MANIFOLD–LIKE MATCHBOX MANIFOLDS

ALEX CLARK, STEVEN HURDER, AND OLGA LUKINA

ABSTRACT. A matchbox manifold is a generalized lamination, which is a continuum whose arc-components define the leaves of a foliation of the space. The main result of this paper implies that a matchbox manifold which is manifold-like must be homeomorphic to a weak solenoid.

1. INTRODUCTION

A *continuum* is a compact, connected, and non-empty metrizable space. The notion of a manifold-like continuum is derived from the notion of an ϵ-map, which was introduced by Alexandroff [5]:

DEFINITION 1.1. *Let X be a metric space, Y a topological space and $\epsilon > 0$ a constant. Then a map $f: X \to Y$ is said to be an ϵ-map if f is a continuous surjection and for each point $y \in Y$, the inverse image $f^{-1}(y)$ has diameter less than ϵ. A metric space X is said to be Y–like, for some topological space Y, if for every $\epsilon > 0$, there is an ϵ-map $f_\epsilon: X \to Y$.*

For example, a space X is *circle-like* if it is Y-like, where $Y = \mathbb{S}^1$ is the circle. More generally, let

$$(1) \qquad \mathcal{M}(n) = \{M \mid M \text{ is a closed connected manifold of dimension } n\}.$$

DEFINITION 1.2. *A continuum X is said to be* manifold-like, *if there exists $n \geq 1$ such that for every $\epsilon > 0$, there exists $M_\epsilon \in \mathcal{M}(n)$ and an ϵ-map $f_\epsilon: X \to M_\epsilon$.*

The study of the properties of ϵ-maps and manifold-like continua has a long history in the study of the topology of spaces. Eilenberg showed in [19] that an ϵ-map, for $\epsilon > 0$ sufficiently small, admits a left approximate inverse. Ganea studied the properties of compact, locally connected manifold-like ANR's of dimension n in [20], and showed that such a space has the homotopy type of a closed n-manifold. Deleanu [12, 13] showed that an n-dimensional connected polyhedron which is manifold-like is a closed pseudo-manifold. Bob Edwards gave in his 1978 ICM address [18] an overview of the further applications of ϵ-approximations and homeomorphisms.

Mardešić and Segal [26, 27] studied the properties of manifold-like connected polyhedron, and gave conditions under which such spaces must be a topological manifold. These authors used a technique of approximation of the given continuum by an inverse limit of spaces, and noted that their results do not apply to a continuum which is not locally connected, such as the dyadic solenoid.

The goal of this work is to characterize a class of manifold-like continua for which the Mardešić and Segal results do not apply. These are the matchbox manifolds as studied by the authors in [9, 10, 11], and discussed below. Our study of matchbox manifolds in these works was inspired by a result of Bing in [6]. Recall that a topological space X is *homogeneous* if for every $x, y \in X$, there exists a homeomorphism $h: X \to X$ such that $h(x) = y$.

THEOREM 1.3. *Let X be a homogeneous, circle-like continuum that contains an arc. Then either X is homeomorphic to the circle $\mathbb{S}^1$, or to an inverse limit of coverings of $\mathbb{S}^1$.*

This results inspired the subsequent works by McCord [29], Thomas [33], Hagopian [22], Mislove and Rogers [30], and Aarts, Hagopian and Oversteegen [3], all for 1-dimensional flow spaces.

In this work, we give extensions of Theorem 1.3 to continua with higher dimensional arc-components. We first recall two notions which are required to formulate our results. A weak solenoid $\mathcal{S}_\mathcal{P}$ is the inverse limit space of a sequence of covering maps of finite degree greater than one,

$$(2) \qquad\qquad \mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\},$$

where M_ℓ is a compact connected manifold without boundary. The collection of maps $\mathcal{P}$ is called a *presentation* for $\mathcal{S}_\mathcal{P}$. A weak solenoid $\mathcal{S}_\mathcal{P}$ is *regular* if the presentation $\mathcal{P}$ can be chosen so that for each $\ell \geq 0$, the composition $p_\ell^0 \equiv p_1 \circ \cdots p_\ell \colon M_\ell \to M_0$ is a regular covering map; that is, the fundamental group of M_ℓ injects onto a normal subgroup of the fundamental group of M_0 under the map induced by the covering projection p_ℓ^0. A weak solenoid which is not regular is said to be *irregular*. A *Vietoris solenoid* [34, 36] is a 1-dimensional regular solenoid, where each M_ℓ is a circle, as arises in the conclusion of Theorem 1.3.

A matchbox manifold $\mathfrak{M}$ is a continuum equipped with a decomposition $\mathcal{F}$ into leaves of constant dimension, so that the pair $(\mathfrak{M}, \mathcal{F})$ is a foliated space in the sense of [31], for which the local transversals to the foliation are totally disconnected. In particular, the leaves of $\mathcal{F}$ are the path connected components of $\mathfrak{M}$. A matchbox manifold with 2-dimensional leaves is a lamination by surfaces in the sense of Ghys [21] and Lyubich and Minsky [25], while Sullivan called them "solenoidal spaces" in [32, 35]. The terminology "matchbox manifold" follows the usage introduced in [1, 2, 4]. A Vietoris solenoid is a 1-dimensional matchbox manifold, and more generally, McCord showed in [29] that n-dimensional solenoids are examples of n-dimensional matchbox manifolds.

Next, recall a result of the first two authors. A matchbox manifold is said to be *equicontinuous* if the holonomy pseudogroup $\mathcal{G}_\mathcal{F}$ associated to the foliation $\mathcal{F}$ (see Section 3) defines an equicontinuous action on its transversal space, as defined in Definition 3.3.

THEOREM 1.4. [9, Theorems 1.2 and 1.4] *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a weak solenoid, and in particular is manifold-like. Moreover, if $\mathfrak{M}$ is homogeneous, then $\mathfrak{M}$ is homeomorphic to a regular solenoid.*

The main result of this paper, as follows, yields a generalization of Theorem 1.3 to higher dimensional matchbox manifolds.

THEOREM 1.5. *A manifold-like matchbox manifold $\mathfrak{M}$ is equicontinuous.*

Theorems 1.4 and 1.5 then yield the following partial converse to Theorem 1.4:

COROLLARY 1.6. *A manifold-like matchbox manifold $\mathfrak{M}$ is homeomorphic to a weak solenoid.*

The hypothesis that a matchbox manifold $\mathfrak{M}$ is manifold-like does not imply that $\mathfrak{M}$ is homogeneous, as the "discriminant obstruction" to homogeneity for a weak solenoid is supported in arbitrarily small open neighborhoods of points in $\mathfrak{M}$. The discriminant invariant was introduced and studied in the works in [15, 16, 17, 24].

The remainder of this paper is organized as follows. In Section 2 we recall the definitions of foliated spaces and matchbox manifolds, and give some of their basic properties. Particular care is taken to introduce various metric estimates related to the geometry of the leaves of the foliation, and to its dynamical properties. In Section 3 we recall the construction of the holonomy along leafwise paths.

In Section 4, we prove Theorem 1.5, using the path lifting property for ϵ-maps from a matchbox manifold to a compact manifold. This is the key technical tool, which is used to show that the foliation $\mathcal{F}$ on $\mathfrak{M}$ must be equicontinuous.

The Appendix A contains the proof of a technical result, Proposition 4.1 below. The proof of this result does not appear to be in the literature, and may even have a simpler proof than is given here. However, the result is essential for the proof of Theorem 1.5, so is included for completeness.

2. Foliated spaces and matchbox manifolds

We recall some background concepts used in the proof of our main theorems.

2.1. **Matchbox manifolds.** We first recall the definition of a matchbox manifold.

DEFINITION 2.1. *A* matchbox manifold of dimension n *is a continuum* $\mathfrak{M}$*, such that there exists a compact, separable, totally disconnected metric space* $\mathfrak{X}$*, and for each* $x \in \mathfrak{M}$ *there is a compact subset* $\mathfrak{T}_x \subset \mathfrak{X}$*, an open subset* $U_x \subset \mathfrak{M}$*, and a homeomorphism* $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ *defined on the closure* $\overline{U}_x$ *in* $\mathfrak{M}$*, such that* $\varphi_x(x) = (0, w_x)$ *where* $w_x \in int(\mathfrak{T}_x)$*. Moreover, it is assumed that each* φ_x *admits an extension to a foliated homeomorphism* $\widehat{\varphi}_x \colon \widehat{U}_x \to (-2,2)^n \times \mathfrak{T}_x$ *where* $\widehat{U}_x \subset \mathfrak{M}$ *is an open subset such that* $\overline{U}_x \subset \widehat{U}_x$*. The space* $\mathfrak{T}_x$ *is called the* local transverse model *at* x*.*

The assumption that the transversals $\mathfrak{T}_x$ are totally disconnected implies that the local charts φ_x satisfy the compatibility axioms of foliation charts for a foliated space, as in [8, 9, 31].

Let $\pi_x \colon \overline{U}_x \to \mathfrak{T}_x$ denote the composition of φ_x with projection onto the second factor.

Also introduce the transversal maps $\tau_x \colon \mathfrak{T}_x \to \mathcal{T}_x \subset \mathfrak{M}$, defined for $w \in \mathfrak{T}_x$ by $\tau_x(w) = \varphi_x^{-1}(0, w)$. The subspace $\mathcal{T}_x$ is given the metric $d_{\mathcal{T}_x}$ which is the restriction of the metric $d_{\mathfrak{M}}$.

