Internal and Surface Waves in a Two-Layer Fluid

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THESIS

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Dedicated to my advisers, family, friends and the continuity of scientific achievement.

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SUMMARY

In order to investigate the validity of the rigid-lid approximation, two fluid systems, a free-surface system and a rigid-lid system, are compared. Both the free-surface and rigid-lid systems have waves propagating at the interface between two layers of immiscible fluids of different densities, a deep bottom with featureless topography and a surface boundary. The surface boundary of the free-surface system is considered free, whereas the surface boundary of the rigid-lid system is a rigid-lid. The full (Euler) models for these situations are reduced to systems of linear evolutionary equations posed spatially on two dimensional Euclidean space. Compatible initial-boundary-valued problems for these systems of equations are compared in order to obtain sharp estimates of upper fluid layer depths and time scales over which the rigid-lid approximation is valid.

CHAPTER 1

INTRODUCTION

The mathematical theory of waves on the interface of a stratified, immiscible, two-fluid system of different densities is of interest because it is the simplest example of internal wave propagation and because of the difficult modeling, mathematical and numerical issues that arise in the analysis of this system (see [1] and [2]). A survey article by Helfrinch and Melville [3] provides an overview of the properties of internal solitary waves, along with a vast bibliography. An extensive collection of measurements and synthetic aperture radar (SAR) images of various regions of the ocean can be found at [4]. This data shows how varied can be the propagation of internal waves. The variety is reflected in the mathematical models for such geophysical processes. Internal waves are also generated in the atmosphere, for example when winds are blowing over mountain ranges (see [5]).

One of the principle assumptions made in the oceanographic community investigating internal wave motion, is that if waves are deep enough relative to the wavelength of the disturbance, their interactions with the free surface are not significant and thus the top of the fluid can be viewed as an impenetrable, bounding surface. This is the *rigid-lid* assumption and is considered valid when the pycnocline is far from the top (see [6] [7]). Nonlinear models for the interface using this assumption can be found, for example, in [1], [8], [9], [10] and [11].

This dissertation will investigate the validity of the rigid-lid assumption using a methodology inspired by Bona, Chen and Saut [12] and Bona, Pritchard, and Scott [13] by comparing appropriate initial-value problems. In particular, consideration will be given to the case of surface waves and internal waves at the interface of two sheets of fluid, one on top of the other. The fluids will be modeled by considering their kinematics and applying the principles of mass and momentum conservation. In order to provide a framework for deriving mathematical equations from the principles of oceanography, Bernoulli's equation and also how geometrical conditions constrain mass and energy transport will briefly be discussed.

In particular, comparisons will be made of the linear dynamics of a pair of two-fluid systems, each consisting of a homogeneous fluid of depth d_1 and density ρ_1, and each overlying another homogeneous fluid of depth d_2 and density $\rho_2 > \rho_1$, all in hydrostatic equilibrium. The underlying surface of each two-fluid system will be horizontal and featureless. Because of the density difference, waves can propagate along the interface between the two layers. At the surface, the systems will be distinguished as follows: In the *rigid-lid system*, Π_r, the top layer of fluid is taken to be bounded by a rigid lid, which is to say, an impenetrable bounding surface. In a *free surface system*, Π_f, the surface of the top layer is free and allowed to undergo small displacements. In both cases, it is required that the deviation of the interfaces from hydrostatic equilibrium, denoted by ζ_{free} and ζ_{rigid}, as well as the deviation η from hydrostatic equilibrium of fluid surface in Π_f, be graphs over the bottom, so that breaking waves are excluded from our theory. Definition sketches for the free surface and rigid-lid stratified, two-fluid systems are given in Figure 1. The Π_r and Π_f models provide the framework for our mathematical analysis, (compare [2], [14] and [1]).

The equations of motion for the fluids are written in a coordinate system in which internal and surface waves propagate horizontally over the xy-plane and in which the z-axis follows the direction of gravity and $z = 0$ represents the surface of the upper fluid layer in equilibrium. The motions under consideration are modeled using variables x, y, z and t. Because it is assumed that the interfaces are graphs over the horizontal plane $z = -d = -(d_1 + d_2)$, our models will depend only on the spatial variables $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and time t.

To simplify the analysis, the equations of motion have been linearized around hydrostatic equilibrium and the lower layer is considered infinitely deep, [15], [16], [6] and [17]. By following standard Fourier transform methods used in the analysis of linear partial differential equations, it will be shown that the initial-boundary-value problems give rise to coupled systems of linear, ordinary differential equations for the Fourier transformed interface functions $\hat{\zeta}_{free}$ and $\hat{\zeta}_{rigid}$, $\hat{\eta}$ as well as the velocity potential of the upper layer $\hat{\phi}_1$. In order to estimate the difference $\zeta_{free} - \zeta_{rigid}$ and η, the models Π_r and Π_f are chosen with the same initial data. Moreover, conservation of energy will be established and then utilized to make an estimate of the energy difference between the systems Π_r and Π_f. A mathematical foundation for the oceanographic rigid-lid assumption as well as computable estimates on its range of applicability is thereby established.

Comparison between the rigid-lid and free surface systems are starting to emerge in the literature in the context of Boussinesq/Boussinesq systems and the Korteweg-de Vries (KdV) approximation. Coupling between internal and surface waves have been discussed by Craig, Guyenne and Sulem [2] where they comment on the discrepancies arising from the rigid-lid assumption. One-dimensional, nonlinear, long wave regime models investigated recently by Duchene [14] considered the influence of the rigid-lid assumption on the evolution of the interface.

This dissertation is organized as follows. In Chapter 2, the equations of motion of the full Euler system are presented along with the dimensionless, linearized equations for the Fourier transformed free-surface and rigid-lid systems. In Section 3.1, the general solutions of the initial-value problems for these linearized systems will be obtained. In Section 3.2, the dispersion relations will be studied. In particular, the positive eigenvalues of the free-surface system are given by a fast mode $\omega_F = \sqrt{gk}$, corresponding to surface wave motion, and a slow mode $\omega_S = \omega_F\sqrt{\frac{(\rho_2-\rho_1)\tanh d_1 k}{\rho_2+\rho_1\tanh d_1 k}}$, corresponding to internal wave motion. On the other hand, the rigid-lid system has a positive eigenvalue given by $\omega_R = \omega_F\sqrt{\frac{(\rho_2-\rho_1)\tanh d_1 k}{\rho_1+\rho_2\tanh d_1 k}}$. These elementary modes of vibration are ordered as $\omega_S \leq \omega_R \leq \omega_F$ and ω_R and ω_S are of the same order of magnitude for wavelengths both large and small compared to the depth d_1. In Section 3.3, initial conditions will be addressed and prepared in order to compare the two initial-value problems. Conservation of mechanical energy will be established in Section 3.4 and it will be used to derive a measure for comparing the energy of the two systems. In Chapter 4, estimates will be established for the free surface, interface difference and the energy measure. In particular, sharp L^2 estimates with respect to a small parameter $\epsilon = \lambda/d_1$ are obtained for the free surface and interface difference providing depths d_1 and time scales over which the rigid-lid approximation remains good. It will be shown that for short wave initial data, with respect to the upper fluid depth d_1, the time scale validity of the approximation is exponentially long in ϵ, whereas for long wavelength data, the approximation is valid for times of order $\sqrt{d_1/g}$. Moreover, energy is transported from the lower fluid layer to the upper fluid layer linearly with time.

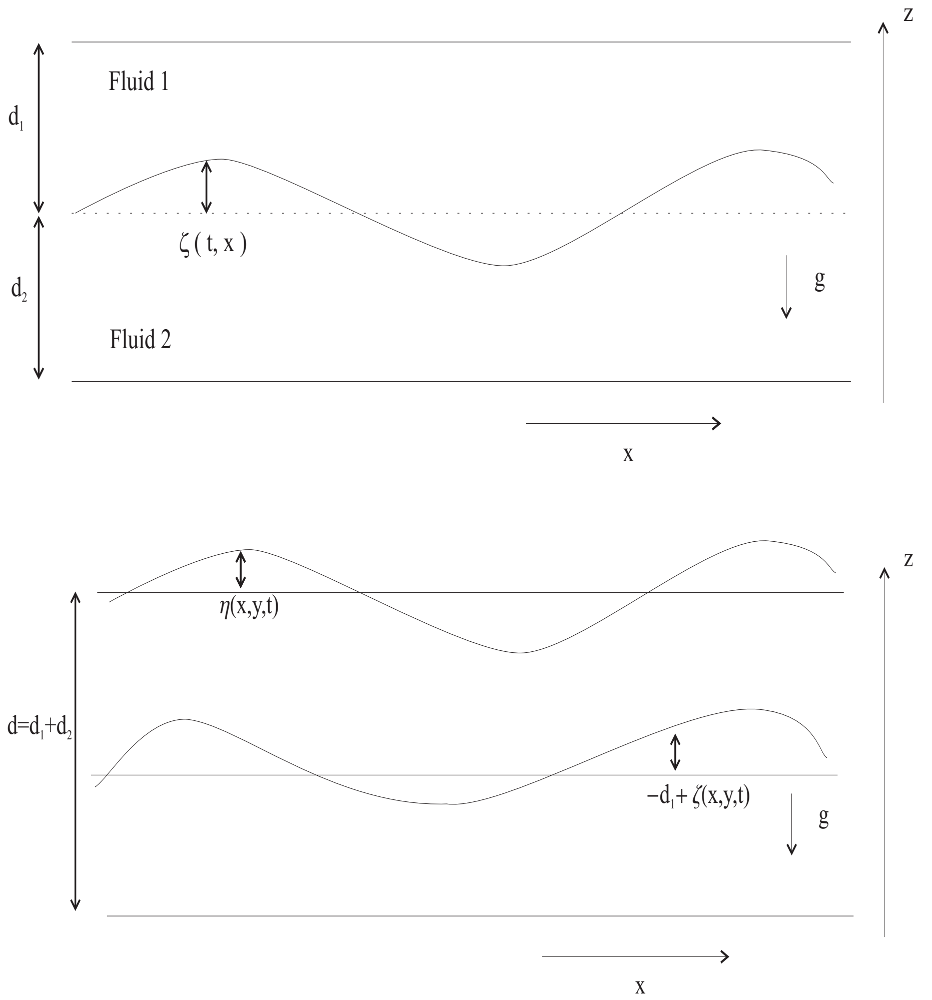

Figure 1. For clarity, the figures are presented in cross section. Rigid-lid model: A homogeneous fluid of density ρ_1 and equilibrium depth d_1 lies above a homogeneous fluid with density $\rho_2 > \rho_1$ and equilibrium depth d_2. The interface between the fluids is described by the graph of $z = -d_1 + \zeta(t, \mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^2$. The rigid-lids are located at the bottom $z = -d = -d_1 - d_2$ and the surface $z = 0$. The force of gravity is acting vertically downward. Free-surface model: Same as the rigid-lid model except the free surface is described by the graph $z = \eta(t, \mathbf{x})$.

