SELF-INTERSECTION POINTS OF GENERALIZED KOCH CURVES

MICHAEL CANTRELL AND JUDITH PALAGALLO

Abstract. We present a two-parameter family of curves $K_{a,\theta}$ that are modifications of the Koch curve. For each $\theta$, there is a corresponding pivotal value $a(\theta)$ of $a$, where $K_{a,\theta}$ is simple if $a > a(\theta)$, and $K_{a,\theta}$ is self-intersecting if $a < a(\theta)$. We find $a(\theta)$, for $\theta \in (0, \pi/3]$, then show that the pivotal-valued curves comprise two classes of self-intersecting curves that we characterize by whether or not $\theta = \pi/n$ for some even positive integer $n > 2$.

In this paper, a curve is the image of a nontrivial continuous function from $I = [0, 1]$ into the plane. The Koch curve, introduced by the Swedish mathematician Helge von Koch in 1904, is an example of a self-similar curve, in which every piece contains a portion similar to the whole figure. It is also a simple curve, the image of a one-to-one, continuous function from $I$ into the plane. Here we discuss a two-parameter family of curves $K_{a,\theta}$ that are modifications of the Koch curve. We are particularly interested in the self-intersection properties of curves that are not simple. The work in this paper generalizes material found in [2]. We note also that Keleti modifies the Koch curve in a different way in [3].

Our generalizations of the Koch curve start with a generic generator $G = G(a, \theta)$ as shown in Figure 1 and allow the angle $\theta$ and the length $a$ to vary. In this article we restrict values to $0 < a < 1/2$ and $0 < \theta \leq \pi/3$. For each $\theta$, there is a corresponding pivotal value $a(\theta)$ of $a$, where $K_{a,\theta}$ is simple if $a > a(\theta)$, and $K_{a,\theta}$ is self-intersecting if $a < a(\theta)$. We provide a method for finding the pivotal values. The pivotal-valued curves comprise two classes of self-intersecting curves.

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that we characterize by whether or not $\theta = \pi/n$ for some even positive integer $n > 2$. The self-intersection sets are composed of double points: points with two inverse images. Our characterization classifies the double points of a given pivotal-valued curve $K_{a,\theta}$ and its lagoons - the bounded components of the complement of the curve in $\mathbb{R}^2$.

1. Construction of generalized Koch curves

We consider the generator $G$ as a path in the complex plane $\mathbb{C} = \mathbb{R}^2$ from $(0, 0)$ to $(1, 0)$ and make use of the constants $C(j)$ and $D(j)$ given in Table 1 to describe this path. The path is composed of four intervals that abut in pairs at three interior nodes $C(1)$, $C(2)$ and $C(3)$ and we define $D(j) = C(j + 1) - C(j)$, $j = 0, 1, 2, 3$. We restrict attention to $0 < a < 1/2$ and $L < 1$. If $L < 1$ then $G$ generates a self-similar curve by a replacement process that we describe below. In some sense, $G$ is a classic generator of fractal images. For example, when $a = 1/3$ and $\theta = \pi/3$, we are in the setting of the Koch curve.

1.1. The iteration process. We begin by defining a natural homeomorphism $f_1$ from the interval $I = [0, 1]$ onto $G$. The mapping $f_1$ is specified by requiring that it map each interval $[j/4, (j + 1)/4]$ in $I$ linearly onto the corresponding interval $[C(j), C(j + 1)]$ in $G$, $j = 0, 1, 2, 3$. Then, starting with $f_1$ we recursively define a sequence of continuous maps $\{f_k\}$ on $I$ which converges uniformly to a continuous

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### Table 1. Values of $C(j)$ and $D(j)$

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(j)$</td>
<td>0</td>
<td>$a$</td>
<td>$a + Le^{i\theta}$</td>
<td>$1 - a$</td>
<td>1</td>
</tr>
<tr>
<td>$D(j)$</td>
<td>$a$</td>
<td>$Le^{i\theta}$</td>
<td>$Le^{-i\theta}$</td>
<td>$a$</td>
<td>$-$</td>
</tr>
</tbody>
</table>
map \( f \) from \( I \) into \( \mathbb{C} \). The **generalized Koch curve** \( K_{a,\theta} \) is defined to be the image \( f(I) \).