For $w \in \mathfrak{T}_x$ the set $\mathcal{P}_x(w) = \pi_x^{-1}(w) \subset \overline{U}_x$ is called a *plaque* for the coordinate chart φ_x. We adopt the notation, for $z \in \overline{U}_x$, that $\mathcal{P}_x(z) = \mathcal{P}_x(\pi_x(z))$, so that $z \in \mathcal{P}_x(z)$. Note that each plaque $\mathcal{P}_x(w)$ is given the topology so that the restriction $\varphi_x \colon \mathcal{P}_x(w) \to [-1,1]^n \times \{w\}$ is a homeomorphism. Then $int(\mathcal{P}_x(w)) = \varphi_x^{-1}((-1,1)^n \times \{w\})$.

Let $U_x = int(\overline{U}_x) = \varphi_x^{-1}((-1,1)^n \times int(\mathfrak{T}_x))$. Note that if $z \in U_x \cap U_y$, then $int(\mathcal{P}_x(z)) \cap int(\mathcal{P}_y(z))$ is an open subset of both $\mathcal{P}_x(z)$ and $\mathcal{P}_y(z)$. The collection of sets

$$\mathcal{V} = \{\varphi_x^{-1}(V \times \{w\}) \mid x \in \mathfrak{M} , \ w \in \mathfrak{T}_x , \ V \subset (-1,1)^n \text{ open}\}$$

forms the basis for the *fine topology* of $\mathfrak{M}$. The connected components of the fine topology are called leaves, and define the foliation $\mathcal{F}$ of $\mathfrak{M}$. In particular, the leaves of the foliation $\mathcal{F}$ of $\mathfrak{M}$ are the path-connected components of $\mathfrak{M}$. For $x \in \mathfrak{M}$, let $L_x \subset \mathfrak{M}$ denote the leaf of $\mathcal{F}$ containing x.

DEFINITION 2.2. *A* smooth matchbox manifold *is a space* $\mathfrak{M}$ *as above, such that there exists a choice of local charts* $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ *such that for all* $x, y \in \mathfrak{M}$ *with* $z \in U_x \cap U_y$*, there exists an open set* $z \in V_z \subset U_x \cap U_y$ *such that* $\mathcal{P}_x(z) \cap V_z$ *and* $\mathcal{P}_y(z) \cap V_z$ *are connected open sets, and the composition*

$$\psi_{x,y;z} \equiv \varphi_y \circ \varphi_x^{-1} \colon \varphi_x(\mathcal{P}_x(z) \cap V_z) \to \varphi_y(\mathcal{P}_y(z) \cap V_z)$$

is a smooth map, where $\varphi_x(\mathcal{P}_x(z) \cap V_z) \subset \mathbb{R}^n \times \{w\} \cong \mathbb{R}^n$ *and* $\varphi_y(\mathcal{P}_y(z) \cap V_z) \subset \mathbb{R}^n \times \{w'\} \cong \mathbb{R}^n$*. The leafwise transition maps* $\psi_{x,y;z}$ *are assumed to depend continuously on* z *in the* C^∞*-topology.*

A map $f \colon \mathfrak{M} \to \mathbb{R}$ is said to be *smooth* if for each flow box $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ and $w \in \mathfrak{T}_x$ the composition $y \mapsto f \circ \varphi_x^{-1}(y, w)$ is a smooth function of $y \in (-1,1)^n$, and depends continuously on w in the C^∞-topology on maps of the plaque coordinates y. As noted in [31] and [8, Chapter 11], this allows one to define smooth partitions of unity, vector bundles, and tensors for smooth foliated spaces. In particular, one can define leafwise Riemannian metrics. We recall a standard result, whose proof for foliated spaces can be found in [8, Theorem 11.4.3].

THEOREM 2.3. *Let* $\mathfrak{M}$ *be a smooth matchbox manifold. Then there exists a leafwise Riemannian metric for* $\mathcal{F}$*, such that for each* $x \in \mathfrak{M}$*, the leaf* L_x *inherits the structure of a complete Riemannian manifold with bounded geometry, and the Riemannian metric and its covariant derivatives depend continuously on* x *.*

Bounded geometry implies, for example, that for each $x \in \mathfrak{M}$, there is a leafwise exponential map $\exp_x^{\mathcal{F}} \colon T_x \mathcal{F} \to L_x$ which is a surjection, and the composition $\exp_x^{\mathcal{F}} \colon T_x \mathcal{F} \to L_x \subset \mathfrak{M}$ depends continuously on x in the compact-open topology on maps. All matchbox manifolds are assumed to be smooth with a given leafwise Riemannian metric, and with a fixed choice of metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$.

2.2. **Metric estimates.** We formulate some relations between the metric properties of a matchbox manifold $\mathfrak{M}$ and the metric properties of the leaves of $\mathcal{F}$. These technical conditions are used in studying the dynamics and geometry of these spaces.

For $x \in \mathfrak{M}$ and $\epsilon > 0$, let $D_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x, y) \leq \epsilon\}$ be the closed ϵ-ball about x in $\mathfrak{M}$, and $B_{\mathfrak{M}}(x, \epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x, y) < \epsilon\}$ the open ϵ-ball about x.

Let $d_{\mathfrak{X}}$ denote the metric on the space $\mathfrak{X}$ in Definition 2.1. For $w \in \mathfrak{X}$ and $\epsilon > 0$, let $D_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w, w') \leq \epsilon\}$ be the closed ϵ-ball about w in $\mathfrak{X}$, and let $B_{\mathfrak{X}}(w, \epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w, w') < \epsilon\}$ be the open ϵ-ball about w.

Each leaf $L \subset \mathfrak{M}$ has a complete path-length metric, induced from the leafwise Riemannian metric:

$$d_{\mathcal{F}}(x, y) = \inf \left\{ \|\gamma\| \mid \gamma \colon [0,1] \to L \text{ is piecewise C}^1 \ , \ \gamma(0) = x \ , \ \gamma(1) = y \ , \ \gamma(t) \in L \quad \forall \ 0 \leq t \leq 1 \right\}$$

where $\|\gamma\|$ denotes the path-length of the piecewise C^1-curve $\gamma(t)$. If $x, y \in \mathfrak{M}$ are not on the same leaf, then set $d_{\mathcal{F}}(x, y) = \infty$.

For each $x \in \mathfrak{M}$ and $r > 0$, let $D_{\mathcal{F}}(x, r) = \{y \in L_x \mid d_{\mathcal{F}}(x, y) \leq r\}$.

For each $x \in \mathfrak{M}$, the Gauss Lemma implies that there exists $\lambda_x > 0$ such that $D_{\mathcal{F}}(x, \lambda_x)$ is a *strongly convex* subset for the metric $d_{\mathcal{F}}$. That is, for any pair of points $y, y' \in D_{\mathcal{F}}(x, \lambda_x)$ there is a unique shortest geodesic segment in L_x joining y and y' and contained in $D_{\mathcal{F}}(x, \lambda_x)$. This standard concept of Riemannian geometry is discussed in detail in [7], and in [14, Chapter 3, Proposition 4.2]. Then for all $0 < \lambda < \lambda_x$ the disk $D_{\mathcal{F}}(x, \lambda)$ is also strongly convex. The leafwise metrics for $\mathcal{F}$ constructed in the proof of Theorem 2.3 have uniformly bounded geometry, and the first and second order covariant derivatives of the metrics depend continuously on the point $x \in \mathfrak{M}$, so by the compactness of $\mathfrak{M}$, we obtain:

LEMMA 2.4. *There exists $\lambda_{\mathcal{F}} > 0$ such that for all $x \in \mathfrak{M}$, $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ is strongly convex.*

2.3. **Regular coverings.** We next formulate the definition of a *regular covering* of a matchbox manifold $\mathfrak{M}$. It follows from standard considerations (see [9]) that a matchbox manifold admits a covering by foliation charts which satisfies additional regularity conditions.

PROPOSITION 2.5. [9] *For a smooth foliated space $\mathfrak{M}$, given $\epsilon_{\mathfrak{M}} > 0$, there exist $\lambda_{\mathcal{F}} > 0$ and a choice of local charts $\varphi_x \colon \overline{U}_x \to [-1, 1]^n \times \mathfrak{T}_x$ with the following properties: For each $x \in \mathfrak{M}$,*

(1) $U_x \equiv int(\overline{U}_x) = \varphi_x^{-1}\left((-1, 1)^n \times \mathfrak{T}_x\right)$, with $\overline{U}_x \subset B_{\mathfrak{M}}(x, \epsilon_{\mathfrak{M}})$.
(2) The plaques of φ_x are strongly convex for the metric $d_{\mathcal{F}}$ with diameter less than $\lambda_{\mathcal{F}}$.