CHAPTER 2

THE EQUATIONS OF MOTION

2.1 Euler Equations for the Two-Fluid System

Let Ω_t be a domain in $\mathbb{R}^3$ that contains an inviscid, incompressible, homogeneous fluid at time t. The system describing the motion of the fluid is the Euler equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P = -g\hat{\mathbf{k}} \quad & \text{in } \Omega_t, \\ \nabla \cdot \mathbf{u} = 0 \qquad & \text{in } \Omega_t, \end{aligned} \tag{2.1}$$

where g denotes the acceleration of gravity, $\mathbf{u} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$ denotes the fluid velocity vector field, and where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ represent the standard orthonormal vectors along the x, y, and z axes in $\mathbb{R}^3$, respectively. Here, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$ is the gradient operator written as a column vector using Cartesian coordinates and P denotes the pressure field. If the initial velocity field is irrotational, so that $\nabla \times \mathbf{u} = 0$, Helmholtz's vorticity principle implies that the velocity field will remain irrotational for all time t. Hence, the velocity field $\mathbf{u}$ is derived from a potential $\phi = \phi(x, y, z, t)$, so that

$$\mathbf{u} = \nabla\phi. \tag{2.2}$$

It follows from (2.1) and (2.2) that ϕ satisfies Laplace's equation

$$\triangle\phi = 0 \tag{2.3}$$

in Ω_t, for all time t.

To completely describe the dynamics of the fluid in the domain Ω_t, conditions must be applied at the boundary. For the purpose of analysis, very particular types of boundary conditions are considered. The first is called a *fixed solid boundary* through which no fluid can flow. In this case, where ρ is the fluid density, the component of mass flux density $\mathbf{F} = \rho\mathbf{u}$ normal to the boundary is zero, which implies the normal component u_n of the velocity is zero. The second type of boundary that will be considered is a *material boundary*, a situation in which no fluid particles are transported across the boundary and particles on the boundary remain on the boundary. The free surface and the interface between two fluid layers are examples of material boundaries. A material boundary surface is usually defined by an equation $G(x, y, z, t) = 0$, so provided that z is defined implicitly in terms of x, y, and t, a particle of fluid initially on the surface will remain on the surface for all time. This is represented analytically by the condition $DG/Dt = 0$ on the boundary, where D/Dt is the material derivative, defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2.4}$$

The interpretation of the material derivative is understood by considering the time rate of change of a fluid variable, say, property γ, in terms of a vector field $\mathbf{x}(t)$ representing the location of fluid particles at time t. The fluid property γ depends on both $\mathbf{x}(t)$ and t, so by the chain rule, the time rate of change of γ for the material element (an element of material moving with the fluid) is given by

$$\frac{d\gamma}{dt} = \frac{\partial\gamma}{\partial t} + \frac{\partial\gamma}{\partial x}\frac{dx}{dt} + \frac{\partial\gamma}{\partial y}\frac{dy}{dt} + \frac{\partial\gamma}{\partial z}\frac{dz}{dt} \equiv \frac{\partial\gamma}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\nabla\gamma. \tag{2.5}$$

The rate of change of position for the material element is the fluid velocity

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} \equiv (u, v, w). \tag{2.6}$$

Thus, for a material element, $d\gamma/dt$ is equal to

$$\frac{d\gamma}{dt} = \frac{\partial\gamma}{\partial t} + u\frac{\partial\gamma}{\partial x} + v\frac{\partial\gamma}{\partial y} + w\frac{\partial\gamma}{\partial z} \equiv \frac{\partial\gamma}{\partial t} + \mathbf{u}\cdot\nabla\gamma = \frac{D\gamma}{Dt}, \tag{2.7}$$

and, in terms of field coordinates, D/Dt gives the rate of change observed following a parcel passing through the point $\mathbf{x}$ at time t.

Notice that a fixed boundary is a special case of the material boundary in which G is independent of time, so that $\mathbf{u}\cdot\nabla G = 0$. The force exerted on any non-solid boundary by an inviscid fluid is due to pressure acting at right angles to the boundary. Since there is a balancing of forces at the boundary, this means that, neglecting surface tension, the pressure must be continuous across the boundary. Use will be made of the continuity of pressure in the sequel to derive matching conditions at the interface of the two fluid layers. This setup can be seen also in [1, pg. 8].

As depicted in the introductory figures, the boundaries for the pair of two-fluid systems $\Omega_{t_{free}}$ and $\Omega_{t_{rigid}}$ consist of a bottom with featureless topography given by the plane $z = -d = -d_1 - d_2$, an interface described by the graph $z = -d_1 + \zeta(x, y, t)$, and a free surface described as the graph $z = \eta(x, y, t)$ in the case of the unrestricted model system and the plane $z = \eta_{rigid} = 0$ in the rigid-lid model system. These domains are also assumed to be unbounded in the horizontal directions. The free surface $\eta(x, y, t)$ becomes an unknown in the problem.

Before continuing the development of the two-fluid system, the case of a homogeneous fluid with a horizontal bottom and free surface will be reviewed briefly following [18]. On the fixed horizontal planes, the condition of impermeability implies $\mathbf{u} \cdot \hat{\mathbf{k}} = 0$, where $\hat{\mathbf{k}}$ is the normal vector of the solid bounding surface. In terms of xyz-coordinates this condition becomes

$$\phi_z = 0 \quad \text{on } z = -d. \tag{2.8}$$

The free surface satisfies the kinematic condition $D(\eta - z)/Dt = 0$, or, in terms of coordinates,

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0 \quad \text{on } z = \eta(x, y, t). \tag{2.9}$$

Moreover, because of the continuity of pressure, the free surface pressure of the fluid must equal atmospheric pressure p_a, which is taken to be zero. Due to the pressure condition, (2.1), and (2.2), the Bernoulli's condition [18, pg. 423] becomes

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + gz = 0, \quad \text{on } z = \eta(x, y, t). \tag{2.10}$$

In summary, the equations for a fluid bounded by a free surface and bottom can be written in terms of unknown functions η and ϕ and are given by (2.3), (2.8), (2.9), and (2.10).

As an example, consider the case of an irrotational, inviscid, incompressible, homogeneous fluid overlying a featureless bottom. Following the development presented in [12], let Ω_t be the domain in $\mathbb{R}^3$ occupied by the fluid at time t. If d denotes the depth of the undisturbed fluid (see Figure 2), then assuming the initial velocity field is irrotational so that $\nabla \times \mathbf{u} = 0$, and hence

$$\mathbf{u} = \nabla\phi \tag{2.11}$$

for some potential function $\phi = \phi(x, y, z, t)$. Thus ϕ satisfies Laplace's equation

$$\triangle\phi = 0. \tag{2.12}$$

View the boundary as consisting of two parts: the fixed surface located at $z = -d$, and the free surface at $z = \eta(x, y, t)$. On the fixed portion of the boundary, the condition of impermeability (no flow through the solid boundary) $\mathbf{u} \cdot \mathbf{n} = 0$ is satisfied with $\mathbf{n} = \mathbf{k}$ being the unit normal to the plane $z = -d$, which is to say

$$\phi_z = 0 \tag{2.13}$$

on $z = -d$. Since the free surface is a material surface, it satisfies the kinematic condition $D(\eta - z)/Dt = 0$. Thus, the equation

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \tag{2.14}$$

is satisfied on $z = \eta(x, y, t)$.

Assuming the pressure on the free surface is equal to the ambient air pressure, it follows from the momentum and incompressability equations that the Bernoulli condition

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + gz = 0 \tag{2.15}$$

on $z = \eta(x, y, t)$ is also satisfied on the free surface.

In summary,

$$\begin{aligned} \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 &\quad \text{in } -d < z < \eta(x, y, t), \\ \phi_z = 0 &\quad \text{on } z = -d, \\ \eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0 &\quad \text{on } z = \eta(x, y, t), \end{aligned} \tag{2.16}$$

and

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + gz = 0 \tag{2.17}$$

on $z = \eta(x, y, t)$ are valid, where the free surface, in its resting state, is given by the equation $z = 0$.

This finishes the review of a homogeneous fluid with horizontal bottom and free surface. For further details on solving the linearized surface wave problem see [18], [19] and [6].

Returning now to the free surface model, depicted in (1). Velocity potentials ϕ_i $(i = 1, 2)$ associated with the upper and lower layers, satisfy

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0 \quad \text{in } \Omega_t^i, \tag{2.18}$$

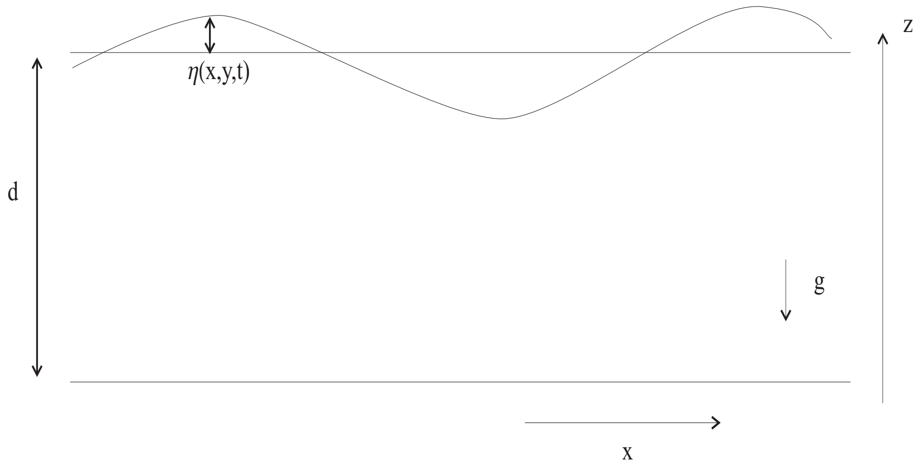

Figure 2. Displacement from equilibrium of a free surface $z = \eta(x, y, t)$ for a fluid of depth d. The z axis points vertically upward and gravity acts vertically downward.

for all time t, where Ω_t^i, represent the region occupied by the fluid i at time t, for $i = 1, 2$. Since the fluids are homogeneous, the densities ρ_i, $i = 1, 2$ are constant so Bernoulli's equation implies

$$\frac{\partial \phi_i}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \phi_i}{\partial x}\right)^2 + \left(\frac{\partial \phi_i}{\partial y}\right)^2 + \left(\frac{\partial \phi_i}{\partial z}\right)^2\right) + gz + \frac{P}{\rho_i} = 0, \tag{2.19}$$

in Ω_t^i and for $i = 1, 2$. These equations have boundary conditions that are specified as follows: In the lower fluid domain, the velocity at the bottom must be horizontal; that is, on $\Gamma_2 := \{z = -d_1 - d_2\}$,

$$\frac{\partial \phi_2}{\partial z} = 0. \tag{2.20}$$

Since $z = \eta(x, y, t)$, the kinematic condition (2.9) and Bernoulli's equation (2.10), evaluated at the free-surface, results in

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi_1}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \phi_1}{\partial z} = 0 \tag{2.21}$$

and

$$\frac{\partial \phi_1}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right) + gz = 0, \tag{2.22}$$

where the last equality results by assuming atmospheric pressure at the free-surface, $P = P_{atm} = 0$. The relation $\frac{\partial \zeta}{\partial t} = \sqrt{1 + |\nabla \zeta|^2} u_n^i$, in which u_n^i denotes the upward normal of the velocity of fluid i at the surface, describes the condition under which a particle on the interface will remain there for all time. That is, no fluid flows across the surface. Since surface tension is not being introduced into the analysis, the normal component of velocity remains continuous at the interface, (see [1, pg.8]). Therefore, for $i = 1, 2$, the expressions

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi_i}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \zeta}{\partial y} - \frac{\partial \phi_i}{\partial z} = 0 \tag{2.23}$$

and

$$\partial_n \phi_1 = \partial_n \phi_2 \tag{2.24}$$

are valid on the interface Γ_t. The two-fluid system satisfies Laplace's equation in each subregion, namely

$$\phi_{1xx} + \phi_{1yy} + \phi_{1zz} = 0 \quad \text{in } -d_1 < z < \eta(x, y, t) \tag{2.25}$$

and

$$\phi_{2xx} + \phi_{2yy} + \phi_{2zz} = 0 \quad \text{in } -d_1 - d_2 < z < -d_1 + \zeta(x, y, t). \tag{2.26}$$

Since there is continuity of pressure across the interface, Bernoulli's equations (2.19) imply

$$\rho_1 \left\{ \phi_{1t} + \frac{1}{2}\left(\phi_{1x}^2 + \phi_{1y}^2 + \phi_{1z}^2\right) + gz \right\} = \rho_2 \left\{ \phi_{2t} + \frac{1}{2}\left(\phi_{2x}^2 + \phi_{2y}^2 + \phi_{2z}^2\right) + gz \right\} \tag{2.27}$$

are valid at the interface $z = -d_1 + \zeta(x, y, t)$.

