta' - \lceil A \rceil) \xrightarrow{\gamma^*} f_*\omega_Y(\Delta') \xrightarrow{\alpha^*} Hom(f_*\mathcal{O}_Y(-\Delta'), \omega_X)$$

Since $\lceil A \rceil$ is effective, γ^* is injective. Because $f_*\omega_Y(\Delta')$ is a rank one sheaf and the composition $\alpha^* \circ \gamma^*$ is an isomorphism, γ^* is in fact isomorphism. This implies that α^* is a quasi isomorphism, so $\alpha : f_*\mathcal{O}_Y(-\Delta') \to \mathbf{R}f_*\mathcal{O}_Y(-\Delta')$ is also a quasi isomorphism. That is, $R^i f_* \mathcal{O}_Y(-\Delta') = 0, \forall i > 0$. □

The following lemma is well known, we include the proof here for completeness.

Lemma 2.2.2. *Let $f : Y \to X$ be a proper morphism from a smooth variety Y, with a reduced simple normal crossing boundary B. Given a f- nef and big line bundle L on Y such that L*

is also f-big at all the lc (log canonical) centers of (Y, B). Then $R^i f_* K_Y(B+L) = 0$ for all $i > 0$. Particularly in our case, $R^i f_* K_Y(\lfloor \Delta_Y \rfloor) = 0$ for all $i > 0$.

Proof. Take an irreducible component $S \subseteq B$, then consider

$$0 \to K_Y((B-S)+L) \to K_Y(B+L) \to K_S(B-S+L) \mid_S \to 0.$$

Since S is a lc center of (Y, B), $L \mid_S$ is f- nef and big on S by assumption. The lc center of $(S, (B-S) \mid_S)$ is the restriction from (Y, B), so $L \mid_S$ is f-big on all of the lc centers of $(S, (B-S) \mid_S)$. As a result, by induction on dimension we conclude that $R^i f_*(K_S(B-S+L) \mid_S) = 0$.

On the other hand, if there is no irreducible component in B then the statement is the classical Kawamata- Viehweg vanishing theorem. So by induction on the number of irreducible components of B, we conclude that $R^i f_* K_Y((B-S)+L) = 0$. Then after pushing forward the exact sequence, we see that $R^i f_* K_Y(B+L) = 0$.

In the case of Theorem 2.3.1, we consider the pair $(Y, \lfloor \Delta \rfloor)$ and the line bundle L is $\mathcal{O}_Y$. Since f is birational, $\mathcal{O}_Y$ is automatically f-nef and big. Since f is isomorphism at the generic point of $\lfloor \Delta \rfloor$, $\mathcal{O}_Y$ is f-big on the lc center. So $R^i f_* K_Y(\lfloor \Delta_Y \rfloor) = 0$ for all $i > 0$.

□

2.3 Applications

2.3.1 A vanishing theorem due to Kovács

In this subsection, we follow Fujino's idea in (17) to give a quick proof of the followin theorem.

Theorem 2.3.1. *(2) Let (X, Δ) be a log canonical pair and let $f : Y \to X$ be a proper birational morphism from a smooth variety Y such that Ex$(f)\cup$ Supp$f_*^{-1}\Delta$ is a simple normal crossing divisor on Y. If we write*

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

and put $E = \sum_{a_i=-1} E_i$, then

$$R^i f_* \mathcal{O}_Y(-E) = 0$$

for every $i > 0$.

Proof. Consider the following maps,

$$Y \xrightarrow{h} Z \xrightarrow{g} X, \quad f = g \circ h : Y \to X$$

where $g : (Z, \Delta_Z) \to (X, \Delta)$ is a dlt modification such that $K_Z + \Delta_Z = g^*(K_X + \Delta)$. And $h : Y \to Z$ is a Szabó resolution such that $K_Y = h^*(K_Z + \Delta_Z) + A - B - \Delta_Y$, where $\Delta_Y = h_*^{-1}\Delta_Z$.

We claim that $R^i f_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor) = 0, \forall i > 0$. By Theorem 2.2.1, $R^i h_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor) = 0, \forall i > 0$. Also note that $h_* \mathcal{O}_Y(\lceil A \rceil - \lfloor \Delta_Y \rfloor) = h_* \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor) = \mathcal{O}_Z(-\lfloor \Delta_Z \rfloor)$. (lemma 12 in (16)). So by Leray spectral sequence, $R^if_*\mathcal{O}_Y(-\lfloor\Delta_Y\rfloor) = R^ig_*\mathcal{O}_Z(-\lfloor\Delta_Z\rfloor)$. Now we claim that $R^ig_*\mathcal{O}_Z(-\lfloor\Delta_Z\rfloor) = 0, \forall i > 0$. For

$$-\lfloor\Delta_Z\rfloor = K_Z - g^*(K_X + \Delta) + \Delta_Z - \lfloor\Delta_Z\rfloor.$$

Since dlt model is $\mathbb{Q}$-factorial, $K_Z + \Delta_Z - \lfloor\Delta_Z\rfloor$ is $\mathbb{Q}$-Gorenstein. So the pair $(Z, \Delta_Z - \lfloor\Delta_Z\rfloor)$ is a klt pair. Then the claim follows from the relative Kawamata-Viewheg vanishing theorem of klt pairs.

Note that $f : Y \to X$ is not a log resolution. To fix the problem, we can blow up centers with simple normal crossing with $\mathrm{Supp}(\Delta_Y + A + B)$. Say the blow up is $\pi : W \to Y$. There are two cases can happen; If we blow up klt locus, it is a Szabó resolution and then the divisor with -1 discrepancy is $\Delta_W = \pi_*^{-1}(\Delta_Y)$. Then $R^i\pi_*\mathcal{O}_W(-\Delta_W) = 0$ by Theorem 2.2.1. If we blow up centers inside non-klt locus, then the divisor with -1 discrepancy may be $\Delta_W = \pi_*^{-1}(\Delta_Y) + F$, where F is the exceptional divisor produced by blow up. Then $R^i\pi_*\mathcal{O}_W(-\Delta_W) = 0$ by direct calculation. In any case we showed that the higher direct image is not changed by these two kinds of blowing up. So we can conclude Kovács's vanishing theorem. □

2.3.2 Depth of isolated log canonical centers

Given a log canonical pair (X, Δ), and an isolated lc center $p \in X$ which is a closed point. Without loss of generality, we assume X is an affine space and p is the only closed point. By definition, we have a log resolution $f : Y \to X$ such that

$$K_Y = f^*(K_X + \Delta) + A - B - E.$$

Here A, B are effective and $\lfloor B \rfloor = 0$, E is the reduced exceptional divisor such that $f(E) = p$.

Theorem 2.3.2. *For* $1 < i < n$, $H_p^i(X, \mathcal{O}_X)$ *is dual to* $H^{n-i}(E, K_E)$ *by Matlis duality. For* $i = n$, $H_p^n(X, \mathcal{O}_X)$ *is dual to* $f_*\mathcal{O}_Y(K_Y + E)$.

Proof. Push forward the following exact sequence on Y,

$$0 \to K_Y \to K_Y(E) \to K_E \to 0.$$

By Grauert-Riemenschneider vanishing theorem, we have $R^{n-i}f_*\mathcal{O}_Y(K_Y+E) \cong \mathrm{H}^{n-i}(E, K_E)$ for $i < n$. So to prove the statement, it suffices to prove the duality between $\mathrm{H}_p^i(X, \mathcal{O}_X)$ and $R^{n-i}f_*\mathcal{O}_Y(K_Y + E) \cong \mathrm{H}^{n-i}(E, K_E)$. To this end, we consider the quasi isomorphism $f_*\mathcal{O}_Y(-E) \cong \mathbf{R}f_*\mathcal{O}_Y(-E)$ implied by Theorem 2.3.1. Apply Grothendieck duailty theorem, we have

$$\mathbf{R}Hom(f_*\mathcal{O}_Y(-E), \omega_X^\bullet) \cong \mathbf{R}Hom(\mathbf{R}f_*\mathcal{O}_Y(-E), \omega_X^\bullet) \cong \mathbf{R}f_*\omega_Y^\bullet(E)$$

Take $-i$th cohomology, we have

$$\mathcal{E}xt^{-i}(f_*\mathcal{O}_Y(-E), \omega_X^\bullet) \cong R^{n-i}f_*\mathcal{O}_Y(K_Y + E) \tag{2.1}$$

By local duality, the left hand side is isomorphic to $\mathrm{Hom}(\mathrm{H}_p^i(f_*\mathcal{O}_Y(-E)), I)$, where I is an injective hull of k .

To prove the statement, we claim that $\mathrm{H}_p^i(f_*\mathcal{O}_Y(-E)) \cong \mathrm{H}_p^i(\mathcal{O}_X)$ for $i > 1$. This follows from the following exact sequence

$$0 \to f_*\mathcal{O}_Y(-E) \to \mathcal{O}_X \to \mathcal{O}_p \to 0,$$

and the fact that $\mathrm{H}_p^i(\mathcal{O}_p) = 0$ iff $i > 0$.