By a standard argument, there exists a finite collection $\{x_1, \ldots, x_\nu\} \subset \mathfrak{M}$ where $\varphi_{x_i}(x_i) = (0, w_{x_i})$ for $w_{x_i} \in \mathfrak{X}$, and regular foliation charts $\varphi_{x_i} \colon \overline{U}_{x_i} \to [-1, 1]^n \times \mathfrak{T}_{x_i}$ satisfying the conditions of Proposition 2.5, which form an open covering of $\mathfrak{M}$. Relabel the various maps and spaces accordingly, so that $\overline{U}_i = \overline{U}_{x_i}$ and $\varphi_i = \varphi_{x_i}$. Accordingly, label the transverse spaces $\mathfrak{T}_i = \mathfrak{T}_{x_i}$ and the projection maps $\pi_i = \pi_{x_i} \colon \overline{U}_i \to \mathfrak{X}_i$. Then the image $\pi_i(U_i \cap U_j) = \mathfrak{T}_{i,j} \subset \mathfrak{T}_i$ is a clopen subset for all $1 \leq i, j \leq \nu$.

We also then have the transversal mappings $\tau_i \colon \mathfrak{T}_i \to \mathcal{T}_i \subset \mathfrak{M}$ for $1 \leq i \leq \nu$.

A *regular covering* of $\mathfrak{M}$ is a finite covering $\mathcal{U} = \{U_1, \ldots, U_\nu\}$ such that for each $1 \leq i \leq \nu$ there is a foliated coordinate map $\varphi_i \colon U_i \to (-1, 1)^n \times \mathfrak{T}_i$ which satisfies the regularity conditions in Proposition 2.5. We assume in the following that a regular foliated covering of $\mathfrak{M}$ has been chosen.

2.4. **More metric estimates.** We introduce lower and upper bounds on the diameters of balls in the leaves of $\mathcal{F}$ with respect to the ambient metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$. To assist with the notation, we use the convention that $\lambda > 0$ will denote a small leafwise distance, and ϵ will denote a small distance in $\mathfrak{M}$. Later when we introduce the target manifold M, we let δ denote a small distance in M.

For $x \in \mathfrak{M}$ and $\epsilon > 0$, let $D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon) \subset L_x$ denote the connected component containing x of the intersection $L_x \cap D_{\mathfrak{M}}(x, \epsilon)$. Define the continuous functions

$$(3) \qquad \rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, x, \epsilon) = \max \{ d_{\mathcal{F}}(x', x) \mid x' \in D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon) \}$$

$$(4) \qquad \rho(d_{\mathfrak{M}}, d_{\mathcal{F}}, x, \lambda) = \max \{ d_{\mathfrak{M}}(x', x) \mid x' \in D_{\mathcal{F}}(x, \lambda) \} \ .$$

Then for all $x \in \mathfrak{M}$, $\epsilon > 0$ and $\lambda > 0$, we have

$$(5) \qquad D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon) \subset D_{\mathcal{F}}(x, \rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, x, \epsilon)) \text{ and } D_{\mathcal{F}}(x, \lambda) \subset D_{\mathfrak{M}}(x, \rho(d_{\mathfrak{M}}, d_{\mathcal{F}}, x, \lambda)) \ .$$

As $\mathfrak{M}$ is compact, we can then define the increasing functions of $\epsilon > 0$ and $\lambda > 0$,

$$(6) \qquad \rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon) = \max \{ \rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, x, \epsilon) \mid x \in \mathfrak{M} \}$$

$$(7) \qquad \rho(d_{\mathfrak{M}}, d_{\mathcal{F}}, \lambda) = \max \{ \rho(d_{\mathfrak{M}}, d_{\mathcal{F}}, x, \lambda) \mid x \in \mathfrak{M} \} \ .$$

Moreover, for $0 < \lambda' < \lambda \leq \lambda_{\mathcal{F}}$ we have the strict inclusion $B_{\mathcal{F}}(x, \lambda') \subset B_{\mathcal{F}}(x, \lambda)$, and thus the function $\rho(d_{\mathfrak{M}}, d_{\mathcal{F}}, \lambda)$ is strictly increasing for $0 < \lambda < \lambda_{\mathcal{F}}$.

Let $\epsilon_{\mathcal{F}}^* = \max \{ \epsilon \mid \rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon) \leq \lambda_{\mathcal{F}} \}$, so that $D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon_{\mathcal{F}}^*) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$ for all $x \in \mathfrak{M}$.

Introduce the continuous function $\lambda_{\mathcal{F}}(\epsilon)$ which is the inverse of $\rho(d_{\mathfrak{M}}, d_{\mathcal{F}}, \lambda)$, for $0 < \lambda \leq \lambda_{\mathcal{F}}$. Thus, $\lambda_{\mathcal{F}}(\epsilon)$ is the largest radius $\lambda \leq \lambda_{\mathcal{F}}$ such that the disk $D_{\mathcal{F}}(x, \lambda)$ in the leafwise metric is contained in the ball $B_{\mathfrak{M}}(x, \epsilon)$ for all $x \in \mathfrak{M}$. Combining the above definitions, we obtain that for all $x \in \mathfrak{M}$,

$$(8) \qquad D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}(\epsilon)) \subset D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \cap D_{\mathfrak{M}}(x, \epsilon) \ .$$

Choose $\epsilon_0 > 0$ so that $\rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon_0) \leq \lambda_{\mathcal{F}}/2$, and set $\lambda_0 = \lambda_{\mathcal{F}}(\epsilon_0)$. Then by the definition of $\rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon_0)$ and the inclusions (8), for all $x \in \mathfrak{M}$ we have the inclusions

$$(9) \qquad D_{\mathcal{F}}(x, \lambda_0) \subset D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon_0) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}/2) \cap B_{\mathfrak{M}}(x, \epsilon_0) \ .$$

Next, choose $\epsilon_1 > 0$ so that $\rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon_1) \leq \lambda_0/10$, and let $\lambda_1 = \lambda_{\mathcal{F}}(\epsilon_1)$. Then for all $x \in \mathfrak{M}$,

$$(10) \qquad D_{\mathcal{F}}(x, \lambda_1) \subset D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon_1) \subset D_{\mathcal{F}}(x, \lambda_0/10) \cap B_{\mathfrak{M}}(x, \epsilon_1) \ .$$

This choice of ϵ_1 will be recalled in Section 4 and Appendix A.

A matchbox manifold $\mathfrak{M}$ is *minimal* if every leaf of $\mathcal{F}$ is dense.

3. Holonomy

The holonomy pseudogroup of a smooth foliated manifold $(M, \mathcal{F})$ generalizes the induced dynamical systems associated to a section of a flow. The holonomy pseudogroup for a matchbox manifold $(\mathfrak{M}, \mathcal{F})$ is defined analogously to the smooth case.

3.1. The foliation pseudo-star group.
Let $\mathcal{U} = \{ \varphi_i \colon \overline{U}_i \to [-1, 1]^n \times \mathfrak{T}_i \mid 1 \leq i \leq \nu \}$ be a regular covering of $\mathfrak{M}$ as in Section 2.3. A pair of indices (i, j), $1 \leq i, j \leq \nu$, is said to be *admissible* if the *open* coordinate charts satisfy $U_i \cap U_j \neq \emptyset$. For (i, j) admissible, define clopen subsets $\mathfrak{D}_{i,j} = \pi_i(U_i \cap U_j) \subset \mathfrak{T}_i \subset \mathfrak{X}$. The convexity of foliation charts imply that plaques are either disjoint, or have connected intersection. This implies that there is a well-defined homeomorphism $h_{j,i} \colon \mathfrak{D}_{i,j} \to \mathfrak{D}_{j,i}$ with domain $\mathfrak{D}(h_{j,i}) = \mathfrak{D}_{i,j}$ and range $R(h_{j,i}) = \mathfrak{D}_{j,i}$.

The maps $\mathcal{G}_{\mathcal{F}}^{(1)} = \{ h_{j,i} \mid (i, j) \text{ admissible} \}$ are the transverse change of coordinates defined by the foliation charts. By definition they satisfy $h_{i,i} = Id$, $h_{i,j}^{-1} = h_{j,i}$, and if $U_i \cap U_j \cap U_k \neq \emptyset$ then $h_{k,j} \circ h_{j,i} = h_{k,i}$ on their common domain of definition. The *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$ is the topological pseudogroup modeled on $\mathfrak{X}$ generated by the elements of $\mathcal{G}_{\mathcal{F}}^{(1)}$. The elements of $\mathcal{G}_{\mathcal{F}}$ have a standard description in terms of the "holonomy along paths", which we next describe.

A sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ is *admissible*, if each pair $(i_{\ell-1}, i_\ell)$ is admissible for $1 \leq \ell \leq \alpha$, and the composition

$$(11) \qquad h_{\mathcal{I}} = h_{i_\alpha, i_{\alpha-1}} \circ \cdots \circ h_{i_1, i_0}$$

has non-empty domain. The domain $\mathfrak{D}_{\mathcal{I}}$ of $h_{\mathcal{I}}$ is the *maximal clopen subset* of $\mathfrak{D}_{i_0} \subset \mathfrak{T}_{i_0}$ for which the compositions are defined.

For the study of the dynamical properties of $\mathcal{F}$, it is necessary to introduce the collection of maps $\mathcal{G}_{\mathcal{F}}^* \subset \mathcal{G}_{\mathcal{F}}$, defined as follows. Given any open subset $U \subset \mathfrak{D}_{\mathcal{I}}$ we obtain a new element $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}$ by restriction. Then set

(12) $$\mathcal{G}_{\mathcal{F}}^* = \{ h_{\mathcal{I}}|U \mid \mathcal{I} \text{ admissible \& } U \subset \mathfrak{D}_{\mathcal{I}} \} \subset \mathcal{G}_{\mathcal{F}} \ .$$

That is, $\mathcal{G}_{\mathcal{F}}^*$ consists of all possible restrictions of homeomorphisms of the form (11) to open subsets of their domains. However, in the definition of $\mathcal{G}_{\mathcal{F}}^*$ one does not allow arbitrary unions of local homeomorphisms, unless such homeomorphisms can be obtained by restrictions to open subsets of maximal domains of words in the elements in $\mathcal{G}_0$. The collection of maps $\mathcal{G}_{\mathcal{F}}^*$ is closed under the operations of compositions, taking inverses, and restrictions to open sets, and is called a pseudo⋆group in the literature [28].