2.2 Linearized Euler Equations for the Two-Fluid System

Linearizing the equations of motion about hydrostatic equilibrium results in the following system of equations:

$$\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0 \qquad \text{for } i = 1, 2, \tag{2.28}$$

$$\frac{\partial \phi_2}{\partial z} = 0 \qquad \text{at } z = -d_1 - d_2 = -d, \tag{2.29}$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \eta}{\partial t} \qquad \text{at } z = 0, \tag{2.30}$$

$$\frac{\partial \phi_1}{\partial t} + g\eta = 0 \qquad \text{at } z = 0, \tag{2.31}$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \qquad \text{at } z = -d_1, \tag{2.32}$$

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad \text{at } z = -d_1, \text{ and} \tag{2.33}$$

$$\mu \frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t} = g(1 - \mu)\zeta \qquad \text{at } z = -d_1, \tag{2.34}$$

where $\mu = \rho_1 / \rho_2$.

In the sequel, eigenvalues and eigenvectors to the linear evolution equations corresponding to Π_f and Π_r will be constructed simultaneously. To this end, notice that the equations governing these systems are the same except for the fact that $\eta = \eta_t = 0$ in the case of the rigid-lid system. These equalities are seen to be equivalent to the introduction of a parameter $R = 0, 1$ in (2.30) and (2.31) so that

$$R\eta_t - \phi_{1z} = 0 \qquad \text{and} \tag{2.35}$$

$$R\phi_{1t} + \eta = 0, \tag{2.36}$$

where the free-surface and rigid-lid systems correspond to taking $R = 1$ and $R = 0$, respectively.

Remark: The introduction of R into the discussion is of practical value since the solutions of the free surface and rigid-lid systems can be found concurrently.

Let L and T be length and time scales, respectively, and represent dimensionless quantities occurring in the above linear system of equations by tilde of the respective quantities, that is, $x = L\tilde{x}$, $y = L\tilde{y}$, $z = L\tilde{z}$ and $t = T\tilde{t}$. Then dimensional analysis implies relations of the form

$$[\eta] = [\zeta] = L, \tag{2.37}$$

$$[\phi] = \frac{L^2}{T} \equiv \phi_0, \qquad \text{and} \tag{2.38}$$

$$[\partial_z \phi] = \frac{\phi_0}{L}, \tag{2.39}$$

where the bracket notation denotes the dimension of the enclosed quantity. Similar relations hold for the other quantities occurring in (2.28) through (2.34). By choosing a characteristic distance scale d_1 and time scale related to the choice of d_1, rewriting the aforementioned system in terms of the tilde variables gives rise to a natural choice of the time scale T. That is, by using characteristic time and spacial scales

$$T = \sqrt{d_1/g} \qquad \text{and} \tag{2.40}$$

$$L = d_1, \tag{2.41}$$

respectively, the non-dimensionalized, linear, stratified free-surface and rigid lid systems can be written as

$$\tilde{\nabla}^2 \tilde{\phi}_1 = 0 \qquad \text{for } (\tilde{\mathbf{x}}, \tilde{z}) \in \mathbb{R}^2 \times (-1, 0) \tag{2.42}$$

$$\tilde{\nabla}^2 \tilde{\phi}_2 = 0 \qquad \text{for } (\tilde{\mathbf{x}}, \tilde{z}) \in \mathbb{R}^2 \times (-(1+\Gamma), -1) \tag{2.43}$$

$$R\tilde{\eta}_{\tilde{t}} - \tilde{\phi}_{1\tilde{z}} = 0 \qquad \text{at } \tilde{z} = 0 \tag{2.44}$$

$$R\tilde{\phi}_{1\tilde{t}} + \tilde{\eta} = 0 \qquad \text{at } \tilde{z} = 0 \tag{2.45}$$

$$\mu\tilde{\phi}_{1\tilde{t}} - \tilde{\phi}_{2\tilde{t}} = (1-\mu)\tilde{\zeta} \qquad \text{at } \tilde{z} = -1 \tag{2.46}$$

$$\tilde{\zeta}_{\tilde{t}} - \tilde{\phi}_{1\tilde{z}} = 0 \qquad \text{at } \tilde{z} = -1 \tag{2.47}$$

$$\tilde{\phi}_{1\tilde{z}} = \tilde{\phi}_{2\tilde{z}} \qquad \text{at } \tilde{z} = -1 \tag{2.48}$$

$$\tilde{\phi}_{2\tilde{z}} = 0 \qquad \text{at } \tilde{z} = -(1+\Gamma), \tag{2.49}$$

where $\Gamma = d_2/d_1 \to \infty$. For notational clarity, the tildes will be dropped in the sequel. Of note, Gill in [6] discusses oceanic density layers caused by thermal heating over deep water so that $1-\mu \ll 1$ and $\Gamma \gg 1$.

2.3 Conservation of Energy for the Linearized Two-Fluid System

Theorem 2.3.1. *Let ϕ_i, for $i = 1, 2$, η and ζ be the velocity potentials for the upper and lower domains, free-surface and interface for the stratified two-fluid system, respectively. Let $\Psi = (\zeta, \phi_1, \eta, \phi_2)$, assume initially that these functions are in L^2 and define the total energy by*

$$\|\Psi\|_E = \frac{\mu}{2}\int_{\mathbb{R}^2}\int_{-1}^{0} |\nabla\phi_1|^2 \, dzd\mathbf{x} + \frac{\mu}{2}\int_{\mathbb{R}^2} \eta^2 \, d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^2}\int_{-(1+\Gamma)}^{-1} |\nabla\phi_2|^2 \, dzd\mathbf{x} + \frac{(1-\mu)}{2}\int_{\mathbb{R}^2} \zeta^2 \, d\mathbf{x}. \quad (2.50)$$

Then the energy is a conserved quantity for the systems Π_f and Π_r.

Proof. Taking the derivative of the energy (compare [2]) with respect to time formally gives

$$\frac{d}{dt}\|\Psi\|_E = \mu\int\int \nabla\phi_{1t}\cdot\nabla\phi_1 + \mu\int \eta_t\eta + \int\int \nabla\phi_{2t}\cdot\nabla\phi_2 + (1-\mu)\int \eta_t\eta. \quad (2.51)$$

The vector identity $\nabla\phi_{it}\cdot\phi_i = \nabla\cdot(\phi_{it}\nabla\phi_i) - \phi_{it}\nabla^2\phi_i$ and the fact that the velocity potentials ϕ_i are harmonic for $i = 1, 2$ together imply

$$\frac{d}{dt}\|\Psi\|_E = \mu\int\int \nabla\cdot(\phi_{1t}\nabla\phi_1) + \mu\int \eta_t\eta + \int\int \nabla\cdot(\phi_{2t}\nabla\phi_2) + (1-\mu)\int \eta_t\eta. \quad (2.52)$$

Under the physical assumption that $\phi_i(\mathbf{x}, z, t) \to 0$ as $|\mathbf{x}| \to \infty$, the divergence theorem, (2.44), (2.46) and (2.49) together imply

$$\frac{d}{dt}\|\Psi\|_E = \mu \int_{\mathbb{R}^2} \left(R(\eta_t \phi_{1t} - \eta_t \phi_{1t})\right)\, d\mathbf{x} + \int_{\mathbb{R}^2} \left(\phi_{2t}\phi_{2z} - \mu\phi_{1t}\phi_{1z} + (1-\mu)\phi_{2z}\zeta\right)\, d\mathbf{x}, \qquad (2.53)$$

where the first and second integrals on the right-hand side of this equation are being evaluated at $z = 0$ and $z = -1$, respectively.

Using (2.46) and (2.48) in this expression for the rate of change of energy gives the desired result,

$$\frac{d}{dt}\|\Psi\|_E = 0. \qquad (2.54)$$

□

CHAPTER 3

INITIAL VALUE PROBLEM FOR THE STRATIFIED TWO-FLUID SYSTEM

This section begins by first finding the general solutions to the stratified, two-fluid free and rigid-lid systems in the limit $\Gamma \to \infty$. Then, by choosing compatible initial data, relations among the coefficients for these general solutions will be found.

The initial-boundary value problem for the stratified two-fluid system (2.42) through (2.49) can be solved by constructing a system of equations that would relate η and ζ. To this end, take the horizontal Fourier transforms of these equation while noticing, from the theory of linear, ordinary differential equations, the general solutions $\hat{\phi}_i$ for $i = 1, 2$ of the horizontal Fourier transform (over x, y) equations (2.42) and (2.43) have representations

$$\hat{\phi}_1 = A(k,t)\cosh kz + B(k,t)\sinh kz \qquad \text{and} \tag{3.1}$$

$$\hat{\phi}_2 = C(k,t)\cosh kz + D(k,t)\sinh kz, \tag{3.2}$$

where $k = \sqrt{\mathbf{k}\cdot\mathbf{k}} = |\mathbf{k}|$ with $k = (k_1, k_2)$.