1.1.1. **Formulas for** \( f_k \) **and** \( f \).** Because the generator is composed of 4 intervals, we use base 4 representations for numbers in \( I \). Therefore, \( x \in I \) has representation

\[
x = 0.x_1x_2x_3 \ldots \text{base } 4 = \frac{x_1}{4} + \frac{x_2}{4^2} + \frac{x_3}{4^3} + \ldots
\]

where each \( x_i \) is a nonnegative integer less than 4.

For \( x = 0.x_1x_2x_3 \ldots \text{base } 4 = \sum_{j \geq 1}^{x_j} 4^j \) a left shift operator is defined by \( lx = 0.x_2x_3 \ldots \text{base } 4 \). Using the values given in Table 1 we define

\[
f_1(x) = C(x_1) + D(x_1)lx
\]

and

\[
f_{k+1}(x) = C(x_1) + D(x_1)f_k(lx) \text{ for } k \geq 1.
\]

New paths in the plane \( f_k(I) \) are generated with each iteration in the process. As an illustration, Table 2 shows \( f_2(I) \) and \( f_3(I) \) for a curve with \( \theta = \pi/3 \).

Furthermore,

\[
\|f_{k+1} - f_k\| = D\|f_k - f_{k-1}\|
\]

where

\[
D = \max\{\|D(k)\| : k \in \{0, 1, 2, 3\}\} = \max\{a, L\} < 1.
\]

Consequently, \( \{f_k\} \) is a Cauchy sequence of continuous functions and therefore converges to a continuous function \( f \) on \( I \), and \( f \) satisfies the following functional equation

\[
f(x) = C(x_1) + D(x_1)f(lx), x \in I.
\]

The images in Table 3 indicate that, although each generator is a simple path from \((0,0)\) to \((1,0)\) (Visually, the path does not cross itself.), the resulting paths \( K_{a,\theta} \) may have points of self-intersection.
1.1.2. **Visualizing the process.** We see that \( f_{k+1} (I) \) places a scaled, rotated, and translated copy of the generator onto each line segment forming \( f_k (I) \). Consequently, endpoints of any line segment at any stage will always be endpoints of future intervals.

We will verify that in the case where \( \theta = \pi/n \) for some even positive integer \( n > 2 \), the pivotal curve \( K_{a,\theta} \) contains a Cantor set of double points along the vertical line \( \ell \) with equation \( x = a \left( 1 - \frac{a}{2} \right) \). For all other values of \( \theta \) the pivotal curve in the region near \( \ell \) has one double point on \( \ell \). By symmetry and self-similarity the pivotal curves have infinitely many double points. Table 4 shows \( f_2 (I) \), an approximation to \( f (I) \), and a zoom-in of the approximation near the line \( \ell \) for the cases \( \theta = \pi/3, \pi/4, \pi/5, \) and \( \pi/6 \).

### 2. Description of \( K_{a,\theta} \)

The functional equation for \( f \) given in equation (1.2) suggests that each part of \( K_{a,\theta} \) is a suitably reduced, translated, and rotated copy of itself. Specifically, \( K_{a,\theta} = T_0 \cup T_1 \cup T_2 \cup T_3 \), where

\[
T_j = f \left( \left[ 0, j, 0, j + \frac{1}{4} \right] \right)
\]
and each $T_j$ is composed of four parts $T_j = T_{j0} \cup T_{j1} \cup T_{j2} \cup T_{j3}$ with

$$T_{jk} = f \left( \left[ 0, jk, 0, jk + \frac{1}{4} \right] \right), \quad k = 0, 1, 2, 3.$$

Inductively, each $T_{j_1j_2...j_n}$ is composed of four parts

$$T_{j_1j_2...j_n} = \bigcup_{j_{n+1}=a}^{3} T_{j_1j_2...j_nj_{n+1}}.$$

In particular, $T_0$ and $T_3$ are $a$-sized copies of $K_{a,\theta}$; $T_1$ is an $L$-sized copy of $K_{a,\theta}$, rotated by angle $\theta$ about $(0,0)$ in a counterclockwise direction and then translated so that $(0,0)$ maps to $C(1)$; $T_2$ is an $L$-sized copy of $K_{a,\theta}$ rotated by angle $\theta$ about $(0,0)$ in a clockwise direction and then translated so that $(0,0)$ maps to $C(2)$.