For $g \in \mathcal{G}_{\mathcal{F}}^*$ denote its domain by $\mathfrak{D}(g) \subset \mathfrak{X}$, then its range is the clopen set $\mathfrak{R}(g) = g(\mathfrak{D}(g)) \subset \mathfrak{X}$.

3.2. **Admissible chains.** Given an admissible sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ and any $0 \leq \ell \leq \alpha$, the truncated sequence $\mathcal{I}_\ell = (i_0, i_1, \ldots, i_\ell)$ is again admissible, and we introduce the holonomy map defined by the composition of the first ℓ generators appearing in $h_{\mathcal{I}}$,

(13) $$h_{\mathcal{I}_\ell} = h_{i_\ell, i_{\ell-1}} \circ \cdots \circ h_{i_1, i_0} \ .$$

Given $w \in \mathfrak{D}(h_{\mathcal{I}})$ we adopt the notation $w_\ell = h_{\mathcal{I}_\ell}(w) \in \mathfrak{T}_{i_\ell}$. So $w_0 = w$ and $h_{\mathcal{I}}(w) = w_\alpha$.

Given $w \in \mathfrak{D}(h_{\mathcal{I}})$, let $x_0 = \tau_{i_0}(w_0) \in L_{x_0}$. Introduce the *plaque chain*

(14) $$\mathcal{P}_{\mathcal{I}}(w) = \{ \mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \ldots, \mathcal{P}_{i_\alpha}(w_\alpha) \} \ .$$

For each $1 \leq i \leq \nu$, define $\mathcal{T}_i = \varphi_i^{-1}(0, \mathfrak{T}_i) \subset \overline{U}_i \subset \mathfrak{M}$ which is a compact local transversal to $\mathcal{F}$.

Intuitively, a plaque chain $\mathcal{P}_{\mathcal{I}}(w)$ is a sequence of successively overlapping convex "tiles" in L_0 starting at $x_0 = \tau_{i_0}(w_0)$, ending at $x_\alpha = \tau_{i_\alpha}(w_\alpha)$, and with each $\mathcal{P}_{i_\ell}(w_\ell)$ "centered" on the point $x_\ell = \tau_{i_\ell}(w_\ell)$. Recall that $\mathcal{P}_{i_\ell}(x_\ell) = \mathcal{P}_{i_\ell}(w_\ell)$, so we also adopt the notation $\mathcal{P}_{\mathcal{I}}(x) \equiv \mathcal{P}_{\mathcal{I}}(w)$.

A *leafwise path* is a continuous map $\gamma \colon [0,1] \to \mathfrak{M}$ such that there is a leaf L of $\mathcal{F}$ for which $\gamma(t) \in L$ for all $0 \leq t \leq 1$. In the following, we will assume that all paths are piecewise differentiable.

Let γ be a leafwise path, and $\mathcal{I}$ be an admissible sequence. For $w \in \mathfrak{D}(h_{\mathcal{I}})$, we say that $(\mathcal{I}, w)$ *covers* γ, if the domain of γ admits a partition $0 = s_0 < s_1 < \cdots < s_\alpha = 1$ such that the plaque chain $\mathcal{P}_{\mathcal{I}}(w_0) = \{ \mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \ldots, \mathcal{P}_{i_\alpha}(w_\alpha) \}$ satisfies

(15) $$\gamma([s_\ell, s_{\ell+1}]) \subset int(\mathcal{P}_{i_\ell}(w_\ell)) \ , \ 0 \leq \ell < \alpha, \quad \& \quad \gamma(1) \in int(\mathcal{P}_{i_\alpha}(w_\alpha)).$$

The map $h_{\mathcal{I}}$ is said to define the holonomy of $\mathcal{F}$ along the path γ, and satisfies $h_{\mathcal{I}}(w_0) = \pi_{i_\alpha}(\gamma(1))$.

Given two admissible sequences, $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ and $\mathcal{J} = (j_0, j_1, \ldots, j_\beta)$, such that both $(\mathcal{I}, w_0)$ and $(\mathcal{J}, v_0)$ cover the leafwise path $\gamma \colon [0,1] \to \mathfrak{M}$, then

$$\gamma(0) \in int(\mathcal{P}_{i_0}(w_0)) \cap int(\mathcal{P}_{j_0}(v_0)) \quad , \quad \gamma(1) \in int(\mathcal{P}_{i_\alpha}(w_\alpha)) \cap int(\mathcal{P}_{j_\beta}(v_\beta))$$

Thus both (i_0, j_0) and (i_α, j_β) are admissible, and $v_0 = h_{j_0, i_0}(w_0)$, $w_\alpha = h_{i_\alpha, j_\beta}(v_\beta)$.

The proof of the following standard observation can be found in [9].

PROPOSITION 3.1. [9] *The maps $h_{\mathcal{I}}$ and $h_{i_\alpha, j_\beta} \circ h_{\mathcal{J}} \circ h_{j_0, i_0}$ agree on their common domains.*

Two leafwise paths $\gamma, \gamma' \colon [0,1] \to \mathfrak{M}$ are *homotopic* if there exists a family of leafwise paths $\gamma_s \colon [0,1] \to \mathfrak{M}$ with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$. We are most interested in the special case when $\gamma(0) = \gamma'(0) = x$ and $\gamma(1) = \gamma'(1) = y$. Then γ and γ' are *homotopic relative endpoints*, or *endpoint-homotopic*, if they are homotopic with $\gamma_s(0) = x$ for all $0 \leq s \leq 1$, and similarly $\gamma_s(1) = y$ for all $0 \leq s \leq 1$. Thus, the family of curves $\{ \gamma_s(t) \mid 0 \leq s \leq 1 \}$ are all contained in a common leaf L_x. We then have the following result.

LEMMA 3.2. [9] *Let* $\gamma, \gamma' \colon [0,1] \to \mathfrak{M}$ *be endpoint-homotopic leafwise paths. Then their holonomy maps* h_γ *and* $h_{\gamma'}$ *agree on some open subset* $U \subset \mathfrak{D}(h_\gamma) \cap \mathfrak{D}(h_{\gamma'}) \subset \mathfrak{T}_*$.

Finally, we recall the definition of an equicontinuous pseudogroup.

DEFINITION 3.3. *The action of the pseudogroup* $\mathcal{G}_\mathcal{F}$ *on* $\mathfrak{X}$ *is* equicontinuous *if for all* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $g \in \mathcal{G}_\mathcal{F}^*$, *if* $w, w' \in \mathfrak{D}(g)$ *and* $d_\mathfrak{X}(w, w') < \delta$, *then* $d_\mathfrak{X}(g(w), g(w')) < \epsilon$. *Thus,* $\mathcal{G}_\mathcal{F}^*$ *is equicontinuous as a family of local group actions.*

Further properties of the pseudogroup $\mathcal{G}_\mathcal{F}$ for a matchbox manifold are discussed in [9, 10, 23].

4. EQUICONTINUOUS HOLONOMY

In this section, we give the proof of Theorem 1.5. A key point in the proof is based on the path lifting property for ϵ-maps between a matchbox manifold $\mathfrak{M}$ and a target manifold M. The philosophy of path lifting is folklore, as observed by Bob Edwards in his 1978 ICM address on ϵ-approximations and homeomorphisms [18, Section 4]. We develop this technique in the context of matchbox manifolds, making use of standards results for ϵ-maps along with properties of the holonomy maps.

Let $\mathfrak{M}$ be a manifold-like matchbox manifold. Let $\mathcal{U} = \{\varphi_i \colon \overline{U}_i \to [-1, 1]^n \times \mathfrak{T}_i \mid 1 \leq i \leq \nu\}$ be a regular covering of $\mathfrak{M}$ as in Section 2.3. Let $\mathcal{G}_\mathcal{F}^*$ be the pseudo$\star$group associated to the regular covering $\mathcal{U}$ as in Section 3.1. We must show that the conditions of Definition 3.3 are satisfied: given $\epsilon > 0$, we must show there exists $\delta > 0$ such that for each admissible chain $\mathcal{I}$ with holonomy $h_\mathcal{I}$, if $w, w' \in \mathfrak{D}(h_\mathcal{I}) \subset \mathfrak{X}$ and $d_\mathfrak{X}(w, w') < \delta$, then $d_\mathfrak{X}(h_\mathcal{I}(w), h_\mathcal{I}(w')) < \epsilon$.

Recall that $\epsilon_\mathcal{U} > 0$ denotes a Lebesgue number for the covering $\mathcal{U}$ of $\mathfrak{M}$. That is, for each $x \in \mathfrak{M}$, there exists some index $1 \leq i \leq \nu$ such that $B_\mathfrak{M}(x, \epsilon_\mathcal{U}) \subset U_i$.