3.1 General Solution for the Stratified Two-Fluid Systems

Theorem 3.1.1. *Let* $(\hat{\zeta}, \hat{\phi}_1, \hat{\eta}, \hat{\phi}_2)$ *represent the general solution to the Fourier transformed system* (2.42) *through* (2.49). *Let* $A(k,t)$ *and* $B(k,t)$ *be functions given above, so that once determined,* $\hat{\phi}_1$ *is known. Then*

$$(\hat{\zeta}, A, \hat{\eta}, B)^T_{free} = F_+ e^{i\omega_F t}\mathbf{e}_{i\omega_F} + F_- e^{-i\omega_F t}\mathbf{e}_{-i\omega_F} + S_+ e^{i\omega_S t}\mathbf{e}_{i\omega_S} + S_- e^{-i\omega_S t}\mathbf{e}_{-i\omega_S} \tag{3.3}$$

and

$$(\hat{\zeta}, A, 0, 0)^T_{rigid} = R_+ e^{i\omega_R t}\mathbf{e}_{i\omega_R} + R_- e^{-i\omega_R t}\mathbf{e}_{-i\omega_R} \tag{3.4}$$

for the free-surface and rigid-lid systems, respectively, where

$$\hat{\phi}_1 = A(k,t)\cosh kz + B(k,t)\sinh kz \tag{3.5}$$

$$\hat{\phi}_2 = C(k,t)\cosh kz + D(k,t)\sinh kz \tag{3.6}$$

$$\hat{\zeta} = \hat{\zeta}(k,t) \qquad \textit{and} \tag{3.7}$$

$$\hat{\eta} = \hat{\eta}(k,t). \tag{3.8}$$

The positive frequencies are given by

$$\omega_F = \sqrt{k}, \tag{3.9}$$

$$\omega_S = \sqrt{\frac{k(1-\mu)\tanh k}{1+\mu\tanh k}} \qquad \text{and} \tag{3.10}$$

$$\omega_R = \sqrt{\frac{k(1-\mu)\tanh k}{\mu+\tanh k}}. \tag{3.11}$$

The associated eigenvectors for $i\omega_F$ *and* $i\omega_S$ *are*

$$\mathbf{e}_{i\omega_F} = \left(1, \frac{-i\omega_F}{u(k)}, \frac{-\omega_F^2}{u(k)}, \frac{-i\omega_F^3}{u(k)}\right)^T, \tag{3.12}$$

$$\mathbf{e}_{i\omega_S} = \left(1, \frac{-i\omega_S}{v(k)}, \frac{-\omega_S^2}{v(k)}, \frac{-i\omega_S^3}{v(k)}\right)^T \tag{3.13}$$

where

$$u(k) = \cosh k(k\tanh k - \omega_F^2) \tag{3.14}$$

$$v(k) = \cosh k(k\tanh k - \omega_S^2) \tag{3.15}$$

for the free-surface system. The eigenvector associated with ω_R *for the rigid-lid system is*

$$\mathbf{e}_{i\omega_R} = \left(1, \frac{-i\omega_R}{k\sinh k}, 0, 0\right)^T \tag{3.16}$$

and the relations $\mathbf{e}_{-i\omega} = \mathbf{e}_{i\omega}^*$ *are satisfied for* $\omega = \omega_F$, ω_S *and* ω_R.

Proof. The Fourier transform of (2.42) and (2.43) gives rise to a one parameter family of second-order, ordinary differential equations, parameterized by k given by

$$\hat{\phi}_{izz} - k^2\hat{\phi}_i = 0. \tag{3.17}$$

Since ϕ_i, η and ζ are considered physical quantities, they are real-valued functions, so elementary solutions can be represented by

$$\hat{\phi}_1 = A(k,t)\cosh kz + B(k,t)\sinh kz \tag{3.18}$$

$$\hat{\phi}_2 = C(k,t)\cosh kz + D(k,t)\sinh kz \tag{3.19}$$

$$\hat{\zeta} = \hat{\zeta}(k,t) \qquad \text{and} \tag{3.20}$$

$$\hat{\eta} = \hat{\eta}(k,t). \tag{3.21}$$

The Fourier transform of (2.49) evaluated at the bottom implies

$$D = \tanh[k(1+\Gamma)]C. \tag{3.22}$$

In the limit $\Gamma \to \infty$, this expression reduces to $C = D$. The Fourier transform of (2.48) evaluated at the interface $z = -1$ together with the previous result implies $\hat{\phi}_2$ can be written in terms of A and B namely,

$$\hat{\phi}_2 = \frac{\cosh kz + \sinh kz}{1 - \tanh k}\left(B - A\tanh k\right). \tag{3.23}$$

Similarly, the Fourier transforms of (2.44) through (2.47) implies, for each fixed k, the time-dependent, first-order system of ODE's depending on the parameter R

$$\frac{d}{dt}\begin{pmatrix}\hat{\zeta}\\ AR\\ \hat{\eta}R\\ BR\end{pmatrix} = \begin{pmatrix} 0 & -k\sinh k & 0 & k\cosh k\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & k\\ \frac{-(1-\mu)R}{\cosh k(1+\mu\tanh k)} & 0 & \frac{-(\mu+\tanh k)}{1+\mu\tanh k} & 0\end{pmatrix}\begin{pmatrix}\hat{\zeta}\\ A\\ \hat{\eta}\\ B\end{pmatrix}. \tag{3.24}$$

Denoting this matrix by L, the characteristic polynomial, after expanding the determinant by minors, is

$$\det(L-\lambda I) = \lambda^4 R(1+\mu\tanh k) + \lambda^2 k(\mu+\tanh k+(1-\mu)R) + k^2(1-\mu)\tanh k \tag{3.25}$$

where I is the 4×4 identity matrix (see also [17] and [20]). The eigenvectors corresponding to eigenvalues λ are found by solving the homogeneous system

$$(L-\lambda I)\mathbf{e}_\lambda = 0. \tag{3.26}$$

Elementary row reduction results in the eigenvectors

$$\mathbf{e}_\lambda = \left(1, \frac{-\lambda}{a(k,\lambda)}, \frac{R\lambda^2}{a(k,\lambda)}, \frac{R\lambda^3}{a(k,\lambda)}\right)^T \tag{3.27}$$

where

$$a(k, \lambda) = (k \tanh k + R\lambda^2) \cosh k. \tag{3.28}$$

The proof will be complete by showing the roots of the characteristic polynomial occur in complex conjugate pairs $\lambda = \pm i\omega_F$ and $\lambda = \pm i\omega_S$ for $R = 1$ and $\lambda = \pm i\omega_R$ for $R = 0$.

Lemma 3.1.2. *The roots of the characteristic polynomial are purely imaginary, occurring in complex conjugate pairs $\lambda = \pm i\omega$, where the positive ω's are*

$$\omega_F = \sqrt{k} \tag{3.29}$$

and

$$\omega_S = \sqrt{\frac{k(1-\mu)\tanh k}{1 + \mu \tanh k}} \tag{3.30}$$

for $R = 1$ and

$$\omega_R = \sqrt{\frac{k(1-\mu)\tanh k}{\mu + \tanh k}}. \tag{3.31}$$

for $R = 0$.

Proof. If $\lambda = i\omega$, then (3.25) becomes

$$\omega^4 R(1 + \mu \tanh k) - \omega^2 k(\mu + \tanh k + (1-\mu)R) + k^2(1-\mu)\tanh k = 0. \tag{3.32}$$

In case $R = 0$, for $k \geq 0$, writing $\omega = \pm\omega_R$ implies

$$\omega_R = \sqrt{\frac{k(1-\mu)\tanh k}{\mu + \tanh k}} \geq 0. \tag{3.33}$$

In case $R = 1$, it will be now shown that the roots of (3.32) are real. To this end let $\tau = \tanh k$, divide (3.32) by k^2 and define $\sigma = \omega^2/k$ then

$$\sigma^2(\mu\tau + 1) - \sigma(\tau + 1) + (1-\mu)\tau = 0. \tag{3.34}$$

The discriminant of this quadratic is

$$(\tau + 1)^2 - 4(1-\mu)\tau(1+\mu\tau) = (\tau - 1)^2 + 4\tau - 4(1-\mu)\tau^2 \tag{3.35}$$

$$= (\tau - 1)^2 + 4\mu\tau(1 - (1-\mu)\tau) > 0. \tag{3.36}$$

Therefore ω^2 is a real number. The quadratic in ω^2

$$a\omega^4 + b\omega^2 + c = 0 \tag{3.37}$$

has roots ω_1^2 and ω_2^2 such that $\omega_1^2\omega_2^2 = c/a$ and $\omega_1^2+\omega_2^2 = -b/a$ where $a = \mu\tau+1$, $b = -k(\tau+1)$ and $c = k^2(1-\mu)\tau$. This implies $c/a > 0$ and $-b/a > 0$ so $\omega_1^2, \omega_2^2 \geq 0$. Therefore ω_1 and ω_2 are real numbers. The quadratic formula implies

$$\begin{aligned} \sigma &= \frac{\tau+1 \pm \sqrt{(1+\tau)^2 - 4(\mu\tau+1)(1-\mu)\tau}}{2(\mu\tau+1)} \\ &= \frac{\tau+1 \pm \sqrt{(\tau(2\mu-1)+1)^2}}{2(\mu\tau+1)} \end{aligned} \tag{3.38}$$

so the roots are given by $\sigma_1 = 1$ and $\sigma_2 = (1-\mu)\tau/(1+\mu\tau)$. In other words, the dispersion relation for the free-surface system is

$$(\omega^2 - k)\left(\omega^2(1+\mu\tanh k) - k(1-\mu)\tanh k\right) = 0 \tag{3.39}$$

with positive frequencies given by

$$\omega_F = \sqrt{k} \qquad \text{and} \tag{3.40}$$

$$\omega_S = \sqrt{\frac{k(1-\mu)\tanh k}{1+\mu\tanh k}}. \tag{3.41}$$

The subscripts F and S correspond to wave motion with **fast** and **slow** phase velocities, respectively, as will be seen in the next section. □

□

3.2 Dispersion Relation of the Stratified Two-Fluid System

Lemma 3.2.1. *For $k \geq 0$, $\omega_S \leq \omega_R \leq \omega_F$.*

Proof. By definition of ω_R

$$\omega_R = \omega_F \sqrt{\frac{(1-\mu)\tanh k}{\mu + \tanh k}}. \tag{3.42}$$

If

$$f(k) \equiv \sqrt{\frac{(1-\mu)\tanh k}{\mu + \tanh k}} = \sqrt{\frac{1-\mu}{1+\mu\coth k}}, \tag{3.43}$$

then since $\coth k$ is monotonically decreasing, $f(k)$ is monotonically increasing and

$$\sup_{k \geq 0} f(k) = \sqrt{\frac{1-\mu}{1+\mu}} \leq 1. \tag{3.44}$$

This implies $\omega_R \leq \omega_F$ for $k \geq 0$.

Define the function $q(k)$ by

$$q(k) = \frac{\omega_S}{\omega_R} = \sqrt{\frac{\mu + \tanh k}{1 + \mu \tanh k}} \geq 0, \tag{3.45}$$

so that

$$q^2 = \frac{\mu \cosh k + \sinh k}{\cosh k + \mu \sinh k} = \frac{1 - \alpha e^{-2k}}{1 + \alpha e^{-2k}} \tag{3.46}$$

where $\alpha = (1-\mu)/(1+\mu)$. Taking the derivative of q^2 with respect to k gives

$$\frac{dq^2}{dk} = \frac{2\alpha e^{-2k}}{(1 + \alpha e^{-2k})^2} > 0. \tag{3.47}$$

Since the composition of monotonically increasing functions is monotonically increasing, $q(k)$ is monotonic with

$$\lim_{k\to 0} q(k) = \sqrt{\mu} \qquad \text{and} \tag{3.48}$$

$$\lim_{k\to\infty} q(k) = \sup_{k\geq 0} q(k) = 1. \tag{3.49}$$

□

Graphs of the frequencies ω_S, ω_R and ω_F for the case $\mu = 0.4$ are given in the figure below.