□

Corollary 2.3.3. *For any integer $3 \leq t \leq n$, we have depth$_p\mathcal{O}_X \geq t$ if and only if $H^{i-1}(E, \mathcal{O}_E) = 0, \forall 1 < i < t$. (Note that by assumption X is normal, so we know depth$_p\mathcal{O}_X$ is at least two.) Moreover, $f_*\omega_Y \cong \omega_X$ if and only if $H^{n-1}(E, \mathcal{O}_E) = 0$.*

Proof. In the proof of Theorem 2.3.2, we showed $\mathrm{H}^i_p(f_*\mathcal{O}_Y(-E)) \cong \mathrm{H}^i_p(\mathcal{O}_X)$ for $i > 1$. Also since by assumption depth ${}_p\mathcal{O}_X$, which implies $\mathrm{H}^0_p(\mathcal{O}_X) = \mathrm{H}^1_p(\mathcal{O}_X) = 0$. So for any integer $3 \le t \le n$, we have

$$\begin{aligned}
\mathrm{depth}_p\mathcal{O}_X \ge t &\Leftrightarrow \mathrm{H}^i_p(X, \mathcal{O}_X) = 0, \forall i < t \\
&\Leftrightarrow \mathrm{H}^i_p(X, f_*\mathcal{O}_Y(-E)) = 0, \forall 1 < i < t \\
&\Leftrightarrow \mathrm{H}^{n-i}(E, K_E) = 0, \forall 1 < i < t \text{ (Matlis duality and Equation (2.1))} \\
&\Leftrightarrow \mathrm{H}^{i-1}(E, \mathcal{O}_E) = 0, \forall 1 < i < t. \text{ (Serre Duality)}
\end{aligned}$$

For the second statement, we note that $H^n_p(X, \mathcal{O}_X)$ is dual to $\mathcal{E}xt^{-n}(\mathcal{O}_X, \omega^\bullet_X) = \omega_X$. With the second statement of Theorem 2.3.2, we see that $\omega_X \cong f_*\mathcal{O}_Y(K_Y + E)$. By the exact sequence

$$0 \longrightarrow f_*\mathcal{O}_Y(K_Y) \longrightarrow f_*\mathcal{O}_Y(K_Y + E) \longrightarrow H^0(E, K_E) \longrightarrow 0,$$

we see that $f_*\omega_Y \cong f_*\mathcal{O}_Y(K_Y + E) \cong \omega_X$ if and only if $H^0(E, K_E) = 0$. Then the statement follows from Serre duality on E.

□

Remark 2.3.4. *The cohomology group $H^i(E, \mathcal{O}_E)$ is independent of resolution, because $H^i(E, \mathcal{O}_E) \cong R^i f_*\mathcal{O}_Y$ by Kovács vanishing theorem. And that $R^i f_*\mathcal{O}_Y$ is well known to be independent of resolution.*

Corollary 2.3.5. *(Proposition 4.7 (8)) Given a closed isolated lc center p of a pair (X, Δ), then X is Cohen-Macaulay at p if and only if $H^i(E, \mathcal{O}_E) = 0, \forall 0 < i < n-1$.*

2.3.3 Examples

In this section we use results in the previous section to calculate depth of some log canonical singularities. The examples we consider are cones over some smooth variety, we follow the construction in Section 3.1 of (11). First we recall some facts about cones.

Given a smooth variety $X \subset \mathbb{P}^n$, let L denote $\mathcal{O}_X(1)$. Then we have the following natural map

$$\bigoplus_{m \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \longrightarrow \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$$

The affine cone over X with respect to L is then defined as

$$C_a(X, L) := \operatorname{Spec}_k \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$$

By exercise II.5.14 in (18), $\bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$ is integral closure of the homogeneous coordinate ring of X. As a result, $C_a(X, L)$ is a normal variety with isolated singular point at the vertex.

Now we construct a natural resolution of $C_a(X, L)$. Note that we have the following map,

$$\bigoplus_{m \geq 0} H^0(X, L^{\otimes m}) \longrightarrow \bigoplus_{m \geq 0} L^{\otimes m} \longrightarrow 0.$$

Take spectrum of this sequence, we have

$$BC_a(X,L) := \mathrm{Spec}_X \bigoplus_{m\geq 0} L^{\otimes m} \longrightarrow \mathrm{Spec}_k \bigoplus_{m\geq 0} H^0(X, L^{\otimes m}) =: C_a(X,L).$$

Note that $BC_a(X,L)$ is a smooth variety and the map $f : BC_a(X,L) \to C_a(X,L)$ is an resolution with exceptional divisor E isomorphic to X. Now we calculate the Jacobian of f. First note that by construction there is a natural map $\pi : BC_a(X,L) \to X$ which is an $\mathbb{A}^1$ bundle. By adjunction formula we see that

$$K_{BC_a(X,L)} + E = \pi^* K_X. \tag{2.2}$$

On the other hand, by Proposition 3.14 in (11) we see that for some $m > 0$,

$$\begin{aligned} mK_{C_a(X,L)} \text{ is Cartier } &\Leftrightarrow \ mK_{C_a(X,L)} = 0 \text{ (Since } \mathrm{Pic}(C_a(X,L) = 0)\\ &\Leftrightarrow \ mK_{C_a(X,L)\setminus 0} = 0\\ &\Leftrightarrow \ \mathcal{O}_X(mK_X) \cong L^{\otimes mr} \text{ for some r (Since } Cl(C_a(X,L)) \cong Cl(X)/<L>) \end{aligned}$$

Combining with Equation (2.2) and the fact that $\pi^* L = \mathcal{O}_{BC_a(X,L)}(-E)$, we see that

$$K_{BC_a(X,L)} + (1+r)E = 0 = f^* K_{C_a(X,L)}.$$

As a result, as long as X has trivial canonical divisor, $C_a(X, L)$ is log canonical. If X is an Abelian variety, since $H^i(X, \mathcal{O}_X) \neq 0, \forall i$. By corollary 2.3.3 the cone $C_a(X, L)$ is never Cohen-Macaulay unless $\dim X = 1$. If X is a K3 surface, in particular $H^1(X, \mathcal{O}_X) = 0$. So by corollary 2.3.3, $C_a(X, L)$ has depth equal to three at the vertex. In particular, it is Cohen-Macaulay.

2.3.4 Normal isolated Du Bois singularity

The notion of Du Bois singularities is a generalization of the notion of rational singularities. For a proper scheme of finite type X there exists a complex $\underline{\Omega}^{\bullet}_X$, which is an analogue of De Rham complex. It comes with a natural map $\mathcal{O}_X \to \underline{\Omega}^0_X$ and we say that X has Du Bois singularities if this map is a quasi isomorphism.

In this subsection we consider the case where (X, p) is a normal isolated Du Bois singularity of dimension n, and $f : Y \to X$ is a log resolution such that f is an isomorphism outside of p. We claim that the idea in the previous subsection can be applied to this case. The crucial fact we need is the following,

Theorem 2.3.6. *(Theorem 6.1 in (19)) Take a log resolution $f : Y \to X$ as above, and let E be the reduced preimage of p. Then (X, p) is a normal Du Bois singularity if and only if the natural map*

$$R^i f_* \mathcal{O}_Y \to R^i f_* \mathcal{O}_E$$

is an isomorphism for all $i > 0$.

This theorem implies that $R^i f_* \mathcal{O}_Y(-E) = 0, \forall i > 0$. That is

$$f_* \mathcal{O}_Y(-E) \cong \mathbf{R} f_* \mathcal{O}_Y(-E)$$

Then exactly the same proof as in previous section yields

Theorem 2.3.7. *Given (X, p) is a normal isolated Du Bois singularity of dimension n. For $1 < i < n$, $H^i_p(X, \mathcal{O}_X)$ is dual to $H^{n-i}(E, K_E)$ by Matlis duality. For $i = n$, $H^n_p(X, \mathcal{O}_X)$ is dual to $f_* \mathcal{O}_Y(K_Y + E)$. In particular, $f_* \mathcal{O}_Y(K_Y + E) \cong \omega_X \cong \mathcal{O}(K_X)$.*

Then the corollaries in the previous section also hold.

Remark 2.3.8. *The last statement has been proved in (7) (the Claim in Theorem 2.3).*

CHAPTER 3

A VANISHING THEOREM FOR LOG CANONICAL PAIRS

In this chapter we prove a Nadel vanishing type theorem for log canonical pairs. It partially generalizes the main vanishing theorem in (3), in which the varieties they consider is locally complete intersection with rational singularities. Here we only require the varieties being normal.

The result in the paper of de Fernex and Ein is applied to Castelnuovo-Munford regularity problem of ideal sheaves in projective spaces. More precisely, they consider a locally complete intersection subscheme V of codimension e in $\mathbb{P}^n$. By inversion of adjunction, if V is log canonical then $(\mathbb{P}^n, e \cdot V)$ is a log canonical pair. Then a direct application of their theorem to the ideal sheaf $\mathcal{I}_V$ gives a lower bound of regularity of $\mathcal{I}_V$. Their result generalized a main theorem in (4), and also contains a result in (20) as a special case.

Here we do not follow de Fernex and Ein's idea to consider regularity problems. Instead, we emphasize the method we used to generalise their theorem. The main ingredient of our method is a torsion freeness theorem due to Ambro and Fujino, which is a generalization of Kollár's torsion freeness theorem to log canonical setting.

The results in this chapter has been published by the author in (21).

3.1 Preliminaries

We first recall a more general notion of singularities of pair. Recall that we say (X, Δ) is a pair if X is normal and $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein. More generally, we consider $(X, \Delta; eZ)$ where Z is any subscheme of X and e is a positive integer. Take a log resolution $f : Y \to X$ such that the support of $\mathrm{Exc(f)} \cup f_*^{-1}\Delta \cup \mathcal{I}_Z \cdot \mathcal{O}_Y$ is a union of simple normal crossing divisors.

Definition 3.1.1. *Given a resolution $f : Y \to X$ as above, suppose $\mathcal{I}_Z \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$, we let $div(\mathcal{I}_Z \cdot \mathcal{O}_Y)$ denote $-F$. In particular, $e \cdot div(\mathcal{I}_Z \cdot \mathcal{O}_Y) = -eF$.*

A pair $(X, \Delta; eZ)$ is called log canonical if for any log resolution as above we can write

$$K_Y - f^*(K_X + \Delta) + e \cdot div(\mathcal{I}_Z \cdot \mathcal{O}_Y) = P - N$$

where both P and N are effective, without common components and all coefficients of components of N are less or equal to one.

The following theorem due to Ambro-Fujino is essential to the proof of theorem 3.2.1. See (22) and (23) Theorem 6.3.

Theorem 3.1.2. *Let (Y, Δ) be a simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on Y. Let $f : Y \to X$ be a proper morphism between algebraic varieties and let L be a Cartier divisor on Y such that $L - (K_Y + \Delta)$ is f-semi-ample.*

(i). every associated prime of $R^q f_* O_Y(L)$ is the generic point of the

f-image of some stratum of (Y, Δ).