Since $\overline{U}_i$ is compact for each $1 \leq i \leq \nu$, there exists a uniform modulus of continuity function $\rho_\mathcal{U}(\epsilon) > 0$ for the projections π_i: let $\epsilon > 0$, then $\rho_\mathcal{U}(\epsilon)$ is the largest value such that

$$(16) \qquad B_\mathfrak{M}(x, \rho_\mathcal{U}(\epsilon)) \cap \overline{U}_i \subset \pi_i^{-1}(B_{\mathfrak{X}_i}(\pi_i(x), \epsilon)) \text{ for all } x \in \overline{U}_i \ .$$

For ϵ_1 as defined in Section 2.2, define $\epsilon_\mathcal{F}$ by

$$(17) \qquad 0 < \epsilon_\mathcal{F} = \min \{\rho_\mathcal{U}(\epsilon), \epsilon_1/2, \epsilon_\mathcal{U}/4\} \ .$$

4.1. Continuity estimates.
Choose an $\epsilon_\mathcal{F}$-map $f \colon \mathfrak{M} \to M$ onto the compact topological manifold M, where for simplicity we omit the subscript $\epsilon_\mathcal{F}$ in the notation for f. Let d_M be a metric on M. For $w \in M$ and $\delta > 0$, let $B_M(w, \delta) = \{w' \in M \mid d_M(w', w) < \delta\}$ denote the open disk in M of radius δ, and $D_M(w, \delta) = \{w' \in M \mid d_M(w', w) \leq \delta\}$ denote the closed disk in M of radius δ.

Assume that d_M is chosen so that there exists a constant $\delta_M > 0$ such that for all $w \in M$ and $0 < \delta \leq \delta_M$, the disk $D_M(w, \delta)$ is homeomorphic to a disk in $\mathbb{R}^n$. For example, if M is a smooth Riemannian manifold, then let $\delta_M > 0$ be such that each disk $D_M(w, \delta_M)$ is strongly convex.

Since $\mathfrak{M}$ is compact, there exists a uniform modulus of continuity function $\epsilon_f(\delta) > 0$ for f: for $\delta > 0$, the constant $\epsilon_f(\delta)$ is the largest value such that

$$(18) \qquad B_\mathfrak{M}(x, \epsilon_f(\delta)) \subset f^{-1}(B_M(f(x), \delta)) \text{ for all } x \in \mathfrak{M} \ .$$

Let $\delta_f^* > 0$ be the largest radius such that $\epsilon_f(\delta) \leq \epsilon_\mathcal{F}$ for all $0 < \delta \leq \delta_f^*$.

Set $\lambda_{\mathcal{F}, f}(\delta) = \lambda_\mathcal{F}(\epsilon_f(\delta))$, which is well-defined for $0 < \delta \leq \delta_f^*$. Recall from Section 2.4 that $\lambda_{\mathcal{F}, f}(\delta)$ is then the largest radius $\lambda \leq \lambda_\mathcal{F}$ such that the disk $D_\mathcal{F}(x, \lambda)$ in the leafwise metric is contained in the ball $B_\mathfrak{M}(x, \epsilon_f(\delta))$ for all $x \in \mathfrak{M}$.

Combining (8) and (18) we obtain a leafwise modulus of continuity for f:

$$(19) \qquad D_\mathcal{F}(x, \lambda_{\mathcal{F}, f}(\delta)) \subset B_\mathfrak{M}(x, \epsilon_f(\delta)) \subset f^{-1}(B_M(f(x), \delta)) \quad \text{for all } 0 < \delta \leq \delta_f^* \ , \ x \in \mathfrak{M} \ .$$

As f is an $\epsilon_{\mathcal{F}}$-map, we have that $f^{-1}(f(x)) \subset B_{\mathfrak{M}}(x, \epsilon_{\mathcal{F}})$ for all $x \in \mathfrak{M}$. As M is compact and $f^{-1}(f(x))$ is a compact set with diameter at most $\epsilon_{\mathcal{F}}$ for all $x \in \mathfrak{M}$, there exists $0 < \delta_1 \leq \delta_M/10$ so that for all $x \in \mathfrak{M}$, we have

$$(20) \qquad f^{-1}(D_M(f(x), \delta_1)) \subset B_{\mathfrak{M}}(x, 2\epsilon_{\mathcal{F}}) = B_{\mathfrak{M}}(x, \epsilon_1) \ .$$

Let $\lambda_2 = \lambda_{\mathcal{F}, f}(\delta_1)$; that is, λ_2 is then is the largest radius $\lambda \leq \lambda_{\mathcal{F}}$ such that the disk $D_{\mathcal{F}}(x, \lambda)$ in the leafwise metric is contained in the ball $B_{\mathfrak{M}}(x, \delta_1)$ for all $x \in \mathfrak{M}$.

Then for all $x \in \mathfrak{M}$, by (19), (10), and the choice of δ_1 we have

$$(21) \qquad D_{\mathcal{F}}(x, \lambda_2) \subset f^{-1}(D_M(f(x), \delta_1)) \cap D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \subset D_{\mathfrak{M}}(x, \epsilon_1) \cap D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \subset D_{\mathcal{F}}(x, \lambda_0/10) \ .$$

Finally, we require a basic result concerning ϵ-maps on matchbox manifolds, which is a type of converse to the inclusions in (21), and whose proof is in the spirit of the work by Eilenberg [19]. The proof of the following is deferred to Appendix A.

PROPOSITION 4.1. *Let $\mathfrak{M}$ be a matchbox manifold with leafwise Riemannian metric on $\mathcal{F}$. Then there exists $\epsilon_{\mathcal{F}} > 0$ such that, if $f: \mathfrak{M} \to M$ is an $\epsilon_{\mathcal{F}}$-map to a compact manifold M, then for $x_0 \in \mathfrak{M}$ with $w_0 = f(x_0)$, we have $B_M(w_0, \delta_1) \subset f(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2))$.*

4.2. **Local lifting property.** We next establish a technical result used in the proof of Theorem 1.5. For δ_1 as chosen above so that (20) holds, set $\epsilon_1' = \epsilon_f(\delta_1)$.

LEMMA 4.2. *Let $f: \mathfrak{M} \to M$ be an $\epsilon_{\mathcal{F}}$-map. Let $x_0 \in \mathfrak{M}$ and suppose that $B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{U}}) \subset U_i$ for some $1 \leq i \leq \nu$. Then for $z \in B_{\mathfrak{M}}(x_0, \epsilon_1')$ and $y \in D_{\mathcal{F}}(x, \lambda_2) \subset \mathcal{P}_i(x)$, we have $f^{-1}(f(y)) \cap \mathcal{P}_i(z) \neq \emptyset$.*

Proof. Set $w_0 = f(x_0) \in M$, then by (18) we have that $B_{\mathfrak{M}}(x_0, \epsilon_1') \subset f^{-1}(B_M(w_0, \delta_1)) \subset U_i$ so that $f(z) \in B_M(w_0, \delta_1)$. Moreover, by Proposition 4.1, we have that $B_M(w_0, \delta_1) \subset f(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2))$. Then choose $x_z \in D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2)$ with $f(x_z) = f(z)$.

Let $\gamma_y: [0,1] \to \mathcal{P}_i(x_0)$ be the geodesic path in $\mathcal{P}_i(x_0)$ with $\gamma_y(0) = x_z$ and $\gamma_y(1) = y$.

Let $0 \leq s_* \leq 1$ be the largest value such that

$$f^{-1}(f((\gamma_y(s))) \cap \mathcal{P}_i(z) \neq \emptyset \quad \text{for all} \quad 0 \leq s \leq s_* \ .$$

We claim that $s_* = 1$. Suppose that $s_* < 1$, then we show this yields a contradiction.

Set $x_* = \gamma_y(s_*)$ and $w_* = f(x_*)$. Then there exists $z_* \in f^{-1}(w_*) \cap \mathcal{P}_i(z)$ by the definition of s_*. Note that $x_* \in B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{F}}) \subset B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{U}}/4)$ by the choice of λ_2 and the fact that $D_{\mathcal{F}}(x_0, \lambda_2)$ is strongly convex. By the choice of f and $\epsilon_{\mathcal{F}}$ in (17), we have $d_{\mathfrak{M}}(x_*, z_*) < \epsilon_{\mathcal{F}} < \epsilon_{\mathcal{U}}/4$. Thus $d_{\mathfrak{M}}(x_0, z_*) < \epsilon_{\mathcal{U}}/2$ and hence $B_{\mathfrak{M}}(z_*, \epsilon_{\mathcal{F}}) \subset B_{\mathfrak{M}}(x_0, \epsilon_{\mathcal{U}}) \subset U_i$. It then follows from the choice of λ_2 and the above observations that

$$(22) \qquad D_{\mathcal{F}}(z_*, \lambda_2) \subset B_{\mathfrak{M}}(z_*, \epsilon_{\mathcal{F}}) \subset U_i \ .$$

The value of $\epsilon_{\mathcal{F}} > 0$ is less than or equal to the choice $\epsilon_1/2$ for this constant in the proof of Proposition 4.1 in Appendix A, and thus $B_M(w_*, \delta_1) \subset f(D_{\mathcal{F}}(z_*, \lambda_{\mathcal{F}}/2))$. The assumption that $s_* < 1$ implies that for $s_* \leq s < 1$ sufficiently small so that $f(\gamma_z(s)) \in B_M(w_*, \delta_1)$, we have that

$$\mathcal{P}_i(z) \cap f^{-1}(f(\gamma_z(s))) = D_{\mathcal{F}}(z_*, \lambda_{\mathcal{F}}/2) \cap f^{-1}(f(\gamma_z(s))) \neq \emptyset$$

which contradicts the choice of s_*. $\qquad \square$

We next extend the conclusion of Lemma 4.2 from paths contained in a coordinate chart, to leafwise paths defined by a plaque chain of arbitrary length.