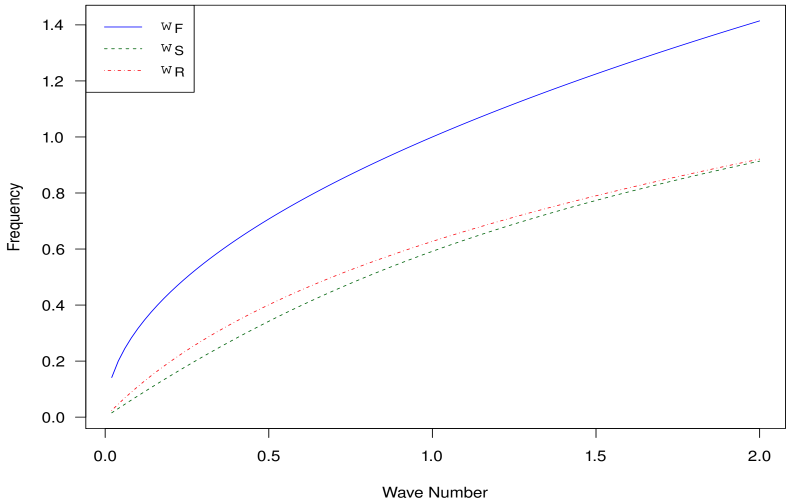

Figure 3. Positive frequencies for the free-surface ω_F, ω_S and rigid-lid ω_R as a function of k for the ratio of upper and lower fluid densities $\mu = \rho_1/\rho_2 = 0.4$.

Lemma 3.2.2. *The frequencies ω_S, ω_R are $O(k)$ and $O(\sqrt{k})$ as $k \to 0$ and $k \to \infty$, respectively.*

Proof. From the definition of ω_S and ω_R

$$\lim_{k\to\infty} \frac{\omega_S}{\sqrt{k}} = \sqrt{\frac{(1-\mu)\tanh k}{1+\mu\tanh k}} = \sqrt{\frac{1-\mu}{1+\mu}} \tag{3.50}$$

and

$$\lim_{k\to\infty} \frac{\omega_R}{\sqrt{k}} = \sqrt{\frac{1-\mu}{1+\mu}} \tag{3.51}$$

so $\omega_S = O(\sqrt{k}), \omega_R = O(\sqrt{k})$ as $k \to \infty$. Since

$$\frac{\omega_S}{k} = \sqrt{\frac{1-\mu}{1+\mu\tanh k}}\sqrt{\frac{\tanh k}{k}} \tag{3.52}$$

and

$$\lim_{k\to 0} \sqrt{\frac{\tanh k}{k}} = 1, \tag{3.53}$$

it follows that

$$\lim_{k\to 0} \omega_S/k = \sqrt{1-\mu} \tag{3.54}$$

so that $\omega_S = O(k)$ as $k \to 0$. Similarly,

$$\lim_{k\to 0} \frac{\omega_R}{k} = \sqrt{\frac{1-\mu}{\mu}} \lim_{k\to 0} \sqrt{\frac{\tanh k}{k}} = \sqrt{\frac{1-\mu}{\mu}} \tag{3.55}$$

and $\omega_R = O(k)$ as $k \to 0$. ☐

3.3 Initial Value Problem for the Free-Surface System

By equation (3.3), the Fourier transformed solutions for the free-surface system can be written as

$$\hat{\zeta}_{free}(k,t) = F_+ e^{i\omega_F t} + F_- e^{-i\omega_F t} + S_+ e^{i\omega_S t} + S_- e^{-i\omega_S t}, \tag{3.56}$$

$$A_{free}(k,t) = \frac{i\omega_F}{u}\left(F_- e^{-i\omega_F t} - F_+ e^{i\omega_F t}\right) + \frac{i\omega_S}{v}\left(S_- e^{-i\omega_S t} - S_+ e^{i\omega_S t}\right), \tag{3.57}$$

$$\hat{\eta}_{free}(k,t) = \frac{-\omega_F^2}{u}\left(F_+ e^{i\omega_F t} + F_- e^{-i\omega_F t}\right) + \frac{-\omega_S^2}{v}\left(S_+ e^{i\omega_S t} + S_- e^{-i\omega_S t}\right) \quad \text{and} \tag{3.58}$$

$$B_{free}(k,t) = \frac{i\omega_F^3}{u}\left(F_- e^{-i\omega_F t} - F_+ e^{i\omega_F t}\right) + \frac{i\omega_S^3}{v}\left(S_- e^{-i\omega_S t} - S_+ e^{i\omega_S t}\right) \tag{3.59}$$

where $F_\pm$ and $S_\pm \in \mathbb{C}$.

In order to compare systems Π_f and Π_r, compatible initial conditions are required. In particular, since $\eta_{rigid} = \partial_t \eta_{rigid} \equiv 0$ for $t \geq 0$, initial data is chosen so that

$$\eta_{free}(\mathbf{x},0) = 0, \tag{3.60}$$

$$\partial_t \eta_{free}(\mathbf{x},0) = 0, \tag{3.61}$$

$$\zeta_{free}(\mathbf{x},0) = \zeta_{rigid}(\mathbf{x},0) \equiv \zeta_0(\mathbf{x}) \quad \text{and} \tag{3.62}$$

$$\partial_t \zeta_{free}(\mathbf{x},0) = \partial_t \zeta_{rigid}(\mathbf{x},0) \equiv \zeta_1(\mathbf{x}). \tag{3.63}$$

The derivative of $\hat{\phi}_1$ in (3.18) with respect to z evaluated at $z=0$ implies $\hat{\phi}_1 = kB(t,k) = \hat{\eta}_t$ so $\partial_t \hat{\eta}_{free}(k,0) = 0$ is equivalent to $B(k,0) = 0$. Thus there are four equations for $F_\pm$ and $S_\pm$

in terms of the two initial data which are zero and two initial data which will be chosen to correspond to initial data of the rigid-lid system.

The first two conditions imply the relations

$$\frac{\omega_F^2}{u}(F_+ + F_-) + \frac{\omega_S^2}{v}(S_+ + S_-) = 0 \qquad \text{and} \tag{3.64}$$
$$\frac{\omega_F^3}{u}(F_+ - F_-) + \frac{\omega_S^3}{v}(S_+ - S_-) = 0, \tag{3.65}$$

which are equations for $F_\pm$ in terms of $S_\pm$. Their solution is

$$F_+ = \frac{-ur^2}{2v}\left((S_+ + S_-) + r(S_+ - S_-)\right) \tag{3.66}$$
$$F_- = \frac{-ur^2}{2v}\left((S_+ + S_-) - r(S_+ - S_-)\right) \tag{3.67}$$

where r is defined by

$$r(k) = \frac{\omega_S}{\omega_F} = \sqrt{\frac{(1-\mu)\tanh k}{1+\mu\tanh k}} = \sqrt{\frac{1-\mu}{\mu+\coth k}}. \tag{3.68}$$

Notice that $r(k)$ is monotonically decreasing with $0 \leq r < 1$ and

$$\sup_{k\geq 0} r(k) = \sqrt{\frac{1-\mu}{1+\mu}}. \tag{3.69}$$

The relations (3.66), the last two initial conditions and the definitions of u and v together imply

$$S_+ = s_0\hat{\zeta}_0 - is_1\hat{\zeta}_1 \qquad \text{and} \tag{3.70}$$

$$S_- = s_0\hat{\zeta}_0 + is_1\hat{\zeta}_1 \tag{3.71}$$

where

$$s_0 = \frac{v}{2(v-ur^2)} = \frac{\tanh k - r^2}{2k\tanh k(1-r^2)} \tag{3.72}$$

$$= \frac{\mu(\tanh k + 1)}{2k(1-r^2)(1+\mu\tanh k)} \qquad \text{and} \tag{3.73}$$

$$s_1 = \frac{s_0}{\omega_S}. \tag{3.74}$$

Finally, substituting $F_\pm$ into the definitions for $\hat{\eta}_{free}$, $\hat{\zeta}_{free}$ and collecting terms yields

$$\hat{\eta}_{free}(k,t) = \mathcal{F}(t)S_+ + \mathcal{F}^*(t)S_- \qquad \text{and} \tag{3.75}$$

$$\hat{\zeta}_{free}(k,t) = \mathcal{G}(t)S_+ + \mathcal{G}^*(t)S_- \tag{3.76}$$

where

$$\mathcal{F}(t) = \frac{-\omega_S^2 e^{i\omega_S t}}{2v}\Big((1+r)(1-e^{i(\omega_F-\omega_S)t}) + (1-r)(1-e^{-i(\omega_F+\omega_S)t})\Big) \qquad \text{and} \tag{3.77}$$

$$\mathcal{G}(t) = \frac{-ur^2}{2v}\left((1+r)e^{i\omega_F t} + (1-r)e^{-i\omega_F t}\right) + e^{i\omega_S t}. \tag{3.78}$$

3.4 Initial Value Problem for the Rigid-Lid System

The general solution of the initial value problem for the rigid-lid system is given by (3.4) and written in terms of coordinate functions as

$$\hat{\zeta}_{rigid}(k,t) = R_+ e^{i\omega_R t} + R_- e^{-i\omega_R t} \qquad \text{and} \tag{3.79}$$

$$A_{rigid}(k,t) = \frac{i\omega_R}{k\sinh k}\left(R_- e^{-i\omega_R t} - R_+ e^{i\omega_R t}\right) \tag{3.80}$$

$$= \frac{-\hat{\zeta}_t}{k\sinh k}, \tag{3.81}$$

where the last equality follows by taking the time derivative of the first equation. Evaluating these expressions at $t = 0$ gives two equations expressing $R_\pm$ in terms of the initial data $\hat{\zeta}_0$ and $\hat{\zeta}_1$ whose solutions are easily shown to be

$$R_+ = \frac{1}{2}\left(\hat{\zeta}_0 - i\frac{\hat{\zeta}_1}{\omega_R}\right) \qquad \text{and} \tag{3.82}$$

$$R_- = \frac{1}{2}\left(\hat{\zeta}_0 + i\frac{\hat{\zeta}_1}{\omega_R}\right). \tag{3.83}$$

Equivalently, to ease the initial calculations when comparing with the free-surface system, $R_\pm$ can be written in terms of $S_\pm$ by recognizing that for prepared initial data

$$R_+ + R_- = \mathcal{G}(0)S_+ + \mathcal{G}^*(0)S_- \qquad \text{and} \tag{3.84}$$

$$i\omega_R(R_+ - R_-) = \dot{\mathcal{G}}(0)S_+ + \dot{\mathcal{G}}^*(0)S_-. \tag{3.85}$$