(ii). let $\pi : X \to V$ be a projective morphism to an algebraic variety V such that

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^*H$$

for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H on X. Then $R^q f \mathcal{O}_Y(L)$ is π-acyclic, that is,

$$R^p\pi_* R^q f_* \mathcal{O}_Y(L) = 0$$

for every $p > 0$ and $q \geq 0$.

From now on we consider a pair $(X, \Delta; eZ)$, where $Z \subset X$ is a pure-dimensional reduced subscheme of codimension e. We also require that all of the irreducible components of Z are not contained in either $\mathrm{sing}(X)$ or $\mathrm{supp}(\Delta)$. We prove the following lemma needed later.

Lemma 3.1.3. *There is a log resolution $f : Y \to (X, \Delta; eZ)$ such that,*

$$K_Y - f^*(K_X + \Delta) + e \cdot div(\mathcal{I}_Z \cdot \mathcal{O}_Y) = P - N - E_Z,$$

where both P and N are effective, and E_Z is the sum of all components of the support of $\mathcal{I}_Z \cdot \mathcal{O}_Y$ mapping to the generic points of Z. And the support of P and N does not contain any divisor mapping to any generic points of Z. In particular, $f(E_Z) = Z$.

Proof. Since no component of Z is contained in $\mathrm{sing}(X)$, we can take factorizing resolution of Z in X. Moreover, since no component of Z is contained in $\mathrm{supp}(\Delta)$, by Corollary 3.2 in (24) we can take this resolution to be a log resolution of (X, Δ). So we have a log resolution

$f : Y_1 \to (X, \Delta)$ such that

(i) f_1 is an isomorphism over the generic points of Z.

(ii)$\mathcal{I}_Z \cdot \mathcal{O}_{Y_1} = \mathcal{I}_Z \cdot \mathcal{L}$, where $\mathcal{L}$ is a line bundle.

(iii)The strict transform $Z_1 \subset Y_1$ of Z is smooth and $Z_1 \cup \mathrm{exc}(f_1) \cup \mathrm{supp}(f_1^{-1}\Delta)$ is simple normal crossing.

Next we take the blow up of Y_1 along Z_1 and denote it by $f_2 : Y_2 \to Y_1$, and let Z_2 be the exceptional divisor of f_2 . In summary, we have the following diagram.

$$\begin{array}{ccc} Z_2 & \hookrightarrow & Y_2 \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ Z_1 & \hookrightarrow & Y_1 \\ \downarrow & & \downarrow{\scriptstyle f_1} \\ Z & \hookrightarrow & X \end{array}$$

Let $K_{Y_1} - f_1^*(K_X + \Delta) = \sum a_i E_i$. Since f_1 is an isomorphism over the generic points of Z, none of the E_i's is mapping to generic points of Z. On the other hand, f_2 is an blowing up of smooth variety along smooth subscheme of codimension e, we have $K_{Y_2} - f_2^* K_{Y_1} = (e-1)Z_2$.

Let $\mathcal{I}_Z \cdot \mathcal{O}_{Y_2} = \mathcal{O}_{Y_2}(-Z_2 - F)$ where Z_2 is mapped to generic points of Z and F is mapped to non-generic points. Then we have

$$\begin{aligned}
& K_{Y_2} - f^*(K_X + \Delta) + e \cdot div(\mathcal{I}_Z \cdot \mathcal{O}_{Y_2}) \\
& = K_{Y_2} - f_2^* K_{Y_1} + f_2^*(K_{Y_1} - f_1^*(K_X + \Delta)) \underbrace{-e \cdot Z_2 - e \cdot F}_{e \cdot div(\mathcal{I}_Z \cdot \mathcal{O}_{Y_2})} \\
& = (e-1) \cdot Z_2 + f_2^*(\sum a_i E_i) - e \cdot Z_2 - e \cdot F \\
& := P - N - Z_2
\end{aligned}$$

Since none of the E_i's passes through the generic point of Z_1, the support of $f_2^*(\sum a_i E_i)$ does not contain any component of Z_2. In conclusion, the support of P and N does not contain any divisor dominating any component of Z. □

3.2 Main Theorem

Theorem 3.2.1. *Suppose the pair $(X, \Delta; eZ)$ is log canonical. Then for any nef line bundles A and M, such that $A \otimes \mathcal{O}(-K_X - \Delta)$ is ample and $M \otimes \mathcal{I}_Z^{\otimes e}$ is globally generated, we have*

$$H^i(X, A \otimes M \otimes \mathcal{I}_Z) = 0 \;\; for\, i > 0.$$

Proof. Let $h : \hat{X} \to X$ be the normalization of blowing up of X along Z, and line bundle $\mathcal{O}_{\hat{X}}(1) = \mathcal{I}_Z \cdot \mathcal{O}_{\hat{X}}$. Note that $\mathcal{O}_{\hat{X}}(1)$ is h-ample and $h_* \mathcal{O}_{\hat{X}}(1) = \bar{\mathcal{I}}_Z = \mathcal{I}_Z$ since $\mathcal{I}_Z$ is a radical ideal.

Take a log resolution $f : Y \to (X, \Delta; eZ)$ as in lemma 3.1.3. Since the inverse image $\mathcal{I}_Z \cdot \mathcal{O}_Y$ is invertible and Y is smooth, we see that f factors through h, with morphism $g : Y \to \hat{X}$. We also note that $g^*\mathcal{O}_{\hat{X}}(1) = \mathcal{I}_Z \cdot \mathcal{O}_Y$. Then by the log canonical assumption we have

$$K_Y - f^*(K_X + \Delta) + e \cdot \operatorname{div}(g^*\mathcal{O}_{\hat{X}}(1)) = P - B - E - E_Z, \tag{3.1}$$

where P is effective, $\lfloor B \rfloor = 0$, E is reduced and E_Z is the components of the support of $\mathcal{I}_Z \cdot \mathcal{O}_Y$ mapping to the generic points of Z. Note that the support of P does not dominate any component of Z by lemma 3.1.3.

To apply theorem 3.1.2, we go as following. First we define $\Delta_Y = \lceil P \rceil - P + B + E$, then (Y, Δ_Y) is a simple normal crossing pair. Then we define L by adding $f^*A + f^*M + \Delta_Y$ to both hand sides of equation (3.1), that is,

$$L := f^*A + f^*M + K_Y - f^*(K_X + \Delta) + e \cdot \operatorname{div}(g^*\mathcal{O}_{\hat{X}}(1)) + \Delta_Y = f^*A + f^*M - E_Z + \lceil P \rceil$$

Note that then L is a line bundle.

To apply Theorem 3.1.2 (2), we calculate

$$\begin{aligned} L - (K_Y + \Delta_Y) &= f^*A - f^*(K_X + \Delta) + f^*M + e \cdot \operatorname{div}(g^*\mathcal{O}_{\hat{X}}(1)) \\ &= g^*(\underbrace{h^*A - h^*(K_X + \Delta)}_{\text{semiample}} + \underbrace{h^*M + e \cdot \mathcal{O}_{\hat{X}}(1)}_{\text{globally genarated}}) \\ &:= g^*H \end{aligned}$$

because by assumption $A - K_X$ is ample and $M \otimes \mathcal{I}_Z^{\otimes e}$ is globally generated. In particular, H is semiample. Plus the fact that $H.C > 0$ for every curve C on $\hat{X}$ (easy to check), we conclude that H is in fact an ample line bundle on $\hat{X}$. Then by Theorem 3.1.2 (2), we have $H^i(\hat{X}, g_*L) = 0$ for $i > 0$. Moreover, H is also h-ample. So by loc. cit. we have $R^j h_* g_* L = 0, \forall j > 0$. As a result,

$$0 = H^i(\hat{X}, g_*L) = H^i(X, h_*g_*L), \forall i > 0. \tag{3.2}$$

The theorem follows from this equation and the next claim.

Claim 3.2.2. *$h_*g_*L = f_*L = A \otimes M \otimes \mathcal{I}_Z$.*

Proof. Since $L = f^*A + f^*M - E_Z + \lceil P \rceil$, by projection formula it suffices to prove

$$h_*\mathcal{O}_{\hat{X}}(1) = h_*(g_*\mathcal{O}_Y(\lceil P \rceil - E_Z)). \tag{3.3}$$

To this aim, note that

$$\mathcal{I}_Z = h_*\mathcal{O}_{\hat{X}}(1) \subset h_*g_*(\mathcal{O}_Y(\lceil P \rceil) + g^*\mathcal{O}_{\hat{X}}(1)) \subset h_*g_*\mathcal{O}_Y(\lceil P \rceil - E_Z) \subset \mathcal{O}_X$$

In other words, $h_*g_*\mathcal{O}_Y(\lceil P \rceil - E_Z)$ is an ideal sheaf on X. By Lemma 3.1.3, $\lceil P \rceil$ and E_Z do not have common components and $\lceil P \rceil$ is an effective exceptional divisor, so

$$h_*g_*\mathcal{O}_Y(\lceil P \rceil - E_Z) = f_*\mathcal{O}_Y(\lceil P \rceil - E_Z) = f_*\mathcal{O}_Y(-E_Z).$$

Note that $f_*\mathcal{O}_Y(-E_Z)$ is an ideal sheaf determines a scheme set theoretically equal to Z. So we have

$$\mathcal{I}_Z \subset f_*\mathcal{O}_Y(-E_Z) \subset \sqrt{\mathcal{I}_Z}.$$

Hence we have equation (3.3) and the claim. □

□

With same notations as Theorem 3.2.1,

Theorem 3.2.3. *The conclusion of Theorem 3.2.1 is still true if* $(X, \Delta; eZ)$ *is log canonical except at finitely many points.*

Proof. Apply Lemma 3.1.3, we have the following equation similar to equation (3.1.3),

$$K_Y - f^*(K_X + \Delta) + e \cdot \mathrm{div}(\mathcal{I}_Z \cdot \mathcal{O}_Y) = P - N - E_Z - F.$$

The only difference is that there is a term $-F$ denoting the exceptional over the non-log canonical points. We remark that by the assumption F is non-reduced.