Let $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ be an admissible chain with associated holonomy map $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}^*$ and $w_0 \in \mathfrak{D}(h_{\mathcal{I}})$. As in Section 3.2, we associate to the pair $(\mathcal{I}, w_0)$ the plaque chain $\mathcal{P}_{\mathcal{I}}(w_0) = \{\mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \ldots, \mathcal{P}_{i_\alpha}(w_\alpha)\}$ given in (14).

Next introduce a plaque chain $\mathcal{J} = (j_0, j_1, \ldots, j_\beta)$ which is a refinement of $\mathcal{I}$ at w_0 and is chosen with respect to the leaf distance constant $\lambda_2 > 0$ which was defined so that the inclusions in (21)

hold. By Proposition 3.1, its associated holonomy map $h_{\mathcal{J}}$ at w_0 agrees with the holonomy map $h_{\mathcal{I}}$ at w_0 on their common domains.

Let $\gamma \colon [0, \alpha] \to L_{x_0} \subset \mathfrak{M}$ be the leafwise piecewise geodesic associated to the plaque chain $\mathcal{P}_{\mathcal{I}}(x_0)$. That is, $\gamma \colon [0, \alpha] \to L_{x_0}$ is the concatenation of geodesic segments $\{\gamma_\ell \mid 0 \leq \ell \leq \alpha - 1\}$ in the plaques of the covering $\mathcal{U}$, where $\gamma_\ell \colon [\ell, \ell + 1] \to \mathcal{P}_{i_\ell}(w_\ell)$ satisfies

$$\gamma_\ell(\ell) = x_\ell = \tau_{i_\ell}(w_\ell) \in \mathcal{T}_{i_\ell} \quad , \quad \gamma_\ell(\ell + 1) = x_{\ell+1} = \tau_{i_{\ell+1}}(w_{i_{\ell+1}}) \in \mathcal{T}_{i_{\ell+1}} .$$

Introduce a subdivision of the interval $[0, \alpha]$, given by $0 = s_0 < s_1 < s_2 < \cdots < s_\beta = \alpha$, where there is an increasing subsequence $\{\ell \mid 0 \leq \ell \leq \alpha = s_\beta\}$. For notational convenience, set $s_{-1} = s_0 = 0$ and $s_{\beta+1} = s_\beta = \alpha$. Then set $\xi_\ell = \gamma(s_\ell)$ for $-1 \leq \ell \leq \beta + 1$, and we choose the subdivision so that for each $0 \leq \ell \leq \beta$,

$$(23) \qquad\qquad d_{\mathcal{F}}(\gamma(s), \xi_\ell) < \lambda_2 \text{ for } s_{\ell-1} \leq s \leq s_{\ell+1} .$$

For each $0 \leq \ell \leq \beta$, choose an index $1 \leq j_\ell \leq \nu$ so that $B_{\mathfrak{M}}(\xi_\ell, \epsilon_{\mathcal{U}}) \subset U_{j_\ell}$. It then follows by the choice of $\epsilon_{\mathcal{F}}$, λ_2 and (21) that for each $0 \leq \ell \leq \beta$, we have

$$(24) \qquad \gamma(s) \in D_{\mathcal{F}}(\xi_\ell, \lambda_2) \subset B_{\mathfrak{M}}(\xi_\ell, \epsilon_{\mathcal{F}}) \subset B_{\mathfrak{M}}(\xi_\ell, \epsilon_{\mathcal{U}}/4) \quad \text{for all } s_{\ell-1} \leq s \leq s_{\ell+1} .$$

Moreover, $d_{\mathcal{F}}(\xi_\ell, \xi_{\ell+1}) < \lambda_2$ implies that

$$(25) \qquad\qquad \xi_{\ell+1} \in D_{\mathcal{F}}(\xi_\ell, \lambda_2) \cap D_{\mathcal{F}}(\xi_{\ell+1}, \lambda_2) \subset \mathcal{P}_{j_\ell}(\xi_\ell) \cap \mathcal{P}_{j_{\ell+1}}(\xi_{\ell+1}) .$$

Thus $\mathcal{J} = (j_0, j_1, \ldots, j_\beta)$ is an admissible sequence, and $\mathcal{P}_{\mathcal{J}}(w_0) = \{\mathcal{P}_{j_0}(\xi_0), \mathcal{P}_{j_1}(\xi_1), \ldots, \mathcal{P}_{j_\beta}(\xi_\beta)\}$ defines a holonomy map $h_{\mathcal{J}}$ at w_0.

Now let $\epsilon_{\mathcal{F}} > 0$ be as above, $\xi_0 = x_0 \in \mathcal{T}_{j_0}$ for the plaque chain $\mathcal{J}$ as chosen above, and suppose that $\mathcal{P}_{j_0}(z_0) \cap f^{-1}(f(\xi_0)) \neq \emptyset$ for some $z_0 \in \mathfrak{T}_{j_0}$. Then $d_{\mathcal{F}}(\xi_0, \xi_1) < \lambda_2$ by (23), so by (25) we have $\xi_1 \in \mathcal{P}_{j_0}(\xi_0) \cap \mathcal{P}_{j_1}(\xi_1)$. Hence by Lemma 4.2 there exists $z_1' \in f^{-1}(f(\xi_1)) \cap \mathcal{P}_{j_0}(z_0)$.

Note that $d_{\mathfrak{M}}(\xi_1, z_1') \leq \epsilon_{\mathcal{F}} \leq \epsilon_{\mathcal{U}}/4$, so $z_1' \in B_{\mathfrak{M}}(\xi_1, \epsilon_{\mathcal{U}}/4) \subset U_{j_1}$. Thus, there exists $z_1 \in \mathcal{T}_{j_1}$ such that $z_1' \in \mathcal{P}_{j_1}(z_1)$ and hence $\mathcal{P}_{j_0}(z_0) \cap \mathcal{P}_{j_1}(z_1) \neq \emptyset$.

We now repeat the application of Lemma 4.2 to the new basepoint ξ_1, and then continue recursively to obtain a sequence of points $\{z_\ell \in \mathcal{T}_{j_\ell} \mid 0 \leq \ell \leq \beta\}$ such that for $0 < \ell \leq \beta$ we have:

- $\mathcal{P}_{j_{\ell-1}}(z_{\ell-1}) \cap \mathcal{P}_{j_\ell}(z_\ell) \neq \emptyset$,
- $z_\ell' \in f^{-1}(f(\xi_\ell)) \cap \mathcal{P}_{j_\ell}(z_\ell)$.

Recall that $\pi_i \colon \overline{U}_i \to \mathfrak{T}_i$ for $1 \leq i \leq \nu$ is the transverse projection to the model space $\mathfrak{T}_i$. Then the above shows that for $w_0 = \pi_{j_0}(z_0)$ and $w_\beta = \pi_{j_\beta}(z_\beta)$ we have $w_0 \in \mathfrak{D}(h_{\mathcal{J}})$ and $h_{\mathcal{J}}(w_0) = w_\beta$.

4.3. **Proof of Theorem 1.5.** We can now complete the proof of Theorem 1.5. We have assumed that $\epsilon > 0$ is given, and $\epsilon_{\mathcal{F}} > 0$ is defined as in (17). Then choose an $\epsilon_{\mathcal{F}}$-map f as in Section 4.1. Let $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}^*$ be as in Section 4.2, and $\mathcal{P}_{\mathcal{J}}$ the path chain constructed above from $\mathcal{I}$.

For each $1 \leq i \leq \nu$ the transversal map $\tau_i \colon \mathfrak{T}_i \to \mathcal{T}_i$ is a homeomorphism of compact spaces, and the metric $d_{\mathcal{T}_i}$ on the subspace $\mathcal{T}_i \subset \overline{U}_i \subset \mathfrak{M}$ was defined in Section 2.1 as the restriction of $d_{\mathfrak{M}}$.

Recall that $\epsilon_1' = \epsilon_f(\delta_1)$ was defined in Section 4.2 and used in the hypothesis of Lemma 4.2. By the uniform continuity of the maps τ_i, there exists $\delta > 0$ such that for all $1 \leq i \leq \nu$ and $w \in \mathfrak{T}_i$,

$$(26) \qquad\qquad B_{\mathfrak{T}_i}(w, \delta) \subset \tau_i^{-1}(B_{\mathcal{T}_i}(\tau_i(w), \epsilon_1')) .$$

It thus follows from the above results that $h_{\mathcal{J}}(B_{\mathfrak{T}_{j_0}}(w, \delta)) \subset B_{\mathfrak{T}_{j_\beta}}(h_{\mathcal{J}}(w), \epsilon)$, as was to be shown.

APPENDIX A. LOCAL SURJECTIVITY FOR ϵ-MAPS

In this appendix, we give a technical result concerning ϵ-maps.