Solving for $R_\pm$ gives

$$R_+ = \frac{v - ur^2}{2v}\left(S_+ + S_- + q(S_+ - S_-)\right) \qquad \text{and} \tag{3.86}$$

$$R_- = \frac{v - ur^2}{2v}\left(S_+ + S_- - q(S_+ - S_-)\right) \tag{3.87}$$

where q is the function defined by

$$q(k) \equiv \frac{\omega_S}{\omega_R} = \sqrt{\frac{\mu + \tanh k}{1 + \mu \tanh k}}. \tag{3.88}$$

Using this result in the definition for $\hat{\zeta}_{rigid}$ gives a decomposition into time-dependent and independent parts, namely

$$\hat{\zeta}_{rigid} = \mathcal{H}(t)S_+ + \mathcal{H}^*(t)S_- \tag{3.89}$$

where

$$\mathcal{H}(t) = \frac{v - ur^2}{2v}\left((1+q)e^{i\omega_R t} + (1-q)e^{-i\omega_R t}\right). \tag{3.90}$$

The derivative with respect to time and definition of q implies

$$\dot{\mathcal{H}}(0) = i\omega_R\left(\frac{v - ur^2}{2v}\right)(1 + q - (1-q)) = i\omega_S\left(\frac{v - ur^2}{2v}\right). \tag{3.91}$$

3.5 Interface Difference for the Free-Surface and Rigid-Lid Systems

The results of the previous sections can be used to calculate the difference of $\hat{\zeta}_{free}$ and $\hat{\zeta}_{rigid}$ which will be used in the next chapter to compare the Π_f and Π_r systems, namely

$$\hat{\zeta}_{free}(k,t) - \hat{\zeta}_{rigid}(k,t) = \mathcal{J}(t)S_+(k) + \mathcal{J}^*(t)S_-(k) \tag{3.92}$$

where $\mathcal{J}(t) = \mathcal{G}(t) - \mathcal{H}(t)$ so that

$$\begin{aligned}\mathcal{J}(t) = \frac{-ur^2}{2v}\left((1+r)e^{i\omega_F t} + (1-r)e^{-i\omega_F t}\right) + e^{i\omega_S t} \\ - \left(\frac{v-ur^2}{2v}\right)\left((1+q)e^{i\omega_R t} + (1-q)e^{-i\omega_R t}\right).\end{aligned} \tag{3.93}$$

Straightforward algebraic computations imply

$$\begin{aligned}\mathcal{J}(t) = e^{i\omega_S t}\left(\frac{1+q}{2}\left(1-e^{i(\omega_R-\omega_S)t}\right) + \frac{1-q}{2}\left(1-e^{-i(\omega_R+\omega_S)t}\right)\right) \\ - \frac{ur^2}{2v}\left(e^{i\omega_R t}(e^{i(\omega_F-\omega_R)t}-1) + e^{-i\omega_R t}(e^{-i(\omega_F-\omega_R)t}-1)\right) \\ - \frac{ur^2}{2v}\left(qe^{-i\omega_R t}(1-e^{2i\omega_R t}) + re^{i\omega_F t}(1-e^{-2i\omega_F t})\right).\end{aligned} \tag{3.94}$$

Moreover, evaluating $\mathcal{G}$, $\mathcal{H}$ and their derivatives at $t=0$ implies they are equal,

$$\mathcal{G}(0) = \mathcal{H}(0) = \frac{v-ur^2}{v} \qquad \text{and} \tag{3.95}$$

$$\dot{\mathcal{G}}(0) = \dot{\mathcal{H}}(0) = i\omega_S\left(\frac{v-ur^2}{v}\right) \tag{3.96}$$

so that

$$\mathcal{J}(0) = \dot{\mathcal{J}}(0) = 0. \tag{3.97}$$

In the sequel, the following consequence of the order calculations of the frequencies will be useful.

Lemma 3.5.1. *The coefficient s_0 in (3.70) is $O(k^{-1})$ as k tends to 0 and ∞, and s_1 is $O(k^{-2})$ as $k \to 0$ and $O(k^{-3/2})$ as $k \to \infty$.*

Proof. From the definition of r, $(1 - r^2)(1 + \mu \tanh k) = 1 - \tanh k + 2\mu \tanh k$, so

$$s_0 = \frac{\mu(1 + \tanh k)}{2k(1 - \tanh k + 2\mu \tanh k)}. \tag{3.98}$$

Thus

$$\lim_{k \to 0} s_0 k = \frac{\mu(1 + \tanh k)}{2(1 - \tanh k + 2\mu \tanh k)} = \frac{\mu}{2}, \tag{3.99}$$

$$\lim_{k \to \infty} s_0 k = 1 \tag{3.100}$$

and thus $s_0 = O(k^{-1})$ as k tends to both 0 and ∞. Moreover, since $\omega_S = O(k)$ and $O(\sqrt{k})$ as k tends to 0 and ∞, respectively, $\omega_S^{-1} = O(k^{-1})$ and $O(k^{-1/2})$. Thus $s_1 = s_0 \omega_S^{-1} = O(k^{-2})$ and $O(k^{-3/2})$ as k tends to 0 and ∞, respectively. □

3.6 Energy of the Stratified Two-Fluid System

In the sequel, a comparison of the energies of the stratified two-fluid free-surface and rigid-lid systems will be made. To this end, decompose the energy given in (2.50) into upper and lower components.

From the equations of motion (2.42) through (2.49), the divergence theorem and the vector identity $\nabla\phi \cdot \nabla\phi = \nabla \cdot (\phi\nabla\phi) - \phi\nabla^2\phi$, a straightforward calculation implies

$$E(t) = E_{upper}(t) + E_{lower}(t) \tag{3.101}$$

where

$$E_{upper}(t) = \frac{\mu}{2}\int_{\mathbb{R}^2} \left((\phi_1\phi_{1z})_{z=0} + \eta^2\right)\, d\mathbf{x} \tag{3.102}$$

$$= \frac{\mu}{2}\int_{\mathbb{R}^2} \left(\frac{1}{2}(\partial_z\phi_1^2)_{z=0} + \eta^2\right)\, d\mathbf{x} \qquad \text{and} \tag{3.103}$$

$$E_{lower}(t) = \frac{1}{2}\int_{\mathbb{R}^2} \phi_{2z}(\phi_2 - \mu\phi_1)_{z=-1}\, d\mathbf{x} + \frac{1-\mu}{2}\int_{\mathbb{R}^2} \zeta^2\, d\mathbf{x}. \tag{3.104}$$

Recall, for $z = 0$ and prepared initial data, the equalities $\eta = \phi_{1z} = 0$ are valid for $t \geq 0$ and $t = 0$ for the rigid-lid and free-surface systems, respectively. Thus one can write $E_{rigid}(t) = E_{lower}(t)$, $E_{free}(0) = E_{lower}(0)$ and $E_{upper}(0) = 0$. From conservation of energy,

$$E_{free}(t) = E_{lower}(t) + E_{upper}(t) = E_{free}(0) \qquad \text{and} \tag{3.105}$$

$$E_{rigid}(t) = E_{rigid}(0) = E_{free}(0) = E_{lower}(0) \tag{3.106}$$

for prepared initial data.

An expression for the amount of energy transported to the upper surface, $E_{upper}(t)$, will be sought. The formula $\hat{\phi}_1 = A(k,t)\cosh(kz) + B(k,t)\sinh(kz)$ implies

$$|\hat{\phi}_1|^2 = |A|^2\cosh^2(kz) + |B|^2\sinh^2(kz) + (AB^* + A^*B)\cosh(kz)\sinh(kz). \tag{3.107}$$

Assuming the derivative with respect to z can be taken outside of the integral, by Parseval's theorem, in order to compute the integral of the first term in the expression for E_{upper}, it suffices to recognize that

$$(\partial_z|\hat{\phi}_1|^2)_{z=0} = k(AB^* + A^*B). \tag{3.108}$$

The definitions of $A(k,t)$ and $B(k,t)$ given in (3.3) imply

$$A(k,t) = \frac{i\omega_F}{u}\left(F_-e^{-i\omega_F t} - F_+e^{i\omega_F t}\right) + \frac{i\omega_S}{v}\left(S_-e^{-i\omega_S t} - S_+e^{i\omega_S t}\right) \quad \text{and} \tag{3.109}$$

$$B(k,t) = \frac{i\omega_F^3}{u}\left(F_-e^{-i\omega_F t} - F_+e^{i\omega_F t}\right) + \frac{i\omega_S^3}{v}\left(S_-e^{-i\omega_S t} - S_+e^{i\omega_S t}\right). \tag{3.110}$$

These relations along with (3.66), after collecting terms $S_\pm$, implies

$$A(k,t) = \mathcal{A}(t)S_+ + \mathcal{A}^*(t)S_- \quad \text{and} \tag{3.111}$$

$$B(k,t) = \mathcal{B}(t)S_+ + \mathcal{B}^*(t)S_- \tag{3.112}$$

where

$$\mathcal{A}(t) = \frac{-i\omega_S}{v}\left(e^{i\omega_S t} + \frac{r}{2}\left((1-r)e^{-i\omega_F t} - (1+r)e^{i\omega_F t}\right)\right) \quad \text{and} \tag{3.113}$$

$$\mathcal{B}(t) = \frac{-i\omega_S^3}{v}\left(e^{i\omega_S t} + \frac{i\omega_F^3 r^2}{2v}\left((1+r)e^{i\omega_F t} - (1-r)e^{-i\omega_F t}\right)\right) \tag{3.114}$$

$$= \frac{i\omega_F^3 r^2 e^{i\omega_F t}}{2v}\left(1 - e^{-2i\omega_F t}\right) + \frac{i\omega_S^3 e^{i\omega_S t}}{2v}\left((e^{i(\omega_F-\omega_S)t} - 1) + (e^{-i(\omega_F+\omega_S)t} - 1)\right). \tag{3.115}$$

Thus the expression for the amount of energy transported to the upper surface, E_{upper} is given by

$$E_{upper}(t) = \frac{\mu}{2}\left(\int_{\mathbb{R}^2} |\hat{\eta}|^2 \, d\mathbf{k} + \frac{1}{2}\int_{\mathbb{R}^2} k(AB^* + A^*B)\, d\mathbf{k}\right), \tag{3.116}$$

where a straightforward calculation shows

$$AB^* + A^*B = 2\left((|S_+|^2 + |S_-|^2)(\mathcal{A}^*\mathcal{B} + \mathcal{A}\mathcal{B}^*) + \mathcal{A}\mathcal{B}S_+S_-^* + \mathcal{A}^*\mathcal{B}^*S_+^*S_-\right). \tag{3.117}$$

In the sequel, equation (3.116) will be used to estimate the amount of energy transported to the upper surface for the free-surface system.

CHAPTER 4

COMPARISONS OF THE FREE-SURFACE AND RIGID-LID SYSTEMS

4.1 An Estimate for the Free Surface

In this chapter estimates will be computed for the free surface, η, difference of the interface between the systems Π_f and Π_r and the amount of energy transported to surface. To this end, the following lemmas will be used.