Let $\mathcal{I}^* = f_*\mathcal{O}_Y(-E_Z - F)$. Then exactly the same argument as in the proof of Theorem 3.2.1 shows

$$H^i(X, A \otimes M \otimes \mathcal{I}^*) = 0, \forall i > 0.$$

On the other hand we have,

$$0 \to \mathcal{I}^* \to \mathcal{I}_Z \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is a sheaf supported on some close points by assumption. Take the long exact sequence we have

$$H^i(X, A \otimes M \otimes \mathcal{I}_Z) = 0, \forall i > 0.$$

□

CHAPTER 4

NON $\mathbb{Q}$-GORENSTEIN SINGULARITIES

In this chapter we study non $\mathbb{Q}$-Gorenstein singularities. Recall that a crucial assumption of log pairs is that there exists a boundary Δ so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (See section 2.1.1). In (6), de Fernex and Hacon proposed a new notion of singularity of pairs. Instead of assuming $K_X + \Delta$ is $\mathbb{Q}$-Cartier, they define the pull back of canonical divisors in an asymptotic way.

Without considering boundaries not only gives a more general way to study singularities, but also has the advantage of being more flexibility. For example, it is will known that a generic determinantal variety has rational singularities. But if we want to prove it is log terminal, we first need to find a suitable boundary since in general determinantal varieties are not $\mathbb{Q}$-Gorenstein.

In this chapter, we use the notions of de Fernex-Hacon to study singularities of non $\mathbb{Q}$-Gorenstein normal varieties such as generic determinantal varieties, Schubert varieties and Richardson varieties. The results in this chapter have been published by the author in (25). We remark that after finishing (25), the author noticed that more general results are obtained by (26) and (27) using different methods.

4.1 Preliminaries

we recall some definitions and prove some lemmas that will be used in the proofs of the main theorems. First, we would like to generalize the notion of relative canonical divisor to normal varieties. The idea and notations will be mainly based on the paper (6).

Definition 4.1.1. *Let $f : Y \to X$ be a morphism between two normal varieties. Since Y is normal, for any prime divisor E there is a valuation $V_E(\cdot)$ defined by the local ring of $\mathcal{O}_{Y,E}$. For any ideal sheaf $\mathcal{I}$ on Y, the valuation $Val_E(\mathcal{I})$ is defined as*

$$Val_E(\mathcal{I}) := min\{V_E(\phi) \mid \phi \in \mathcal{I}(U), U \cap E \neq \emptyset\}$$

For any Weil divisor D on X, we define the natural pull back as

$$f^\natural(D) := div(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y) := \sum_{E \subset Y} Val_E(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y).$$

One of the properties of the natural pull back is that

$$\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}.$$

The natural pull back usually does not have the homogeneity property, i.e, $f^\natural(mK_X) \neq mf^\natural(K_X)$ (See (6) for example). As a result, instead of defining the relative canonical divisor directly, we define the following:

Definition 4.1.2. *The m-th limiting relative canonical* $\mathbb{Q}-$*divisor* $K_{m,Y/X}$ *is*

$$K_{m,Y/X} := K_Y - \frac{1}{m} f^{\natural}(mK_X).$$

Given any prime divisor F *on* Y*, we define the m-th limiting discrepancy of* X *to be*

$$a_{m,F}(X) := \mathrm{ord}_F(K_{m,Y/X})$$

X *is said to be lc (resp. lt) if there is some positive integer* m_0 *such that* $a_{m_0,F}(X) \geq -1$ *(resp.* > -1*) for every prime divisor* F *over* X*.*

Let $g : V \to Y$ and $f : Y \to X$ be two birational morphisms, usually we do not expect that $(fg)^{\natural}(mK_X) = g^{\natural}(f^{\natural}mK_X)$. However, the equality holds when $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ is an invertible sheaf (Lemma 2.7 in (6)). A consequence of this property is the following lemma,

Lemma 4.1.3. *(Lemma 3.5 in (6)) Let* m *be a positive interger, and let* $f : Y \to X$ *be a proper birational morphism from a normal variety* Y *such that* mK_Y *is Cartier and* $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ *is invertible. Then for every birational morphism* $g : V \to Y$ *we have*

$$K_{m,V/X} = K_{m,V/Y} + g^* K_{m,Y/X}$$

In particular, when study the singularities of a given variety X, it suffices to find a log resolution of $(X, \mathcal{O}_X(-mK_X))$. More precisely, we just need to find a proper birational morphism $f : Y \to X$ such that Y is smooth and $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ is the invertible sheaf of a divisor

F, and the exceptional locus is a divisor E such that $F \cup E$ is simple normal crossing. The reason is that for any divisor E over X, by taking a common resolution, we can assume E is on a higher resolution $g : V \to Y$. Then since Y is smooth, the order of $K_{m,V/Y}$ over E must be nonnegative, hence will not affect the type of the singularities of X.

Moreover, for any $m \geq 2$, we can find an *m-compatible* boundary Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and

$$K^{\Delta}_{Y/X} := K_Y + \Delta_Y - f^*(K_X + \Delta) = K_{m,Y/X} \tag{4.1}$$

See (6) for the definition of m-compatible boundary.

So if X is log terminal (resp, log canonical) in the new notion, there is a boundary Δ (m-compatible for some positive integer m) such that (X, Δ) is Kawamata log terminal (resp, log canonical) in the classical sense. On the other hand, for any boundary Δ such that $m(K_X + \Delta)$ is Cartier, we have $K^{\Delta}_{Y/X} \leq K_{m,Y/X}$ (Remark 3.9 in (6)). As a result, we conclude that

Proposition 4.1.4. *(Proposition 7.2 in (6)) X is log canonical (resp. log terminal) if and only if there is a boundary Δ such that (X, Δ) is a log canonical pair (resp. log terminal pair).*

Corollary 4.1.5. *Let X be a normal variety with log terminal singularities, then X has rational singularities.*

Remark 4.1.6. *In (6), the authors also proposed a notion for being canonical and terminal, however, there is an example which is canonical but not log terminal, see (28).*

4.2 Main theorems

Theorem 4.2.1. *Let X be a homogeneous variety with a group variety G acting on it. Let Y, Z be subvarieties of X such that Z is smooth and Y is log canonical (resp. log terminal). For any $g \in G$, denote Y^g the transition of Y by g. Then there exists a non empty open $U \subset G$ such that $\forall g \in U$, $Y^g \times_X Z$ is log canonical (resp. log terminal).*

We begin by proving some lemmas which will be used in the proof of this theorem.

Lemma 4.2.2. *Let $f : X \to Y$ be a surjective, smooth morphism onto a normal variety Y with log canonical (resp. log terminal) singularities, then X is also a normal variety with log canonical (resp. log terminal) singularities.*

Proof. Take a log resolution $g : \tilde{Y} \to Y$ with smooth $\tilde{Y}$ and do base change as shown in the following diagram. Since f is smooth and Y is normal, X is also a normal variety((29) Theorem VII.4.9). Also $\tilde{f}$ is smooth morphism by base change, so we conclude that $\tilde{X}$ is a smooth variety (Theorem VII.5.3 in (29)) and $\tilde{g}$ is a resolution of X.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{g} & Y \end{array}$$

First note that f and $\tilde{f}$ are flat, so it makes sense to pull back Weil divisor (take base change) and the pull back is just the preimage of the divisor. We claim that for any Weil divisor D on Y, we have the following commutative property,

$$\tilde{f}^*(g^\natural D) = \tilde{g}^\natural(f^* D). \tag{4.2}$$

Since $\tilde{f}$ is smooth, pull back of any reduced divisor is still reduced ((29) Theorem 7.4.9). The divisorial parts of ideals $(\mathcal{O}_Y(-D)\cdot\mathcal{O}_{\tilde{Y}})\cdot\mathcal{O}_{\tilde{X}}$ and $(\mathcal{O}(-g^{\natural}D))\cdot\mathcal{O}_{\tilde{X}}$ are equal to each other, which is the pull back of $g^{\natural}D$ by $\tilde{f}$. Also by commutativity of the diagram, $\tilde{g}^{\natural}(f^*D)$ is the divisor determined by the ideal $(\mathcal{O}_Y(-D)\cdot\mathcal{O}_{\tilde{Y}})\cdot\mathcal{O}_{\tilde{X}}$. As a result, we can conclude equation (4.2).

For two different canonical divisors $K_{\tilde{Y}}$, $K'_{\tilde{Y}}$ on $\tilde{Y}$, $g_*(K_{\tilde{Y}})$ is linear equivalent to $g_*(K'_{\tilde{Y}})$. So without loss of generality, we can fix $K_{\tilde{Y}}$ (resp. $K_{\tilde{X}}$) and assume $g_*(K_{\tilde{Y}}) = K_Y$ (resp. $\tilde{g}_*(K_{\tilde{X}}) = K_X$). Since f is smooth, the relative differential $\Omega_{X/Y}$ is free. Let E be a divisor linear equivalent to the highest exterior power of $\Omega_{X/Y}$ and note that E is Cartier. Now we claim that

$$\tilde{g}^{\natural}(mK_X) = m\tilde{g}^*(E) + \tilde{f}^*(g^{\natural}(mK_Y)) \tag{4.3}$$

This is because

$$\begin{aligned}
\tilde{g}^{\natural}(mK_X) &= \tilde{g}^{\natural}(m\tilde{g}_*K_{\tilde{X}}) \\
&= \tilde{g}^{\natural}(m\tilde{g}_*(K_{\tilde{X}} - \tilde{f}^*K_{\tilde{Y}} + \tilde{f}^*K_{\tilde{Y}})) \\
&= \tilde{g}^{\natural}(m\tilde{g}_*(\tilde{g}^*(E) + \tilde{f}^*K_{\tilde{Y}})) && \text{(by base change of relative differential)} \\
&= m\tilde{g}^*(E) + \tilde{g}^{\natural}(m\tilde{g}_*(\tilde{f}^*K_{\tilde{Y}})) && \text{(since } E \text{ is Cartier)} \\
&= m\tilde{g}^*(E) + \tilde{g}^{\natural}(mf^*(g_*K_{\tilde{Y}})) && \text{((30) Proposition 1.7 in chapter one)} \\
&= m\tilde{g}^*(E) + \tilde{g}^{\natural}(mf^*(K_Y)) \\
&= m\tilde{g}^*(E) + \tilde{f}^*(g^{\natural}(mK_Y)) && \text{(by equation(4.2))}
\end{aligned}$$