PROPOSITION A.1. *Let $\mathfrak{M}$ be a matchbox manifold with leafwise Riemannian metric on $\mathcal{F}$. Then there exists $\epsilon_{\mathcal{F}} > 0$ such that, if $f: \mathfrak{M} \to M$ is an $\epsilon_{\mathcal{F}}$-map to a compact manifold M, then there exists $\delta_1 > 0$ such that for $x_0 \in \mathfrak{M}$ with $w_0 = f(x_0)$, we have $D_M(f(x_0), \delta_1) \subset f(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2))$.*

Proof. We use the notations of Section 2.2 above.

Choose $\epsilon_0 > 0$ so that $\rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon_0) \leq \lambda_{\mathcal{F}}/2$, as defined by (3) to be the maximal radius of a leafwise disk about x contained in a closed disk of radius ϵ_0 in $\mathfrak{M}$, for *all* $x \in \mathfrak{M}$.

Set $\lambda_0 = \lambda_{\mathcal{F}}(\epsilon_0)$, which is defined in Section 2.4 to be the largest radius $\lambda \leq \lambda_{\mathcal{F}}$ such that the disk $D_{\mathcal{F}}(x, \lambda)$ is contained in $B_{\mathfrak{M}}(x, \epsilon_0)$, for *all* $x \in \mathfrak{M}$. Then by the definition of $\rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon_0)$ and the inclusions (8), for all $x \in \mathfrak{M}$ we have the inclusions

$$(27) \qquad D_{\mathcal{F}}(x, \lambda_0) \subset D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon_0) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}/2) \cap B_{\mathfrak{M}}(x, \epsilon_0) .$$

Next, choose $\epsilon_1 > 0$ so that $\rho(d_{\mathcal{F}}, d_{\mathfrak{M}}, \epsilon_1) \leq \lambda_0/10$, and let $\lambda_1 = \lambda_{\mathcal{F}}(\epsilon_1)$. Then for all $x \in \mathfrak{M}$,

$$(28) \qquad D_{\mathcal{F}}(x, \lambda_1) \subset D_{\mathcal{F}}(d_{\mathfrak{M}}, x, \epsilon_1) \subset D_{\mathcal{F}}(x, \lambda_0/10) \cap B_{\mathfrak{M}}(x, \epsilon_1) .$$

Set $\epsilon_{\mathcal{F}} = \epsilon_1/2$. This constant is chosen so that the result [19, Section 1, Théorème] by Eilenberg holds uniformly for strongly convex compact subsets of $D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \subset L_x$, as will be shown below.

Let $f: \mathfrak{M} \to M$ be an $\epsilon_{\mathcal{F}}$-map, which is onto the compact manifold M. Let d_M be a metric on M. For $w \in M$ and $\delta > 0$, let $B_M(w, \delta) = \{w' \in M \mid d_M(w', w) < \delta\}$ denote the open disk in M of radius δ, and $D_M(w, \delta) = \{w' \in M \mid d_M(w', w) \leq \delta\}$ denote the closed disk in M of radius δ.

Assume that d_M is chosen so that there exists a constant $\delta_M > 0$ such that for all $w \in M$ and $0 < \delta \leq \delta_M$, the disk $D_M(w, \delta)$ is homeomorphic to a disk in $\mathbb{R}^n$. For example, if M is a Riemannian manifold, then let $\delta_M > 0$ be such that each disk $D_M(w, \delta_M)$ is strongly convex.

Since $\mathfrak{M}$ is compact, there exists a uniform modulus of continuity function $\epsilon_f(\delta) > 0$ for f: for $\delta > 0$, the constant $\epsilon_f(\delta)$ is the largest value such that

$$(29) \qquad B_{\mathfrak{M}}(x, \epsilon_f(\delta)) \subset f^{-1}(B_M(f(x), \delta)) \text{ for all } x \in \mathfrak{M} .$$

Let $\delta_f^* > 0$ be the largest radius such that $\epsilon_f(\delta) \leq \epsilon_{\mathcal{F}}$ for all $0 < \delta \leq \delta_f^*$.

Set $\lambda_{\mathcal{F}, f}(\delta) = \lambda_{\mathcal{F}}(\epsilon_f(\delta))$, which is well-defined for $0 < \delta \leq \delta_f^*$. Combining (8) and (29) we obtain a leafwise modulus of continuity for f:

$$(30) \qquad D_{\mathcal{F}}(x, \lambda_{\mathcal{F}, f}(\delta)) \subset B_{\mathfrak{M}}(x, \epsilon_f(\delta)) \subset f^{-1}(B_M(f(x), \delta)) \quad \text{for all } 0 < \delta \leq \delta_f^* , \ x \in \mathfrak{M} .$$

As f is an $\epsilon_{\mathcal{F}}$-map, we have that $f^{-1}(f(x)) \subset B_{\mathfrak{M}}(x, \epsilon_{\mathcal{F}})$ for all $x \in \mathfrak{M}$. As M is compact and $f^{-1}(f(x))$ is a compact set with diameter at most $\epsilon_{\mathcal{F}}$ for all $x \in \mathfrak{M}$, there exists $0 < \delta_1 \leq \delta_M/10$ so that for all $x \in \mathfrak{M}$, we have

$$(31) \qquad f^{-1}(D_M(f(x), \delta_1)) \subset B_{\mathfrak{M}}(x, 2\epsilon_{\mathcal{F}}) = B_{\mathfrak{M}}(x, \epsilon_1) .$$

Let $\lambda_2 = \lambda_{\mathcal{F}, f}(\delta_1)$ so that for all $x \in \mathfrak{M}$, by (30), (28), and the choice of δ_1 we have

$$(32) \qquad D_{\mathcal{F}}(x, \lambda_2) \subset f^{-1}(D_M(f(x), \delta_1)) \cap D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \subset D_{\mathfrak{M}}(x, \epsilon_1) \cap D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \subset D_{\mathcal{F}}(x, \lambda_0/10) ,$$

$$(33) \qquad f(D_{\mathcal{F}}(x, \lambda_2)) \subset D_M(f(x), \delta_1) \subset D_M(f(x), \delta_M/10) .$$

Set $\lambda_3 = 5\lambda_2$ so that by (32) we have $\lambda_3 \leq \lambda_{\mathcal{F}}/2$.

LEMMA A.2. *Let $x \in \mathfrak{M}$ and $w = f(x) \in M$, then $f(D_{\mathcal{F}}(x, \lambda_3)) \subset D_M(w, \delta_M)$.*

Proof. Let $y \in D_{\mathcal{F}}(x, \lambda_3) \subset D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}/2)$ and let $\sigma: [0, 1] \to D_{\mathcal{F}}(x, \lambda_3)$ be the unique geodesic segment with $\sigma(0) = x$ and $\sigma(1) = y$. Set $x_i = \sigma(i/5)$ for $0 \leq i \leq 5$, then $x_5 = y$. Note that as σ is a geodesic, we have

$$d_{\mathcal{F}}(x_i, x_{i+1}) = d_{\mathcal{F}}(x, y)/5 \leq \lambda_3/5 = \lambda_2 , \ 0 \leq i < 5 .$$

Set $w_i = f(x_i)$ for $0 \leq i \leq 5$.

Then for each $0 \leq i \leq 5$, by (33) we have $D_{\mathcal{F}}(x_i, \lambda_2) \subset f^{-1}(D_M(w_i, \delta_1))$, so that the collection $\{D_M(w_i, \delta_1) \mid 0 \leq i \leq 5\}$ is a covering of the image of σ. Thus, $d_M(w_0, w_5) \leq 10\delta_1 \leq \delta_M$, hence $d_M(w, f(y)) \leq \delta_M$, as was to be shown. $\qquad\square$

For $x \in \mathfrak{M}$ and $0 < \lambda \leq \lambda_{\mathcal{F}}$, introduce the following leafwise sets:
$$\begin{aligned} S_{\mathcal{F}}(x, \lambda) &= \{y \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \mid d_{\mathcal{F}}(y, x) = \lambda\} \\ D_{\mathcal{F}}^*(x, \lambda) &= \{y \in D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \mid 0 < d_{\mathcal{F}}(y, x) \leq \lambda\} \ . \end{aligned}$$
Then we have $S_{\mathcal{F}}(x, \lambda) \subset D_{\mathcal{F}}^*(x, \lambda) \subset D_{\mathcal{F}}(x, \lambda) \subset B_{\mathcal{F}}(x, \lambda_{\mathcal{F}})$.

Now let $x_0 \in \mathfrak{M}$ with $w_0 = f(x_0)$. We claim that $B_M(w_0, \delta_1) \subset f(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2))$. Suppose not, then we show this yields a contraction.

Let $w_2 \in B_M(w_0, \delta_1)$ but $w_2 \notin f(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2))$. Then there exists $0 < \delta_2 < \delta_1$ such that

(34) $$B_M(w_2, \delta_2) \subset B_M(w_0, \delta_1) \quad , \quad B_M(w_2, \delta_2) \cap f(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2)) = \emptyset \ .$$

We consider the maps on Čech cohomology induced by f to obtain the contradiction.