We will also utilize symmetries of the curves. For example, looking at Figure 3 we see that $T_{02}$ and $T_{10}$ are symmetric with respect to the line $\ell$.

2.1. Height Set of $K_{a,\theta}$. A careful examination of the curves $K_{a,\theta}$ and their self-intersection points in Table 4 reveals that the intersection points occur in sets that are scaled and rotated copies of the “top” of the curve. Therefore, it is natural to begin with a study of the height set of $K_{a,\theta}$.
Definition 1. The **height set** of a generalized Koch curve \( K_{a,\theta} \) is \( H_S = \{ z \in f(I) : \text{Im}(z) = \max \{ \text{Im}(z) : z \in f(I) \} \} \). That is, the height set is the set of all points on the curve which attain maximal vertical displacement. The **height** of a generalized Koch curve \( K_{a,\theta} \) is \( H = \max \{ \text{Im}(z) | z \in f(I) \} \).

A point \( x = 0.x_1x_2\ldots \) base 4 in \( I \) maps onto \( H_S \) if and only if each pair \( \{x_{2j-1}, x_{2j}\} \) consists of one 1 and one 2. We thus pick \( P_4 = f(0.21) \in H_S \) as a representative point, and find \( H = \text{Im}(P_4) \). (Throughout we use the overbar to indicate that the digits under the overbar are periodically repeated.)

Applying Equation 1.2 we find

\[
P_4 = f(0.21) = a + Le^{i\theta} + Le^{-i\theta} (f(0.12))
\]

\[
= a + Le^{i\theta} + Le^{-i\theta} (a + Le^{i\theta} (f(0.21)))
\]

\[
= a + Le^{i\theta} + aLe^{-i\theta} + L^2 f(0.21)
\]

Therefore

\[
f(0.21)(1 - L^2) = a + Le^{i\theta} + aLe^{-i\theta}
\]

\[
f(0.21) = \frac{a + Le^{i\theta} + aLe^{-i\theta}}{1 - L^2}
\]

\[
= \frac{1}{1 - L^2} (a + L \cos \theta + aL \cos \theta)
\]

\[
+ i \frac{1}{1 - L^2} (L \sin \theta - aL \sin \theta)
\]

So

\[
H = \left( \frac{1 - a}{1 - L^2} \right) L \sin \theta.
\]

2.2. **Cantor sets.** To examine the self-intersection points of these curves we define a class of Cantor sets that uses our function \( f \) and suffices for our purposes. Recall from Equation 1.2 that \( f \equiv f_a \) depends on \( a \). Set

\[
S = \{ t = 0.t_1t_2t_3\ldots \text{base 4} : t_j \in \{0, 3\}, \; j \geq 1 \}
\]

and note that \( f_a(I) \cap I = f_a(S) \). For \( 0 < a < 1/2 \), put \( C_a = f_a(S) \). For \( t \in S \) we have

\[
f_a(t) = (1 - a) \sum_{j=1}^{\infty} a^{j-1} \frac{t_j}{3}.
\]
For example, \( f_{1/3} (t) = (0.k_1 k_2 \ldots \text{base } 3) \), where \( k_j = 2t_j/3 \in \{0, 2\} \), so \( f_{1/3} (S) \) is the standard (middle-thirds removed) Cantor set \( C \). Consequently, the map \( f_a \circ (f_{1/3})^{-1} \) is a one-to-one, continuous, increasing function from \( C \) onto \( f_a (S) \). Hence \( C_a = f_a (S) \) is homeomorphic to \( C \) so \( C_a \) is a Cantor set, and each \( C_a \) is a closed subset of \( I \). For our purposes, a Cantor set \( \tilde{C}_a \) is a reduced, rotated, and translated copy of some \( C_a \). A point \( t \) is an endpoint of \( \tilde{C}_a \) if and only if \( t \in \tilde{C}_a \) and \( t \) is a boundary point of some component of \( \mathbb{R} - \tilde{C}_a \).