Then we can calculate $K_{m,\tilde{X}/X}$ as following,

$$\begin{aligned} K_{m,\tilde{X}/X} &:= K_{\tilde{X}} - \frac{1}{m} \cdot \tilde{g}^{\natural}(mK_X) \\ &= K_{\tilde{X}} - \frac{1}{m} \cdot (m\tilde{g}^*(E) + \tilde{f}^*(g^{\natural}(mK_Y))) \\ &= K_{\tilde{X}} - \frac{1}{m} \cdot (m(K_{\tilde{X}} - \tilde{f}^* K_{\tilde{Y}}) + \tilde{f}^*(g^{\natural}(mK_Y))) \\ &= \tilde{f}^*(K_{\tilde{Y}} - \frac{1}{m} f^{\natural}(mK_Y)) = \tilde{f}^*(K_{m,\tilde{Y}/Y}) \end{aligned}$$

So If we write $K_{m,\tilde{Y}/Y} = \sum a_i E_i$, where E_i's are simple normal crossing exceptional divisors of g. Then $K_{m,\tilde{X}/X} = \sum a_i E_i'$, where E_i''s are the preimages of E_i's. Note that E_i''s are also reduced since $\tilde{f}$ is smooth. As a result, we conclude that X is lc (resp. lt) is Y is lc (resp. lt).

□

Lemma 4.2.3. *(Bertini) Let X be a normal scheme with lc (resp. lt) singularities. Let f be a morphism from X to a projective variety Y. Then for a general point in Y, the fiber has lc (resp. lt) singularities.*

Proof. First we assume that $dimY = 1$. Take a very ample divisor A on Y, consider the base point free linear system $|f^*A|$ on X. Note that f^*A is an union of fibers and we claim that they have the same singularities as X does.

By the result of Proposition 2.4, there is a $\mathbb{Q}$-Cartier divisor Δ such that (X, Δ) is a lc (resp. lt) pair. Let B be a general member in $|f^*A|$. Then by (31) Proposition 7.7,

$$\text{discrep}(B, \Delta|_B) \geq \text{discrep}(X, \Delta).$$

As a result, B is lc (resp. lt) if X is lc (resp. lt). Since B is an union of fibers, we get the conclusion in this case.

For higher dimension Y, we do the same procedure as above. Then replace X by B, Y by $f(B)$ with reduced scheme structure. Note that $f(B)$ has dimension at most equal to $dimB$, and B has codimension one in X. So in each step, the dimension of Y will drop by at least one. Then by induction, the conclusion follows.

□

The following lemma, which will be used in the proof of the next proposition, is a slight modification of Theorem 5.4 in (6).

Lemma 4.2.4. *Let X be a normal variety and S a Cartier divisor on it. Then for any $m \geq 2$, we can find a boundary Δ on X such that $\Delta|_S$ is a m-compatible boundary of S.*

Proof. Take an effective divisor D such that $K_X - D$ is Cartier, by adjunction formula, $(K_X + S - D)|_S = K_S - D_S$ is also Cartier. Let $f : \tilde{X} \to X$ be a log resolution of $((X, S), \mathcal{O}(-mK_X))$, and we assume it is high enough that f restricts to a resolution $\tilde{f} : \tilde{S} \to S$ as shown by the following diagram,

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & \tilde{X} \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ S & \longrightarrow & X \end{array}$$

Denote $\tilde{f}^{\natural}(mD_S)$ as E_S. Since $K_S - D_S$ is Cartier, by Lemma 2.4 in (6) and definition of natural pull back, we have

$$\begin{aligned}\tilde{f}^{\natural}(mK_S) &= \tilde{f}^{\natural}(m(K_S - D_S) + mD_S) \\ &= \tilde{f}^{*}(m(K_S - D_S)) + E_S\end{aligned}$$

As a result, we can write

$$K_{m,\tilde{S}/S} = K_{\tilde{S}} - \tilde{f}^{*}(K_S - D_S) - \frac{1}{m}E_S$$

On the other hand, take a line bundle L on X positive enough such that both $L \otimes \mathcal{O}_X(-mD)$ and its restriction to S, $L|_S \otimes \mathcal{O}_S(-mD_S)$ are globally generated. Let M be general section of $L \otimes \mathcal{O}_X(-mD)$ such that, as an effective divisor, M is reduced and has no common component with D and S. Define Δ_S as $\frac{1}{m}M|_S$. Let $G = M + mD$ and denote its restriction to S as $G_S := M_S + mD_S$, then $K_S + \Delta_S = K_S - D_S + \frac{1}{m}G_S$ is $\mathbb{Q}$-Cartier. (Note that G_S is Cartier.) We also have the following equalities:

$$\begin{aligned}E_S &= \tilde{f}^{\natural}(mD_S) \\ &= \tilde{f}^{\natural}(mD_S + M_S - M_S) \\ &= \tilde{f}^{\natural}(G_S - M_S) \\ &= \tilde{f}^{*}(G_S) + \tilde{f}^{\natural}(-M_S)\end{aligned}$$

Since we choose M general, $\tilde{f}^{\natural}(-M_S) = \tilde{f}_*^{-1}(-M_S)$.

Then we have

$$
\begin{aligned}
K_{\tilde{S}/S}^{\Delta_S} &:= K_{\tilde{S}} + \tilde{f}_*^{-1}\Delta_S - \tilde{f}^*(K_S + \Delta_S) \\
&= K_{\tilde{S}} + \frac{1}{m}\tilde{f}_*^{-1}(m\Delta_S) - \frac{1}{m}\tilde{f}^*(m(K_S + \Delta_S)) \\
&= K_{\tilde{S}} - \frac{1}{m}\tilde{f}_*^{-1}(-M_S) - \frac{1}{m}\tilde{f}^*(mK_S + M_S - G_S + G_S) \\
&= K_{\tilde{S}} - \frac{1}{m}\tilde{f}_*^{\natural}(-M_S) - \frac{1}{m}\tilde{f}^*(m(K_S - D_S)) - \frac{1}{m}f^*(G_S) \\
&= K_{\tilde{S}} - \tilde{f}^*(K_S - D_S) - \frac{1}{m}E_S
\end{aligned}
$$

Hence $K_{m,\tilde{S}/S} = K_{\tilde{S}/S}^{\Delta_S}$, i.e, $\Delta_S = \Delta|_S$ is a m-compatible boundary of S.

□

Lemma 4.2.5. *Let $f : X \to Z$ be a morphism from a normal variety X to a smooth variety Z. If every fiber of $f$is log canonical (resp. log terminal), then X is log canonical (resp. log terminal).*

Proof. Given any point $x \in X$, let $z \in Z$ such that x belongs to the fiber of z. Assume dim$Z = 1$, z can be thought of as a Catier divisor and so is f^*z. Denote the Weil divisor corresponding to f^*z as S, then by lemma 4.2.4, we can find a m-compatible boundary Δ_S such that $\Delta_S = \Delta|_S$ for some Δ on X. By assumption, (S, Δ_S) is a lc (resp. klt) pair. Using Inversion of adjunction Theorem ((10) Theorem 5.50), we conclude that $(X, \Delta + S)$ is a lc (resp. plt) pair in a neighborhood of x . By Proposition 2.4, this is equivalent to saying that X is lc (resp. lt) near x. The argument applies to every point of X, so the proposition follows in this case.

If Z has higher dimension, say n, since Z is smooth so every close point of Z can be cut out by n local coordinates. In particular, these local coordinates are Cartier divisors, so are their pull backs. As a result, we can apply Lemma 4.2.4 and inversion of adjunction to this case. Then by induction on dimension of Z, the proposition follows.

□

Now We are ready to prove Theorem 4.2.1.

Proof. Consider the morphism $\Phi : G \times Y \to X$ defined by the action of G on X. Given $x \in X$, the preimage $\Phi^{-1}(x)$ by definition is

$$\Phi^{-1}(x) = \{(g, y) \in G \times Y | g \cdot y = x\}$$

Now we consider the map $P : \Phi^{-1}(x) \to Y$ induced by the projection from $G \times Y$ to Y as shown in the following diagram. Since G acts transitively on X, P is a surjective morphism.

$$\begin{array}{ccc} \Phi^{-1}(x) & \xrightarrow{\Phi} & x \in X \\ \downarrow{\scriptstyle P} & & \\ Y & & \end{array}$$

Fix $y \in Y$, we claim that the preimage $P^{-1}(y) \cong \mathrm{stab}_G(y)$. Since G acts transitively on X, so we can find $g' \in G$ such that $g' \cdot x = y$. Then we have,

$$\begin{aligned} P^{-1}(y) &\cong \{g \in G | g \cdot y = x\} \\ &\cong \{g \in G | g' g \cdot y = y\} \\ &\cong \mathrm{stab}_G(y). \end{aligned}$$

Also note that Y is a subvariety of X, and G acts transitively on Y. As a result, every fiber of P is isomorphic to $\mathrm{stab}_G(y)$ for some fixed y. By Theorem III 9.9 and III 10.2 in (18), we conclude that P is a smooth morphism. Then by lemma 4.2.2, $\Phi^{-1}(x)$ is lc (resp. lt).

Next we consider the following diagram,

$$\begin{array}{ccc} W & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle g} & & \downarrow \\ G \times Y & \xrightarrow{\Phi} & X \end{array}$$

where $W = (G \times Y) \times_X Z$. It is easy to see that every fiber of Φ is isomorphic to each other, so Φ is a flat morphism. By base change, h is also flat. This will imply W is S_2 Also Z is smooth, so the singular locus of W lies in the fiber, so W is also R_1. By Serre condition, W is normal. So it makes sense to apply the notion of singularities here. By Proposition 4.2.5, W is lc (resp. lt).