For $0 < \lambda < \lambda_{\mathcal{F}}/2$, introduce the collections of open sets in L_{x_0}:

(35) $$\mathcal{L}(\lambda) = \{B_{\mathcal{F}}(x, \lambda) \mid x \in D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2)\}$$

(36) $$\mathcal{L}^*(\lambda) = \{B_{\mathcal{F}}(x, \lambda) \mid x \in D_{\mathcal{F}}^*(x_0, \lambda_{\mathcal{F}}/2) \text{ such that } B_{\mathcal{F}}(x, \lambda) \subset D_{\mathcal{F}}^*(x_0, \lambda_{\mathcal{F}})\} \ .$$

For each $x \in \mathfrak{M}$, the disk $B_{\mathcal{F}}(x, \lambda) \subset L_x$ is strongly convex, thus $\mathcal{L}(\lambda)$ is a good covering of $D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2)$ in the sense of Čech theory. Let $\|\mathcal{L}(\lambda)\|$ denote the simplicial space which is the geometric realization of the collection $\mathcal{L}(\lambda)$.

For $0 < \lambda < \lambda' < \lambda_{\mathcal{F}}/2$, each open disk $B_{\mathcal{F}}(x, \lambda) \in \mathcal{L}(\lambda)$ is contained in the disk $B_{\mathcal{F}}(x, \lambda') \in \mathcal{L}(\lambda')$ which induces a map between the realizations of their nerve complexes, $\iota \colon \|\mathcal{L}(\lambda)\| \to \|\mathcal{L}(\lambda')\|$, which is a homotopy equivalence as all the sets in the cover are strongly convex. Similarly, the induced map on the nerve complex induces a homotopy equivalence $\iota \colon \|\mathcal{L}^*(\lambda)\| \to \|\mathcal{L}^*(\lambda')\|$.

For $0 < \delta < \delta_2$, where δ_2 was chosen so that (34) holds, introduce the collections of open sets in M:

(37) $$\mathcal{M}_{\delta_2}(\delta) = \{B_M(w, \delta) \mid w \in D_M(w_0, \delta_M) - B_M(w_2, \delta_2)\}$$

(38) $$\mathcal{M}_f(\delta) = \{B_M(w, \delta) \mid w \in f(D_{\mathcal{F}}(x_0, \lambda_3))\}$$

(39) $$\mathcal{M}_S(\delta) = \{B_M(w, \delta) \mid w \in f(S_{\mathcal{F}}(x_0, \lambda_3))\} \ .$$

Recall that by Lemma A.2, $f(S_{\mathcal{F}}(x, \lambda_3)) \subset D_M(w, \delta_M)$, so by the choice of δ_2 we have inclusions $\mathcal{M}_S(\delta) \subset \mathcal{M}_f(\delta) \subset \mathcal{M}^*(\delta)$, and so obtain maps of their simplicial realizations

(40) $$\|\mathcal{M}_S(\delta)\| \longrightarrow \|\mathcal{M}_f(\delta)\| \longrightarrow \|\mathcal{M}_{\delta_2}(\delta)\| \ .$$

As the disk $D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}})$ is strongly convex, there is a natural map $\mathcal{R}_\lambda \colon \|\mathcal{L}(\lambda)\| \to D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2)$ which maps a simplex in the realization $\|\mathcal{L}(\lambda)\|$ to the geodesic simplex in $D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2)$ spanned by its vertices. Then $\mathcal{R}_\lambda$ induces isomorphisms

(41) $$\mathcal{R}_\lambda^* \colon \{0\} \cong H^n(D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}}/2); \mathbb{Z}) \to H^n(\|\mathcal{L}(\lambda)\|; \mathbb{Z})$$

(42) $$\mathcal{R}_\lambda^* \colon \mathbb{Z} \cong H^{n-1}(D_{\mathcal{F}}^*(x_0, \lambda_{\mathcal{F}}/2); \mathbb{Z}) \to H^{n-1}(\|\mathcal{L}^*(\lambda)\|; \mathbb{Z}) \ .$$

Similarly, for $0 < \delta < \delta_2$, there is a continuous map $\mathcal{S}_{\delta_2} \colon \|\mathcal{M}_{\delta_2}(\delta)\| \to D_M(w, \delta_M) - B_M(w_2, \delta_2)$.

Let $0 < \delta_3 \leq \delta_2$ be sufficiently small so that we have an inclusion map

(43) $$\mathcal{S}_{\delta_2}^* \colon \mathbb{Z} \cong H^{n-1}(\{D_M(w_0, \delta_M) - B_M(w_2, \delta_2)\}; \mathbb{Z}) \to H^{n-1}(\|\mathcal{M}_{\delta_2}(\delta_3)\|; \mathbb{Z}) \ .$$

We next consider the maps induced by f on the cohomology groups in (41), (42) and (43).

Let $\lambda_4 = \lambda_{\mathcal{F}, f}(\delta_3)$ so that for all $x \in \mathfrak{M}$, we have the inclusion $f(D_{\mathcal{F}}(x, \lambda_4)) \subset D_M(f(x), \delta_3)$. Then f induces an inclusion map $\mathcal{U}_f \colon \mathcal{L}(\lambda_4) \to \mathcal{M}_{\delta_2}(\delta_3)$, which induces a map of their realizations

(44) $$\|\mathcal{U}_f\| \colon \|\mathcal{L}(\lambda_4)\| \to \|\mathcal{M}_{\delta_2}(\delta_3)\| \ .$$

We have that $0 < \delta_3 \leq \delta_2 < \delta_1$ so that by (31) and (32), for all $x \in \mathfrak{M}$ we have

$$(45) \qquad f^{-1}(D_M(f(x), \delta_3)) \cap D_{\mathcal{F}}(x, \lambda_{\mathcal{F}}) \subset D_{\mathcal{F}}(x, \lambda_0/10) .$$

For each $B_M(w, \delta_3) \in \mathcal{M}_f(\delta_3)$ choose $x \in f^{-1}(w) \cap D_{\mathcal{F}}(x_0, \lambda_{\mathcal{F}})$, then the inclusion (45) holds. Thus, f^{-1} induces an inclusion map $\mathcal{V}_f \colon \mathcal{M}_{\delta_2}(\delta_3) \to \mathcal{L}(\lambda_0/10)$, which in turn induces a map between their realizations

$$(46) \qquad \|\mathcal{V}_f\| \colon \|\mathcal{M}_{\delta_2}(\delta_3)\| \to \|\mathcal{L}(\lambda_0/10)\| .$$

Note that the composition $\|\mathcal{V}_f\| \circ \|\mathcal{U}_f\| \colon \|\mathcal{L}(\lambda_4)\| \to \|\mathcal{L}(\lambda_0/10)\|$ is a homotopy equivalence, as the sets in $\mathcal{L}(\lambda_0/10)$ are strongly convex.

Next, for $0 < \lambda \leq \lambda_0/10$, introduce the collection of open balls centered at points of $S_{\mathcal{F}}(x_0, \lambda_3)$,

$$(47) \qquad \mathcal{L}_S(\lambda) = \{B_{\mathcal{F}}(x, \lambda) \mid x \in S_{\mathcal{F}}(x_0, \lambda_3) \text{ such that } B_{\mathcal{F}}(x, \lambda) \subset D_{\mathcal{F}}^*(x_0, \lambda_{\mathcal{F}})\} .$$

As above, the map f induces maps of geometric realizations

$$(48) \qquad \|\mathcal{U}_f\| \quad : \quad \|\mathcal{L}_S(\lambda_4)\| \to \|\mathcal{M}_S(\delta_3)\|$$

$$(49) \qquad \|\mathcal{V}_f\| \quad : \quad \|\mathcal{M}_S(\delta_3)\| \to \|\mathcal{L}_S(\lambda_0/10)\| ,$$

and the composition $\|\mathcal{V}_f\| \circ \|\mathcal{U}_f\| \colon \|\mathcal{L}_S(\lambda_4)\| \to \|\mathcal{L}_S(\lambda_0/10)\|$ is a homotopy equivalence.

The space $\|\mathcal{L}_S(\lambda_0/10)\|$ is the simplicial realization of a good covering of $S_{\mathcal{F}}(x_0, \lambda_3)$, so we have $H^{n-1}(\|\mathcal{L}_S(\lambda_0/10)\|, \mathbb{Z}) \cong \mathbb{Z}$. Let $\omega \in H^{n-1}(\|\mathcal{L}_S(\lambda_0/10)\|, \mathbb{Z})$ be a choice of a generator.

Note that the compact sets $f(S_{\mathcal{F}}(x, \lambda))$ in M limit to the point $w = f(x)$ as $\lambda \to 0$. It follows that there is a non-trivial class $\|\mathcal{V}_f\|^*(\omega) \in Image\left\{\mathcal{S}_{\delta_2}^*\right\}$ where $\mathcal{S}_{\delta_2}^*$ is given in (43).

On the other hand, $\|\mathcal{V}_f\|^*$ factors through the map

$$\|\mathcal{V}_f\|^* \colon H^{n-1}(\|\mathcal{L}(\lambda_0/10)\|; \mathbb{Z}) \to H^{n-1}(\|\mathcal{M}_{\delta_2}(\delta_3)\|, \mathbb{Z}) .$$

The group $H^{n-1}(\|\mathcal{L}(\lambda_0/10)\|; \mathbb{Z}) \cong \{0\}$, so that $\|\mathcal{V}_f\|^*(\omega) = 0$, which is a contradiction. $\qquad \square$

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Alex Clark, Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

Email address: `Alex.Clark@leicester.ac.uk`

Steven Hurder, Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045

Email address: `hurder@uic.edu`

Olga Lukina, Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045

Email address: `lukina@uic.edu`