Figure 2 suggests the set \( H_S \) is a Cantor set. A point \( b = (0.b_1 b_2 b_3 \ldots \text{base } 4) \in I \) that maps onto \( H_S \) can be identified with \( t = (0.t_1 t_2 t_3 \ldots \text{base } 4) \in S \) by defining a function \( g \) with \( g (b) = t \) where \( t_i = \begin{cases} 0 & \text{if } b_i = 12 \\ 3 & \text{if } b_i = 21 \end{cases} \).

Then \( f \circ g^{-1} \) maps \( S \) onto \( H_S \).

**Proposition 1.** \( H_S \) is a \( W \)-sized copy of Cantor set \( C_{L_2} \), where \( W = \frac{L \cos \theta - L^2 \cos 2\theta}{1 - L^2} \).

**Proof.** Refer to Figure 2. Let \( A_0 = [P_1, P_4] \), where \( P_1 = f (0.1\overline{2}) \) and \( P_4 = f (0.2\overline{1}) \). Set \( P_2 = f (0.12\overline{2}) \) and \( P_3 = f (0.21\overline{1}) \) as shown. Using the definition of \( f \) and Euler’s formula, we have

\[ P_1 = f (0.\overline{12}) = \frac{1}{1 - L^2} (a + aL \cos \theta + L^2 \cos 2\theta) + iH \]
Let $I_1 = [P_1, P_2], I_2 = [P_3, P_4]$ and $A_1 = I_1 \cup I_2$. Repeat the construction on each of $I_1$ and $I_2$. We can use symmetry to find other endpoints of the intervals in the construction. Noting that $W = P_4 - P_1 = \frac{L \cos \theta - L^2 \cos 2\theta}{1 - L^2}$, the length of set $H_S$, and that

$$\frac{P_2 - P_1}{P_4 - P_1} = \frac{\frac{L^3 \cos \theta - L^4 \cos 2\theta}{1 - L^2}}{W} = L^2$$

we conclude that $H_S$ is a $W$-sized copy of $C_{L^2}$ shifted to $P_1$. \qed

3. Pivotal Value

In this section the value of $\theta$ will be fixed, and $K_{a,\theta}$ will refer to a curve with our fixed value of $\theta$, but with a variable; then $T_{j_1j_2...j_n}$ also corresponds to this $K_{a,\theta}$. Recall that $a_0$ is the pivotal value of a curve $K_{a,\theta}$ if for all $a < a_0$, $K_{a,\theta}$ is self-intersecting but for $a > a_0$, $K_{a,\theta}$ is a simple curve. With $\theta \in (0, \frac{\pi}{3}]$ fixed, we will demonstrate how to find the pivotal value $a_0$. Choose $n \in \mathbb{N}$ such that $n\theta \leq \frac{\pi}{3} < (n + 1)\theta$. Define the farthest point(s) right of $K_{a,\theta}$ to be the point(s) $z_r \in T_2$ with maximum horizontal displacement; i.e., $z_r = \{z \in f(I) : \text{Re}(z) = \text{max}\{\text{Re}(z) : z \in f(I)\}\}$. We note that $T_{22...2_n}$ is an $L^n$-sized copy of $K_{a,\theta}$, rotated clockwise $n\theta$ and translated so that (0, 0) maps to

$$f\left(0, \frac{22...2_n}{n}\right) = A = (a + Le^{i\theta}) \left(\frac{1 - L^n e^{-i\theta}}{1 - Le^{-i\theta}}\right)$$

with $x$ coordinate equal to

$$a (1 - L^n \cos \theta) + (1 - a) L (\cos \theta - L^n \cos (n - 1) \theta) - L^2 (\cos 2\theta - L^n \cos (n - 2)\theta)$$

$$(1 + L^2 - 2L \cos \theta).$$

Figure 3 shows $f_5(I)$ for $K_{a,\pi/5}$ where $n = 2$ and $M = z_r$.