Composite with $p : G \times Y \to G$, we have map from W to G. Apply Proposition 4.2.3, we can see that general fibers of the map from W to G are lc (resp. lt). But for any $g \in G$, the fiber is just $Y^g \times_X Z$, thus the proposition is proved.

□

Theorem 4.2.6. *Let $f : Y \to X$ be a small resolution of a normal variety X, such that Y is smooth and $-K_Y$ is relatively nef. Then X is log terminal (in the sense of de Fernex and Hacon). Moreover, there is a boundary Δ such that (X, Δ) is canonical (in the old sense).*

Here small means the exceptional locus is of codimension bigger than two.

Proof. The goal is to prove that for some $m \gg 0$, we have the following isomorphism:

$$(\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y) \cong \mathcal{O}_Y(-mK_Y). \tag{4.4}$$

So the corresponding divisors are exactly the same, or equivalently, $K_{m,Y/X} = 0$. In particular, $(\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y)$ is an invertible sheaf. If $f : Y \to X$ is already a log resolution, then we are done by Lemma 4.1.3 and the argument following it. If not, we do further blowing up. But since Y is smooth, blowing up will just make the relative canonical divisor even more effective. In any case, X is log terminal. Showing equation (4.4) is a local problem, so from now on we assume X is affine.

Let U' and U be open sets such that the restriction map $f|_{U'} : U' \to U$ is an isomorphism. As a result, for any $m \geq 0$ we have the following equalities.

$$\begin{aligned}
\Gamma(Y, \mathcal{O}_Y(-mK_Y)) &\cong \Gamma(U', \mathcal{O}_Y(-mK_Y)|U') \\
&\cong \Gamma(U, \mathcal{O}_X(-mK_X)|U) \\
&\cong \Gamma(X, \mathcal{O}_X(-mK_X))
\end{aligned}$$

The first and last equalities come from the fact that the complements of $U$and U' are of codimension at least two. And the middle one comes from the isomorphism. Since X is affine, $\mathcal{O}_X(-mK_X)$ is naturally generated by global sections for any $m \geq 0$. So if we can show that for some m, $\mathcal{O}_Y(-mK_Y)$ is also generated by global sections, we can conclude equation (1) since both these two sheaves have the same global sections as we proved. To this aim, we need the following lemma:

Lemma 4.2.7. *(Relative Basepoint Free Theorem, c.f. (10)) Let Y be a smooth variety, $f : Y \to X$ be a projective morphism. Let D be an f-nef Cartier divisor and $aD - K_Y$ is f-nef and f-big for some $a \geq 0$, then bD is f-free for some $b \gg 0$.*

In our case, $-K_Y$ is the divisor D in this lemma. It is f-nef by the condition in the statement and f-big because f is a birational morphism so the general fiber is of dimension zero (recall that a divisor D being f-big means that the restriction of D to a general fiber is big). As a result, by the lemma we have a surjective map $f^* f_* \mathcal{O}_Y(-bK_Y) \to \mathcal{O}_Y(-bK_Y)$ for some $b \gg 0$, or equivalently, there is a morphism $Y \to \mathbb{P}_X(f_* \mathcal{O}_Y(-bK_Y))$ determined by the surjective morphism. So we can conclude that the sheaf $\mathcal{O}_Y(-bK_Y)$ is generated by $\Gamma(X, f_* \mathcal{O}_Y(-bK_Y)) \cong \Gamma(Y, \mathcal{O}_Y(-bK_Y))$. Thus the first part of this theorem is proved.

As shown by the above arguments, there is an integer $m > 0$ such that $(\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y)$ is invertible and $K_{m,Y/X} = 0$. Then the second part of this theorem follows directly from Lemma 4.1.3. □

Remark 4.2.8. *The proof of the second part of Theorem 4.2.6 actually proves that X satisfies condition $M_{\geq 0}$ of Chiecchio and Urbinati (32). Roughly speaking, a variety is said to satisfy condition $M_{\geq -1}$ (resp. $M_{>-1}$, resp. $M_{\geq 0}$, resp. $M_{>0}$) if there is an integer m_0 such that the m-th limiting discrepancy $a_{m,F}(X) \geq -1$ (resp. > -1, resp. ≥ 0, resp. > 0) for any prime divisor F over X and $m = m_0$. In Theorem 4.5 of (32), the authors show that condition $M_{\geq -1}$ (resp. $M_{>-1}$, resp. $M_{\geq 0}$, resp. $M_{>0}$) is equivalent to the existence of Δ such that (X, Δ) is log canonical (resp. klt, resp. canonical, resp. terminal.)*

4.3 Applications

4.3.0.1 Generic Determinantal Varieties

In this section, we consider non zero linear maps from a dimension e vector space E to a dimension f vector space F. The collection of all these maps can be naturally represented by $X := \mathbb{P}^{ef-1}$, each point corresponds to a $f \times e$ matrix modulo scalar. Let X_k be the the locus on X where the corresponding map is of rank at most k. It is well known that X_k is normal, and Cohen-Macaulay if it is non empty. In this section, we would like to prove moreover that it is log terminal.

First we would like to construct a resolution of X_k. Consider the following incidence correspondence,

$$Y_k := \{(A, V) \in X \times G(e-k, E) | AV = 0\},$$

where $G(e-k, E)$ is Grassmannian of $e-k$ subspace of E. To simplify notations, let $G(e-k, E)$ be denoted by $\mathbb{G}$ from now on. On $\mathbb{G}$ we have the tautological bundle sequence $0 \to S \to E \to$

$Q \to 0$, where S is the universal sub bundle of rank $e-k$. Then the projection map $p_2 : Y_k \to \mathbb{G}$ can be thought as the total space of projective bundle $\mathbb{P}(Q^* \otimes F)$ over $\mathbb{G}$, so Y_k is smooth.

Now we prove $p_1 : Y_k \to X_k$ is a small resolution. First note that by the construction $p_1 : Y_k \to X$ is a surjective map to X_k. Also since we are considering generic determinantal varieties, X_k is of codimension $(e-k)(f-k)$ in X, and the singular locus is X_{k-1}. For any non-singular point $A \in X_k$, A is of exact rank k, so the fiber in Y_k is an unique point $(A, kerA)$. Since the non-singular locus is an open dense set in X_k, p_1 is a birational map. We also know that X_{k-1} is of codimension $e-k+f-k+1$ in X_k, and over the generic point of X_{k-1}, the fiber is of dimension $e-k$. So the exceptional locus is of codimension bigger than two, i.e., p_1 is a small resolution.

Next we show that the anticanonical bundle $-K_{Y_k}$ on Y_k is nef. To this aim, we calculate the first Chern class of the tangent bundle T_{Y_k}. Note that on Y_k we also have tautology sequence $0 \to U \to Q^* \otimes F \to (Q^* \otimes F)/U \to 0$, U is of rank one. (Here we ignore all the pull back symbol to simplify the expression). Then we can write the first Chern class of T_{Y_k} as

$$\begin{aligned} -c_1(K_{Y_k}) = c_1(T_{Y_k}) &= c_1(T_{\mathbb{G}}) + c_1(T_{Y_k/\mathbb{G}}) \\ &= c_1(S^* \otimes Q) + c_1(U^* \otimes (Q^* \otimes F)/U) \\ &= e \cdot \sigma_1 + (kf-1) \cdot \sigma_2 - f \cdot \sigma_1 + \sigma_2 \\ &= (e-f) \cdot \sigma_1 + kf \cdot \sigma_2 \end{aligned}$$

where σ_1 (resp, σ_2) denote the first Chern class of S^* (resp, U^*). So we have $-K_{Y_k}$ is nef if $e \geq f$ and by Theorem 1.2 X_k is log terminal in this case.

On the other hand, if $e \leq f$ we consider the following resolution,

$$Z_k := \{(A, V) \in X \times G(k, F) | Im(A) \subseteq V\},$$

Over $\mathbb{G} := G(k, F)$, there is a universal sequence $0 \to S \to F$ where S is of rank k. Then Z_k can be seen as total space of the bundle $\mathbb{P}(E^* \otimes S) \to \mathbb{G}$. Following the same argument as above, $p_2 : Z_k \to X_k$ is a small resolution. Moreover, still denote the tautology bundle on $\mathbb{P}(E^* \otimes S)$ by U, we have

$$\begin{aligned} c_1(-K_{Z_k}) = c_1(T_{Z_k}) &= c_1(T_{\mathbb{G}}) + c_1(T_{Z_k/\mathbb{G}}) \\ &= c_1(S^* \otimes Q) + c_1(U^* \otimes (E^* \otimes S)/U) \\ &= f \cdot \sigma_1 + (ke - 1) \cdot \sigma_2 - e \cdot \sigma_1 + \sigma_2 \\ &= (f - e) \cdot \sigma_1 + ke \cdot \sigma_2 \end{aligned}$$

is nef again. So we prove in the case $e \leq f$, Z_k is a resolution of X_k with $-K_{Z_k}$ being nef. In conclusion, by Theorem 4.2.6, the generic determinantal varieties are log terminal, and if we chose suitable boundary divisors they are canonical as pairs.

Remark 4.3.1. *The result that the generic determinantal varieties being log terminal was first proved by Israel Vainsencher in (33), see also section 3 in (34) for explicit calculation of log discrepancies.*

After finishing this work, the author was informed that J-C Hsiao (35) also got the same result independently by different method.