Lemma 4.1.1. *For all real numbers x, $|1 - e^{ix}| \leq |x|$.*

Proof. If $f(x) = |1 - e^{ix}| = \sqrt{2(1 - \cos x)}$ then $f(x) \in [0, 2]$, $f(2\pi n) = 0$, 2π-periodic and thus it suffices to show $1 - x^2/2 \leq \cos x$ for $|x| \leq \pi$. Since all functions involved are even, assume $0 \leq x \leq \pi$. By Taylor's theorem, $\cos(x) = P_n(x) + \frac{1}{n!}\int_0^x \cos^{(n+1)}(t)(x-t)^n\, dt$, where P_n is the Taylor polynomial for cosine of degree n at $c = 0$. For $n = 2$, $\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{2}\int_0^x \sin(t)(x-t)^2\, dt$. Since this last integral is positive the desired inequality holds. □

Lemma 4.1.2. *If $0 < a < \infty$ and $\gamma_a = \max(1, 2^{a-1})$, then the inequality*

$$|\alpha - \beta|^a \leq \gamma_a \left(|\alpha|^a + |\beta|^a\right) \tag{4.1}$$

holds for arbitrary complex numbers α and β.

Lemma 4.1.3. *For nonnegative real numbers k*

$$|S_+| + |S_-| \leq |\hat{\zeta}_0| + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|. \tag{4.2}$$

Proof. The expressions for $S_\pm$ are given by (3.70), namely

$$S_+ = s_0\hat{\zeta}_0 - is_1\hat{\zeta}_1 \qquad \text{and} \tag{4.3}$$

$$S_- = s_0\hat{\zeta}_0 + is_1\hat{\zeta}_1 \tag{4.4}$$

where $s_0 = \mu(\tanh k + 1)/[2k(1-r^2)(1+\mu\tanh k)]$ and $s_1 = s_0/\omega_S$. Consequently,

$$\begin{aligned} |S_+| + |S_-| &\leq 2\left(|s_0\hat{\zeta}_0| + |s_1\hat{\zeta}_1|\right) && (4.5)\\ &= 2|s_0|\left(|\hat{\zeta}_0| + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|\right) && (4.6)\\ &= \left|\frac{v}{v-ur^2}\right|\left(|\hat{\zeta}_0| + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|\right). && (4.7) \end{aligned}$$

The definitions of u and v give the relations

$$v - ur^2 = \omega_F^2(1-r^2)\sinh k \qquad \text{and} \tag{4.8}$$

$$v = \frac{\mu\omega_F^2(1+\tanh k)\sinh k}{1+\mu\tanh k}. \tag{4.9}$$

Since $r^2 = (1-\mu)(\tanh k)/(1+\mu\tanh k)$, the following expression is valid:

$$g(k) \equiv \frac{v}{v-ur^2} = \frac{\mu(1+\tanh k)}{1-\tanh k+2\mu\tanh k} \tag{4.10}$$

where $0 < 1-\tanh k + 2\mu\tanh k \leq 1+2\mu$, since $\tanh k \in [0,1)$. The limits

$$\lim_{k\to 0}\frac{v}{v-ur^2} = \mu \qquad \text{and} \tag{4.11}$$

$$\lim_{k\to\infty}\frac{v}{v-ur^2} = 1, \tag{4.12}$$

along with the previous results imply that $g(k)$ is positive and finite for $k\in[0,\infty)$.

By noticing that

$$g(k) = \frac{\mu(\cosh k+\sinh k)}{\cosh k-\sinh k+2\mu\sinh k} \tag{4.13}$$

$$= \frac{\mu e^k}{e^{-k}+\mu(e^k-e^{-k})} = \frac{\mu e^k}{\mu e^k+(1-\mu)e^{-k}} \tag{4.14}$$

$$= \frac{\mu}{\mu+(1-\mu)e^{-2k}}, \tag{4.15}$$

it is clear that g is monotonically increasing with $\sup_{k\geq 0} g(k) = 1$. □

Lemma 4.1.4. *There exists positive constants α and β such that $\omega_S^{-2} \leq k^{-2}(\alpha+\beta k)$.*

Proof. By the definition of ω_S,

$$\omega_S^{-2} = \frac{1+\mu\tanh k}{k(1-\mu)\tanh k} = \frac{1}{(1-\mu)k^2}\left(k\coth k+\mu k\right). \tag{4.16}$$

Defining the function γ by

$$\gamma(k) \equiv k(\coth k + \mu), \tag{4.17}$$

the lemma will be proved by showing $\gamma(k) \leq 1 + (1+\mu)k$. This result is reasonable since

$$\lim_{k\to 0} \gamma(k) = 1 \qquad \text{and} \tag{4.18}$$

$$\lim_{k\to\infty} \frac{\gamma(k)}{k} = 1 + \mu \tag{4.19}$$

implies the graph of γ has the line $y = (1+\mu)k$ as an oblique asymptote.

Suppose, on the contrary, for some $k > 0$, $\gamma(k) > 1 + (1+\mu)k$. Then $k \coth k > 1 + k$ and

$$k \coth k = k(e^k + e^{-k})/(e^k - e^{-k}) \tag{4.20}$$

would imply $k(e^k + e^{-k}) > (1+k)(e^k - e^{-k})$. Multiplying by e^k implies $1 + 2k > e^{2k}$ which is a contradiction. Therefore, choosing $\alpha = (1-\mu)^{-1}$ and $\beta = (1+\mu)/(1-\mu)$ gives the desired result. $\square$

Lemma 4.1.5. *If $x \geq 0$, then given $a > 0$, there exists $C \geq 0$ such that*

$$C\sqrt{1+x^2} > 1 + ax, \tag{4.21}$$

where $C > \max(1, a)$.

Proof. Defining the function $h(x) = C\sqrt{1+x^2} - 1 - ax$, find C such that $h(x) > 0$ for $x \geq 0$. The derivative is

$$h'(x) = \frac{Cx}{\sqrt{1+x^2}} - a \tag{4.22}$$

and $h(0) = C - 1$. Since $x/\sqrt{1+x^2} < 1$, choosing $C > \max(1, a)$ yields an h which is a positive, monotonically increasing function, thereby giving the desired result. □

Theorem 4.1.6. *Let $s \geq 0$ and $\zeta_0 = \zeta(\mathbf{x}, 0)$ and $\zeta_1 = \partial_t \zeta(\mathbf{x}, 0)$ be initial data for the free-surface and rigid-lid two-fluid systems Π_f and Π_r. Let ξ be a function such that $\hat{\xi} = \hat{\zeta}_1 k^{-1/2}$ and assume ζ_0 and ξ are both L^2 functions. Let $\eta(\mathbf{x}, t)$ and $\zeta(\mathbf{x}, t)$ be solutions for that initial data. Then for $t \in [0, T_0]$, there exists a positive constant C such that*

$$\|\eta(\cdot, t)\|^2_{H^s(\mathbb{R}^2)} \leq Ct^2 \left(\|\zeta_0\|_2^2 + \|\xi\|_2^2\right). \tag{4.23}$$

Proof. Recall from (3.77) that $\hat{\eta}_{free}(t) = \mathcal{F}(t)S_+ + \mathcal{F}^*(t)S_-$ where

$$\mathcal{F}(t) = \frac{-\omega_S^2 e^{i\omega_S t}}{2v} \left((1+r)(1 - e^{i(\omega_F - \omega_S)t}) + (1-r)(1 - e^{-i(\omega_F + \omega_S)t})\right). \tag{4.24}$$

The triangle inequality implies $|\hat{\eta}| \leq 2\left(|S_+| + |S_-|\right)|\mathcal{F}(t)|$. The definitions of r, u, ω_S and ω_F and (4.1.1) gives estimates for $|\mathcal{F}(t)|$ namely,

$$|\mathcal{F}(t)| \leq t\left|\frac{\omega_S^2}{v}\right|\left(\left|\frac{1+r}{2}\right||\omega_F - \omega_S| + \left|\frac{1-r}{2}\right||\omega_F + \omega_S|\right) \tag{4.25}$$

$$= t\left|\frac{\omega_S^2}{v}\right|\left|\frac{\omega_F^2 - \omega_S^2}{\omega_F}\right| \tag{4.26}$$

$$= tv^{-1}\omega_S^2\omega_F(1-r^2), \tag{4.27}$$

where the second equality follows from the fact that $r = \omega_S/\omega_F$. The definition of ω_S and the expression of v given in (4.9) implies

$$tv^{-1}\omega_S^2\omega_F(1-r^2) = \frac{t(1-r^2)(1-\mu)\omega_F}{(1+\tanh k)\cosh k} \tag{4.28}$$

Finally, this result, the definitions of ω_F and $\cosh k$ and (3.69) together imply

$$|\mathcal{F}(t)| \leq \frac{t(1-r^2)(1-\mu)\omega_F}{(1+\tanh k)\cosh k} \leq C_1 t\sqrt{k}e^{-k} \tag{4.29}$$

for some positive constant C_1.

By Lemmas (4.1.2) and (4.1.3)

$$|\hat{\eta}|^2 \leq C_2 t^2 k e^{-2k}\left(|\hat{\zeta}_0|^2 + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2\right) \tag{4.30}$$

so the L^2-based Sobolev norm for $\eta(\mathbf{x}, t)$ satisfies

$$\|\eta\|^2_{H^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |k|^2)^s |\hat{\eta}|^2 \, d\mathbf{k} \tag{4.31}$$

$$\leq C_3 t^2 \int_{\mathbb{R}^2} (1 + |k|^2)^s k e^{-2k} \left(|\hat{\zeta}_0|^2 + \left| \frac{\hat{\zeta}_1}{\omega_S} \right|^2 \right) \, d\mathbf{k}. \tag{4.32}$$

for some positive constant C_3. The exponentially decaying function

$$\Upsilon_{s,n}(k) = (1 + |k|^2)^s e^{-2k} k^n \tag{4.33}$$

is integrable for all $s, n \geq 0$ so that

$$\int_{\mathbb{R}^2} \Upsilon_{s,1}(k) |\hat{\zeta}_0|^2 \, d\mathbf{k} \leq C_4 \|\zeta_0\|_2^2. \tag{4.34}$$

Lemmas (4.1.4) and (4.1.5) together imply

$$\Upsilon_{s,1}(k) \left| \frac{\hat{\zeta}_1}{\omega_S} \right|^2 < C_5 \Upsilon_{s,1}(k) k^{-2} (1 + |k|^2)^{1/2} |\hat{\zeta}_1|^2 \tag{4.35}$$

$$= C_5 k^{-1} e^{-2k} (1 + |k|^2)^{s+1/2} |\hat{\zeta}_1|^2 \tag{4.36}$$

$$= C_5 e^{-2k} (1 + |k|^2)^{s+1/2} \left| \frac{\hat{\zeta}_1}{k^{1/2}} \right|^2 \tag{4.37}$$

where the factor $e^{-2k}(1 + |k|^2)^{s+1/2}$ is bounded for all $s \geq 0$. By defining $\hat{\xi} = k^{-1/2} \hat{\zeta}_1$

$$\int_{\mathbb{R}^2} (1 + |k|^2)^s k e^{-2k} \left| \frac{\hat{\zeta}_1}{\omega_S} \right|^2 \, d\mathbf{k} \leq C_6 \|\xi\|_2^2 \tag{4.38}$$

and the desired result is obtained

$$\|\eta\|_{H^s}^2 \leq Ct^2 \left(\|\zeta_0\|_2^2 + \|\xi\|_2^2\right). \tag{4.39}$$