**Lemma 1.** Let $x_r = \text{Re}(z_r)$. The value $a_0$ is the pivotal value of $K_{a,\theta}$ if and only if $x_r = 1 - (a_0/2)$. 
Proof. We see from Figure 3 that \( T_{02} \) and \( T_{10} \) are symmetric with respect to the vertical line \( \ell \) with equation \( x = a \left( 1 - \frac{a}{2} \right) \). The pivotal value of \( a \) is such that if \( a > a_0 \), neither of the two pieces touches \( \ell \), and if \( a < a_0 \), they both cross \( \ell \). Focusing on \( T_{02} \), we see that \( a_0 \) is the value of \( a \) for which

\[
\max \{ x \mid (x, y) \in T_{02} \} = a \left( 1 - \left( \frac{a}{2} \right) \right).
\]

Since \( T_0 \) is an \( a \)-sized copy of \( K_{a, \theta} \), this is equivalent to solving the equation

\[
\max \{ x \mid (x, y) \in T_2 \} = 1 - \left( \frac{a}{2} \right).
\]

\[\square\]

**Figure 3.** Fifth stage in iteration

We now begin the process of finding an expression for \( x_r \) in terms of \( a \) and \( \theta \). Referring to Figure 4 we note that

\[
x_r = \text{Re} (A) + r_1 + r_2,
\]

and that \( r_2 = L^n H \sin n\theta \), a scaled version of the height \( H \).

Since we are looking at a scaled (by \( L^n \)) and rotated (by \( n\theta \)) copy of \( H_s \), to find the value of \( r_1 \) we refer to Figures 2 and 4 and note that

\[
r_1 = L^n \cos n\theta \cdot (\text{Re} P_4).
\]

Using the expression for \( x_r \), we apply Lemma 1 to conclude that the pivotal value \( a_0 \) is the value of \( a \) that solves

\[
1 - \frac{a}{2} = a \left( 1 - L^n \cos \theta \right) + (1 - a) L \left( \cos \theta - L^n \cos (n - 1) \theta \right) - L^2 \left( \cos 2\theta - L^n \cos (n - 2)\theta \right)
\]

\[
+ L^n H \sin n\theta + \cos (n\theta) \left( a \cos \theta \cdot \frac{L^{n+1}}{1 - L^2} + \frac{1}{2} \frac{L^{n+2}}{1 - L^2} \right).
\]
where $H$ and $L$ are given in terms of $a$ and $\theta$.

We have the following examples of numerical solutions.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$n$</th>
<th>Pivotal value $a_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/6$</td>
<td>3</td>
<td>0.045669...</td>
</tr>
<tr>
<td>$7\pi/36$</td>
<td>2</td>
<td>0.075481...</td>
</tr>
<tr>
<td>$\pi/5$</td>
<td>2</td>
<td>0.081385...</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>2</td>
<td>$1 - \sqrt{3}/2 = 0.133974...$</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>1</td>
<td>1/4</td>
</tr>
</tbody>
</table>

The graph in Figure 5 demonstrates the relationship between $a_0$ and $\theta$, with $\pi/8 \leq \theta \leq \pi/3$. The horizontal lines are $\theta = \pi/8$, $\pi/6$, $\pi/4$. For all values of $a$ and $\theta$ lying above the graph, $K_{a,\theta}$ is not simple. For all values lying below the graph, each $K_{a,\theta}$ is simple.

4. Characterization of double points

We now examine in detail the double points and lagoons of pivotal-valued curves, and show that both are classified by whether or not $\theta = \pi/n$ for some even positive integer $n > 2$. Note that in general, by symmetry, if $f(x)$ is a double point, then $f(1-x)$ is also a double point.

4.1. The case $\theta \neq \frac{\pi}{2n}$ for any $n \in \mathbb{N}$. The process of finding the pivotal value $a_0 = a(\theta)$ suggests where the double points lie. Let $\ell$ again be the line $x = a\left(1 - \frac{a}{2}\right)$, where a self-intersection will occur.
Since \( f\left(0.\overline{2}\right) \) is the farthest point right of \( H_S \) of the whole curve, the farthest point right of the height set of \( T_{02...2} \) is \( f\left(0.022\ldots\overline{2}\right) \). By symmetry,

\[
f\left(0.022\ldots\overline{2}\right) = f\left(0.101\ldots\overline{1}\right)\]

Table 5 shows a portion of each of a sequence of pivotal curves, all displayed in the same window. Each of these curves with \( \theta > \pi/4 \) has one double point on the vertical line \( \ell \), and, of course, this line moves to the right as \( a \) increases. The angle \( \pi/3 \) is special in the sense that the pivotal curve has other double points that, together with the one on \( \ell \), form an equilateral triangle. In the approximation to \( K_{a,\pi/3} \) in Table 3 we see several equilateral triangular shaped regions. (See Chapter 4 of [1] for details.)