4.3.0.2 W_d^r in $\mathbf{Pic}^d(C)$

We first recall the definition and construction of W_d^r.

$$Supp(W_d^r) = \{L \in \mathbf{Pic}^d(C) | h^0(L) \geq r+1\}$$

We give W_d^r a scheme structure in the following way: Consider projection map

$$v : C \times \mathbf{Pic}^d(C) \to \mathbf{Pic}^d(C)$$

Fix a Poincare bundle $\mathcal{L}$ on $C \times \mathbf{Pic}^d(C)$ and take an effective divisor E on C positive enough so that $H^1(C, \mathcal{L}|_C \otimes E) = 0$, or equivalently, $R^1 v_* \mathcal{L}(\Gamma) = 0$ where Γ is defined as $E \times \mathbf{Pic}^d(C)$. Then we have an induced map $\gamma : v_* \mathcal{L}(\Gamma) \to v_* \Gamma$ between two locally free sheaves of rank $d+m+1-g$ and m respectively. W_d^r is defined as the degeneracy locus of $rk(\gamma) \leq m+d-g-r$ in $\mathbf{Pic}^d(C)$. It is of expected dimension $\rho = g-(r+1)(g-d+r)$ and is non empty if $\rho \geq 0$ ((36)).

Denote $v_*\mathcal{L}(\Gamma)$ by K^0 and $v_*\Gamma$ by K^1. There is a natural resolution of W^r_d constructed as the following: Consider the Grassmann bundle $\pi : G(r+1, K^0) \to \mathbf{Pic}^d(C)$, with an universal sequence

$$0 \to S \to \pi^* K^0 \to Q \to 0$$

then the resolution of W^r_d is defined as $G^r_d(C) := Zero(S^* \otimes \pi^* K^1)$. It is smooth by the following theorem,

Theorem 4.3.2. *(Smoothness Theorem (36) V.1.6) Let C be a general curve of genus g. Let d,r be integers such that $d \geq 1$ and $r \geq 0$. Then $G^r_d(C)$ is smooth of dimension ρ.*

Since G^r_d is smooth, we have the exact sequence

$$0 \to T_{G^r_d} \to T_{Gr}|_{G^r_d} \to N_{G^r_d/Gr} \to 0$$

where Gr is the Grassmann bundle $G(r+1, K^0) \to \mathbf{Pic}^d(C)$ and $N_{G^r_d/Gr}$ in this case is $S^* \otimes \pi^* K^1$. So this exact sequence can be translated to

$$0 \to T_{G^r_d} \to (S^* \otimes \pi^* K^0/S)|_{G^r_d} \to S^* \otimes \pi^* K^1 \to 0$$

As a result, $c_1(T_{G^r_d}) = (d+1-g) \cdot c_1(S^*)$, i.e., $T_{G^r_d}$ is relatively nef if $d \geq g-1$. However, by the natural correspondence $L \to K_c \otimes L^\vee$ we have isomorphism $W^r_d \cong W^{g-d+r-1}_{2g-2-d}$. The $d \leq g-1$ case is covered by this isomorphism, so it suffices to assume $d \geq g-1$.

By the Riemann-Roch theorem, for $r \leq d-g$ we have $W^r_d(C) = \mathbf{Pic}^d(C)$, which is smooth , so we can limit ourself to $r > d-g$. In this case, the singular locus of $W^r_d(C)$ is $W^{r+1}_d(C)$, and the dimension will be:

$$\begin{aligned} dim(W^{r+1}_d(C)) &= g-(r+2)(r+1-d+g) \\ &= \rho-(r+1)-(r-d+g)-1 \end{aligned}$$

However the general fiber of π restricted to $W^{r+1}_d(C)$ is of dimension $r+1$, so the exceptional locus of π is of codimension at least two, That is, π is a small resolution. Combining the result and Theorem 1.2, we conclude that

Proposition 4.3.3. *Let C be a general smooth curve, then $W^r_d(C)$ is log terminal. In particular, it has rational singularities.*

4.3.0.3 Schubert varieties in $G(k,n)$

In this section, we give some sufficient conditions for a Schubert variety in $G(k,n)$ to be log terminal. The main step is to find a resolution by tower of Grassmann bundles, and then prove that it satisfies all the conditions we need. For the reader's convenience, a brief introduction to Schubert varieties is given in the first subsection. The details can be found in (37). In the second subsection, we give the construction of a resolution, and then show that it is a small contraction in the last subsection.

We start by recalling some preliminaries of Schbert varieties. Fix a complete flag $F_0 \subset F_1 \subset \cdots \subset F_n$, where F_l is an l dimension vector space for $0 \leq l \leq n$. Then Schubert varieties are characterized by partitions as the following: Given a partition $\lambda = (\mu_1^{i_1}, \mu_2^{i_2} \ldots, \mu_j^{i_j})$ such that $n-k \geq \mu_1 > \ldots > \mu_j \geq 0$ and $\sum_{j=s}^{j} i_s = k$, the Schubert variety X_λ is defined as

$$X_\lambda = \{V \in G(k,n) \mid \dim(V \cap F_{n-k+a_s-\mu_s}) \geq a_s, 1 \leq s \leq j)\} \tag{4.5}$$

where a_s is $\sum_{l=1}^{s} i_l$, in particular, $a_j = k$.

The Schubert cell of X_λ is the locus where all these dimension conditions hold with equalities, it is a smooth dense open set in X_λ. The complement of the Schubert cell is a union of some codimension one subvarieties, $\cup_{t=1}^{t=j} D_t$. In terms of partition, D_t corresponds to the partitions $(\mu_1^{i_1}, \ldots, \mu_{t-1}^{i_{t-1}}, \mu_t + 1, \mu_t^{i_t - 1}, \ldots, \mu_j^{i_j})$. The divisor class group $\mathrm{Cl}(X_\lambda)$ is generated by these *Schubert divisors*, hence we can express the canonical divisor as a linear combination of D_t's. More specifically, it is (c.f (38), Proposition 2.2.8)

$$K_{X_\lambda} = -(n-\mu_1)D_1 - (n-\mu_2-a_1)D_2 - \cdots - (n-\mu_j-a_{j-1})D_j \tag{4.6}$$

For future reference, we write $K_{X_\lambda} = c_1 D_1 + c_2 D_2 + \cdots + c_j D_j$.

Now we recall Bott-Samelson resolution. (see (39), (40)) To illustrate the idea, we assume $j = 3$. X_λ corresponds to partition $\lambda = (\mu_1^{i_1}, \mu_2^{i_2}, \mu_3^{i_3})$. Or equivalently, it is the collection of k dimensional vector subspace V satisfying $\dim(V \cap F_i) \geq a_i$, where F_i is a vector space of dimension $f_i = n - k + a_i - \mu_i$ and $a_i = \sum_{l=1}^{i} i_l$. There are two methods to resolve the singularities :

Method1 :(Resolve by Intersection)

We consider the flag variety $F(a_1, a_2, a_3 : n)$, where every point is a tower $V_1 \subset V_2 \subset V_k \subset F_n$ of dimension a_1, a_2, k and n respectively (by definition $a_3 = k$). The resolution of X is defined by

$$Y := \{(V_1, V_2, V_k) \in F(a_1, a_2, a_3 : n) | V_i \subset F_i, i = 1, 2, 3\}$$

With the natural projection to $G(k, n)$, we have a surjective map $\Pi : Y \to X_\lambda$. Note that Π is one to one when the point in X_λ satisfy those conditions with equality.

To show Y is smooth, we express it as a tower of Grassmann bundles as the following: The first level is $\mathbb{G}_1 := G(a_1, F_1)$, with universal sequence $S_1 \hookrightarrow F_1$. Every point of $\mathbb{G}_1$ represents a vector space V_1. To construct Y as a tower of Grassmann bundle, we need a a_2 dimension vector space V_2 containing V_1. So we consider the Grassmann bundle $G(a_2 - a_1, F_2/S_1)$ over $\mathbb{G}_1$, denote the total space by $\mathbb{G}_2$.

On $\mathbb{G}_2$ there is sequence $S_2 \hookrightarrow F_2/S_1$, here we ignore the pull back sign as in the last section. Let $\mathbf{V_2}$ be an extension of S_2 by S_1, we construct the Grassmann bundle $G(k - a_2, F_n/\mathbf{V}_2)$ over $\mathbb{G}_2$ with universal subbundle S_3. (Here the extension $\mathbf{V}_2$ may not be unique, but since we only care about it's first Chern class, so it will not affect the result even when we assume $\mathbf{V}_2 = S_1 \oplus S_2$). Then if we define $\mathbf{V_3}$ to be the extension of S_3 by $\mathbf{V_2}$, the total space of $\mathbf{V_3}$ over $\mathbb{G}_2$, denoted by $\mathbb{G}_3$, is a resolution of X. And by the description, $\mathbb{G}_3$ is exactly Y.

To show $-K_{\mathbb{G}_3}$ is nef, first we write down the relative tangent bundles as following:

$$\begin{aligned} T_{\mathbb{G}_1} &= S_1^* \otimes F_1/S_1 \\ T_{\mathbb{G}_2/\mathbb{G}_1} &= S_2^* \otimes (F_2/S_1)/S_2 \\ T_{\mathbb{G}_3/\mathbb{G}_2} &= S_3^* \otimes (F_n/\mathbf{V}_2)/S_3 \end{aligned}$$

Also note the inclusion relation:

$$Y \subset F(a_1, a_2, a_3 : n) \subset G(a_1, n) \times G(a_2, n) \times G(k, n)$$

On each of these three Grassmannian there is a line bundle determining the Plüker embedding, denote its first Chern class by $\sigma_i, i = 1, 2, 3$ respectively. Then we have, according to the construction, $c_1(S_1) = -\sigma_1$, $c_1(S_1 \oplus S_2) = c_1(\mathbf{V}_2) = -\sigma_2$ and $c_1(\mathbf{V}_2 \oplus S_3) = c_1(\mathbf{V}_3) = -\sigma_3$. As a result, we have the following equalities,

$$\begin{aligned} c_1(T_{\mathbb{G}_3}) &= c_1(T_{\mathbb{G}_1}) + c_1(T_{\mathbb{G}_2/\mathbb{G}_1}) + c_1(T_{\mathbb{G}_3/\mathbb{G}_2}) \\ &= c_1(S_1^* \otimes F_1/S_1) + c_1(S_2^* \otimes (F_2/S_1)/S_2) + c_1(S_3^* \otimes (F_3/\mathbf{V}_2)/S_3) \\ &= f_1 \cdot \sigma_1 + (\sigma_2 - \sigma_1) \cdot (f_2 - a_1) + \sigma_1 \cdot (a_2 - a_1) + (\sigma_3 - \sigma_2) \cdot (n - a_2) + \sigma_2 \cdot (k - a_2) \\ &= (f_1 - f_2 + a_2) \cdot \sigma_1 + (f_2 - n + k - a_1) \cdot \sigma_2 + (n - a_2) \cdot \sigma_3 \end{aligned}$$

More generally, for general j, then Y is sitting inside $F(a_1, a_2, \ldots, a_j, k : n)$, we can write the first Chern class of $T_Y = T_{\mathbb{G}_{j+1}}$ as

$$c_1(T_Y) = \Sigma_{i=1}^{i=j}(f_i - f_{i+1} + a_{i+1} - a_{i-1}) \cdot \sigma_i \tag{4.7}$$

where $f_{j+1} = n$, $a_{j+1} = k$ and $a_0 = 0$.

Plug in the definition of f_i we see that T_Y is Π relative nef if and only if for $0 \leq i \leq j-1$

$$a_i + \mu_{i+1} - \mu_i - a_{i-1} \geq 0$$

Note that we do not care about the coefficient of σ_j is because $G(a_j, n)$ is in fact $G(k, n)$, and $\Pi : Y \to X_\lambda$ will not contract any positive variety in $G(k, n)$.

On the other hand, following the notation in equation (5), we see that $c_{i+1} - c_i = a_i + \mu_{i+1} - \mu_i - a_{i-1}$. As a result, T_Y is relative nef if $\{c_i\}_{i=1}^{j}$ is a monotone increasing sequence.

Method2 :(Resolve by Union)

To illustrate the idea, we still resolve X_λ with partition $(\mu_1^{i_1}, \mu_2^{i_2}, \mu_3^{i_3})$. Let $b_i = n - \mu_i, i = 1, 2$. Then we consider

$$Y := \{(V_k, W_1, W_2) \in F(k, b_1, b_2 : n) | F_i \subset W_i, i = 1, 2\}$$

Note that these conditions imply that $V_k \oplus F_i \subset W_i$, or equivalently, $dim(V_k \cap F_i) \geq a_i$, which is exactly the conditions on schubert varieties. As a result, with the natural projection to

$G(k,n)$, $\Pi : Y \to X$ is a surjective and generically one-to-one map. To express Y as a tower of Grassmann bundles, we start from $\mathbb{G}_2 := G(b_2 - f_2, F_n/F_2)$ with universal subbundle S_2. On $\mathbb{G}_2$ there is a vector bundle defined as $W_2 := S_2 \oplus F_2$ of rank b_2, then the second level of tower is $G(b_1 - f_1, W_2/F_1)$ with universal subbundle S_1. Denote the total space by $\mathbb{G}_1$ and similarly, there is a rank b_1 bundle $W_1 := S_1 \oplus F_1$ on it. The last level is $\mathbb{G} := G(k, W_1)$. It is easily seen that the total space of the tower is exactly Y. Hence we have a smooth resolution.

Under this construction, the relative tangent bundles are

$$
\begin{aligned}
T_{\mathbb{G}_2} &= S_2^* \otimes (F_n/F_2)/S_2 \\
T_{\mathbb{G}_1/\mathbb{G}_2} &= S_1^* \otimes (W_2/F_1)/S_1 \\
T_{\mathbb{G}/\mathbb{G}_1} &= S^* \otimes W_1/S
\end{aligned}
$$

Also since $Y \subset G(k,n) \times G(b_1,n) \times G(b_2,n)$, denote their very ample line bundle class by σ, σ_1 and σ_2 respectively, then we have

$$
\begin{aligned}
c_1(T_Y) &= c_1(T_{\mathbb{G}_2}) + c_1(T_{\mathbb{G}_1/\mathbb{G}_2}) + c_1(T_{\mathbb{G}/\mathbb{G}_1}) \\
&= (f_n - f_2)\cdot\sigma_2 + (b_2 - f_1)\cdot\sigma_1 - (b_1 - f_1)\cdot\sigma_2 + b_1\cdot\sigma - k\cdot\sigma_1 \\
&= (f_n - f_3 - b_2 + f_2)\cdot\sigma_3 + (b_3 - b_1 + f_1 - f_2)\cdot\sigma_2 + (b_2 - f_1 - k)\cdot\sigma_1 + b_1\cdot\sigma
\end{aligned}
$$

More generally, for general j we have the following formula

$$
c_1(T_Y) = \Sigma_{i=1}^{i=j}(b_{i+1} - b_{i-1} + f_{i-1} - f_i)\cdot\sigma_i + b_1\cdot\sigma \tag{4.8}
$$

where $b_{j+1} = f_n$, $b_0 = k$ and $f_0 = 0$.

Plug in the definition of b_i and f_i we see that T_Y is nef if and only if for $1 \leq i \leq j-1$,

$$-a_i - \mu_{i+1} + \mu_i + a_{i-1} \geq 0$$

Or equivalently, in terms of equation (5), T_Y is nef if $\{c_i\}_{i=1}^j$ is monotone decreasing sequence.

More generally, we can combine both methods to construct a smooth resolution under some condition. If we want to have a tower as $V_1 \subset \cdots \subset V_s \subset V_k \subset W_1 \cdots \subset W_t$ in some flag variety such that, $V_i \subset F_i$ (resp. $F'_j \subset W_j$) for $1 \leq i \leq s$ (resp. $1 \leq j \leq t$), where F_i and F_j are fixed vector space. Then the condition we need to make sure the tower being smooth is

$$F_1 \subset F_2 \subset \cdots F_s \subset F'_1 \subset F'_2 \subset \cdots F_n$$

Under this condition, first we use method 2 to construct a variety Y' being the tower $W_1 \subset \cdots \subset W_t$ satisfying $F'_j \subset W_j$. On Y', there is tower $F_1 \subset F_2 \subset \cdots F_s \subset W_1$, then we can use method 1 to construct a Grassmann tower on it and denote the output by Y. By equation (6) and (7) we can write down the first Chern class of T_Y. Translated into the language of equation (5), we see a sufficient condition for T_Y being nef is that the coefficients of K_{X_λ} satisfying $c_1 \leq \cdots \leq c_s$ and then $c_s \geq \cdots \geq c_j$ for some s between 1 and j.

We claim that the resolutions constructed above are small contractions. The point is that given a Schubert variety corresponding to partition $\lambda = (\mu_1^{i_1}, \mu_2^{i_2} \ldots, \mu_j^{i_j})$, the singular locus is a union of Schubert varieties, corresponding to partition $(\mu_1^{i_1}, \cdots, (\mu_s+1)^{i_s+1}, \mu_{s+1}^{i_{s+1}-1} \ldots, \mu_j^{i_j})$.

For each of these locus, it is of codimension $i_s+1+\mu_s-\mu_{s+1}$. But for both kinds of resolutions, the general fiber is of dimension i_s over these locus. As a result, the exceptional locus is of codimension at least two.

Combining the results in this section with Theorem 4.2.6, we can give a sufficient condition of Schubert varieties to be log terminal. Say the Schubert variety X we are considering is of partition $(\mu_1^{i_1}, \mu_2^{i_2} \ldots, \mu_j^{i_j})$. And denote Schubert divisors with partition $(\mu_1^{i_1}, \cdots, \mu_{t-1}^{i_{t-1}}, (\mu_t + 1), \mu_t^{i_t - 1}, \mu_{t+1}^{i_{t+1}}, \ldots, \mu_j^{i_j})$ by D_t, write the canonical divisor $K_X = c_1 \cdot D_1 + c_2 \cdot D_2 + \cdots c_j \cdot D_j$, then we conclude the following proposition,

Proposition 4.3.4. *If the coefficients $\{c_i\}_{i=1}^j$ satisfy either of the following conditions; Monotone increasing or decreasing, or for some $s, 1 < s < j$, $\{c_i\}_{i=1}^s$ is increasing and $\{c_i\}_{i=s}^j$ is decreasing. Then X is log terminal. Moreover, there is a boundary divisor Δ such that (X, Δ) is a canonical pair.*

We remark that in (27) D. Anderson and A. Stapledon prove that in fact all the Schubert varieties are log terminal. Their result is more general than what we prove in this section. With their result and Theorem 4.2.1, we have an interesting corollary. Before the statement of the corollary we first recall the definition of intersection variety, see (38) and (41).

Definition 4.3.5. *Given Schubert varieties $X_{u_1}, \cdots, X_{u_r} \in G(k,n)$, and let $g_\bullet = (g_1, \cdots, g_r)$ be a general r-tuple of elements in G^r, where G is GL_n. The intersection variety $R(u_1, \cdots, u_r; g_\bullet)$ is defined as the intersection of the translated Schubert varieties $g_i X_{u_i}$:*

$$R(u_1, \cdots, u_r; g_\bullet) = g_1 X_{u_1} \cap \cdots \cap g_r X_{u_r}.$$

Corollary 4.3.6. *If the singular locus of* $X_{u_1}, \cdots, X_{u_r}$ *are disjoint with each other, then the intersection variety* $R(u_1, \cdots, u_r; g_\bullet)$ *is log terminal.*

Proof. It suffices to prove the case where $r = 2$, which is also called *Richardson Variety*. Consider $g_1 X_{u_1}$ and $g_2 X_{u_2}$, their singular locus are disjoint since g_1 and g_2 are general. Let U be the smooth locus of $g_2 X_{u_2}$, then $g_1 X_{u_1} \cap U$ is log terminal by Theorem 1.1. The same is true for $U^c \cap g_1 X_{u_1} = U^c \cap V$, where V is the smooth locus of $g_1 X_{u_1}$. □

Remark 4.3.7. *We noticed that in (26), the authors proved that all Richardson varieties have log terminal singularities.*

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  - Intermediate Algebra, Fall 2008, Spring 2013, UIC
  - Introduction to Differential Equations, Spring 2012, UIC
  - Algebra (Grader), Spring 2014, UIC
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Appendix

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