4.2. The case \( \theta = \pi/n \) for some even positive integer \( n > 2 \). In this case, at the pivotal value \( a(\theta) \), the entire height sets of \( T_{02...2}^{n} \) and \( T_{101...1}^{n-1} \) lie on \( \ell \). Since we showed above that the height sets form Cantor sets, we conclude that in this case the double points form a Cantor set on \( \ell \). Recall that a point \( t = 0.t_1t_2... \) base 4 in \( I \) maps onto \( H_S \) if and only if \( \{t_{2j-1}, t_{2j}\} = \{1, 2\} \). Therefore

\[
f\left(0.022\ldots2x_1x_2\ldots\text{base} 4\right) = f\left(0.101\ldots1y_1y_2\ldots\text{base} 4\right)
\]

if and only if \( \{x_{2j-1}, x_{2j}\} = \{y_{2j-1}, y_{2j}\} = \{1, 2\} \) for all \( j \) and \( x_i = 1 \) if and only if \( y_i = 2 \) for all \( i \).
4.3. Lagoons, components of the complement. We define a lagoon as a bounded connected component of $\mathbb{C} \setminus K_{a,\theta}$. With $\theta \in (0, \frac{\pi}{3}]$ fixed, we examine the lagoons associated with the pivotal curve $K_{a,\theta}$. In all cases we remember that the vertical line $\ell$ with equation $x = 1/2$ is a line of symmetry for $K_{a,\theta}$. All of the lagoons have at least one symmetry - that with respect to $\ell$ or to lines that are rotated and translated copies of $\ell$. If $\theta = \pi/n$ for some even positive integer $n > 2$, there are two classes of lagoons, and within each class all lagoons are geometrically similar. The first class consists of lagoons whose boundary includes two endpoints of the same Cantor set as described in the previous section. For example, along $\ell$, there is a lagoon with boundary points $f(\underbrace{.022\ldots21221}_n) = f(\underbrace{.1011\ldots1211\overline{2}}_{n-1})$ and $f(\underbrace{.022\ldots211\overline{2}}_n) = f(\underbrace{.1011\ldots1122\overline{1}}_{n-1})$. This class of lagoons also possesses symmetry along lines that are rotated and translated copies of $\tilde{\ell}$. In the second class of lagoons the points on the boundary of the lagoon which are double points of $K_{a,\theta}$ are extreme values of some Cantor set. For example, this extreme value would occur at $f(\underbrace{.022\ldots221}_n) = f(\underbrace{.1011\ldots11\overline{2}}_{n-1})$.

If $\theta \neq \pi/(2n)$ for any $n \in \mathbb{N}$, the first class of lagoons does not exist since $K_{a,\theta}$ does not self-intersect along a Cantor set. These lagoons are then all geometrically similar.
Table 6. Evolution of lagoons

Table 6 demonstrates the evolution of lagoons as $\theta$ increases through the interval $[5\pi/36, 5\pi/18]$. In this interval when $\theta \leq \pi/6$, the intersection occurs in $T_{0222}$ and $T_{1011}$. For $\pi/6 < \theta \leq \pi/4$, the intersection occurs in $T_{022}$ and $T_{10}$. The curve seems to pull apart below the point of self-intersection. Then portions of the curve come together above the intersection point and finally just touch at $\theta = \pi/4$. This behavior is repeated in $T_{02}$ and $T_{10}$ for $\pi/4 < \theta \leq \pi/3$.

4.4. Conclusion. We found pivotal values and self-intersection properties for a family of generalized Koch curves $K_{a,\theta}$ when $\theta < \pi/3$. Cantor sets provide a way to describe the boundaries of the pivotal curves. The methodology applied to the case with $\theta > \pi/3$ generates different geometric figures that remain to be explored.

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