□

4.2 Dimensionalized Estimate for the Free Surface

In order to investigate the scope of validity of the rigid-lid assumption the free-surface estimate (4.31) developed in the previous section needs to be written in terms of physical variables. To this end, temporarily denote non-dimensional variables by tilde and recall from (2.37) that

$$\eta = L\tilde{\eta}, \quad \zeta = L\tilde{\zeta}, \quad x = L\tilde{x}, \quad y = L\tilde{y}, \quad t = T\tilde{t}, \tag{4.40}$$

where, in this context, L represents the length dimension whose characteristic scale is d_1 and $T = \sqrt{d_1/g}$ is the characteristic time scale. Moreover, $\mathbf{k} = L^{-1}\tilde{\mathbf{k}}$ and a characteristic velocity is given by $U = LT^{-1} = \sqrt{gd_1}$. In order to dimensionalize

$$\|\tilde{\eta}\|_{H^s(\mathbb{R}^2)}^2 \leq C_3\tilde{t}^2 \int_{\mathbb{R}^2} (1+|\tilde{k}|^2)^s \tilde{k} e^{-2\tilde{k}} \left(|\hat{\tilde{\zeta}}_0|^2 + \left| \frac{\hat{\tilde{\zeta}}_1}{\tilde{\omega}_S} \right|^2 \right) d\tilde{\mathbf{k}}, \tag{4.41}$$

use will be made of the dimensionalized H^s norm

$$||\eta||_{H^s}^2 = \int_{\mathbb{R}^2} \left(1 + d_1^2|\mathbf{k}|^2\right)^s |\hat{\eta}(\mathbf{k})|^2 \, d\mathbf{k}. \tag{4.42}$$

Now $||\eta||_2^2$ has dimensions L^4 due to a length squared from $|\eta|^2$ times the element of area $d\mathbf{x}$ and the H^s norm has these dimensions since L^2-based Sobolev spaces are isometrically isomorphic to scaled L^2-spaces, where the scale factor $(1+|k|^2)^s$ is non-dimensional. Similarly, $||\zeta_0||_2^2$ has dimensions L^4.

In order to calculate $[\hat{\zeta}_0]^2$, use is made of Parseval's theorem

$$\int |\hat{\zeta}_0|_2^2 \, d\mathbf{k} = \int |\zeta_0|^2 \, d\mathbf{x}. \tag{4.43}$$

Matching dimensions on both sides of this equation implies $[\hat{\zeta}_0]^2 L^{-2} = L^4$ or $[\hat{\zeta}_0]^2 = L^6$. Similarly, the dimension of $|\hat{\zeta}_1|^2$ is $L^6 T^{-2}$ and since $[\omega_S]^2 = T^{-2}$, the dimension of $|\hat{\zeta}_1 \omega_S^{-1}|^2$ is thus also L^6.

These observations along with the relation $d\tilde{\mathbf{k}} = d_1^2 d\mathbf{k}$ allow us to rewrite (4.41) in the dimensionalized form

$$\begin{aligned} ||\eta||_{H^s}^2 &\le C(\mu) t^2 \frac{g}{d_1} \int_{\mathbb{R}^2} (1 + d_1^2 k^2)^s d_1 k e^{-2d_1 k} \left(|\hat{\zeta}_0|^2 + \left| \frac{\hat{\zeta}_1}{\omega_S} \right|^2 \right) \, d\mathbf{k} \\ &\le C(\mu) t^2 \frac{g}{d_1} \sup_{k \ge 0} \Theta(k, s, 2) \left(\|\zeta_0\|_2^2 + \|\chi\|_2^2 \right) \end{aligned} \tag{4.44}$$

where χ is such that $\hat{\chi} = \frac{\hat{\zeta}_1}{\omega_S}$ and

$$\Theta(k, s, n) = (1 + d_1^2 k^2)^s d_1 k e^{-n d_1 k} \tag{4.45}$$

is a dimensionless function, integrable for all s and $k \geq 0$. Moreover, the function $\Theta(k,s,n) \geq 0$ is such that $\Theta(0,s,n) = \lim_{k\to\infty} \Theta(k,s,n) = 0$ and numerical calculations show Θ has one maximum at some k_{max}, after which it is monotonically decreasing and as s increases, k_{max} gets larger. That is, near zero, the polynomial factor in Θ dominates over a larger range of k-values until the exponentially decaying term dominates.

The derivative of Θ with respect to k is given by

$$\Theta' = d_1(1+(d_1k)^2)^{s-1}e^{-nd_1k}\left((2s+1)(d_1k)^2 - n(d_1k)^3 - nd_1k + 1\right). \tag{4.46}$$

Thus, it achieves a maximum at some k_{max} which satisfies

$$(2s+1)(d_1k)^2 - n(d_1k)^3 - nd_1k + 1 = 0. \tag{4.47}$$

In principle, this cubic can be solved using the formula of Cardano or by approximating the root using Newton's method.

Consider a simple example of a class of initial data with support of k such that $k \in [2\pi/\lambda, \infty)$, where λ is a given wavelength. Then, by the forgoing discussion, two possibilities arise. In the first case, $2\pi/\lambda \geq k_{max}$ implies

$$\sup_{k\geq\frac{2\pi}{\lambda}} \Theta(k,s,n) = \Theta\left(\frac{2\pi}{\lambda}, s, n\right). \tag{4.48}$$

In the second case, $2\pi/\lambda < k_{max}$ implies

$$\sup_{k \geq \frac{2\pi}{\lambda}} \Theta(k, s, n) = \Theta(k_{max}, s, n). \tag{4.49}$$

By choosing $n = 2$, these observations can then be used to compute explicitly an estimate for $||\eta||_{H^s}^2$.

This process of estimating η will now be illustrated in the special case $s = 0$. That is the L^2 norm of η will be computed. To this end, define the parameter

$$\epsilon = \frac{\lambda}{d_1} \tag{4.50}$$

and variable

$$w = d_1 k. \tag{4.51}$$

The definition of Θ implies

$$\Theta(w, 0, 2) = w e^{-2w} \tag{4.52}$$

with $w \in [2\pi/\epsilon, \infty)$ and the inequality

$$||\eta||_2^2 \leq C' t^2 \frac{g}{d_1} \sup_{w \geq \frac{2\pi}{\epsilon}} \Theta(w, 0, 2) \left(\|\zeta_0\|_2^2 + \|\chi\|_2^2 \right) \tag{4.53}$$

is valid. The derivative of Θ with respect to w is given by

$$\Theta' = (1 - 2w)e^{-2w} \tag{4.54}$$

so that $\Theta' = 0$ when $w_{max} = 1/2$ with $\Theta(w_{max}, 0, 2) = (2e)^{-1}$. If $2\pi/\epsilon < 1/2$ or $4\pi < \epsilon$, then

$$\sup_{w \geq \frac{2\pi}{\epsilon}} \Theta(w, 0, 2) = (2e)^{-1}. \tag{4.55}$$

Therefore,

$$\frac{||\eta||_2^2}{\|\zeta_0\|_2^2 + \|\chi\|_2^2} \leq \frac{C'}{2e}\left(t^2 \frac{g}{d_1}\right) \tag{4.56}$$

$$= C''\left(t^2 \frac{g}{d_1}\right) \ll 1 \tag{4.57}$$

when $4\pi < \epsilon$ or $d_1 < \lambda/4\pi$ and $t \ll \sqrt{d_1/g}$.

On the other hand, if $w_{max} = 1/2 \leq 2\pi/\epsilon$ or $\lambda/4\pi \leq d_1$, then

$$\sup_{w \geq \frac{2\pi}{\epsilon}} \Theta(w, 0, 2) = \frac{2\pi}{\epsilon} e^{-\frac{4\pi}{\epsilon}} \tag{4.58}$$

and $\frac{||\eta||_2^2}{\|\zeta_0\|_2^2 + \|\chi\|_2^2} \ll 1$ when

$$t^2 \ll C''' \frac{d_1 \epsilon}{g} e^{4\pi/\epsilon}. \tag{4.59}$$

Thus, if k is supported on $[2\pi/\lambda, \infty)$, the L^2-norm of η can be made small by a well-defined choice of depth d_1 and time scale $t \in [0, T_0]$.

If the support is at low wave numbers (i.e. has power near $k = 0$), then the first order Taylor's expansion of $\Theta(k, s, n)$ and Lemma 4.1.4 imply

$$\Theta(k, s, n) = d_1 k(1 + d_1^2 k^2)^s e^{-nd_1 k} \approx d_1 k \qquad \text{and} \qquad (4.60)$$

$$\omega_S^{-2} \approx (1 - \mu)^{-1}(d_1 k)^{-2} \qquad (4.61)$$

so that

$$||\eta||_{H_s}^2 \leq Ct^2 \frac{g}{d_1} \int_{\mathbb{R}^2} d_1 k \left(|\hat{\zeta}_0|^2 + (1 - \mu)^{-1} \left| \frac{\hat{\zeta}_1}{k} \right|^2 \right) d\mathbf{k}. \qquad (4.62)$$

By defining time $t_0 = O(\sqrt{d_1/g})$ and

$$I' = \int_{\mathbb{R}^2} d_1 k \left(|\hat{\zeta}_0|^2 + (1 - \mu)^{-1} \left| \frac{\hat{\zeta}_1}{k} \right|^2 \right) d\mathbf{k} \qquad (4.63)$$

then

$$||\eta||_{H_s}^2 \leq \left(\frac{t}{t_0} \right)^2 I' < I'. \qquad (4.64)$$

Thus for low wave numbers, the time for which $||\eta(\cdot, t)||_{H_s}^2$ is comparable with I' is of order $\sqrt{d_1/g}$.

4.3 An Estimate of the Interface Difference

In this section an estimate of the interface difference for the free-surface and rigid-lid systems based on their corresponding prepared initial value problems will be obtained. To this end, the following lemma is useful.

Lemma 4.3.1. *For $k \geq 0$, the equality $\frac{-ur^2}{2v} = \frac{1-\mu}{2\mu} e^{-2k}$ is valid.*

Proof. From the definitions of r, q, u, v, ω_S and ω_F

$$\begin{aligned}\frac{-ur^2}{2v} &= -\frac{1}{2}\left(\frac{\omega_S}{\omega_F}\right)^2 \left(\frac{k\tanh k - \omega_F^2}{k\tanh k - \omega_S^2}\right) \\ &= \frac{-\tanh k(1-\mu)}{2(1+\mu\tanh k)}\left(\frac{(1+\mu\tanh k)(\tanh k - 1)}{\mu\tanh k(1+\tanh k)}\right) \\ &= \frac{(1-\mu)(1-\tanh k)}{2\mu(1+\tanh k)} = \frac{1-\mu}{2\mu} e^{-2k},\end{aligned} \tag{4.65}$$

where the last equality follows from the fact that

$$\frac{1-\tanh k}{1+\tanh k} = \frac{\cosh k - \sinh k}{\cosh k + \sinh k} = e^{-2k}. \tag{4.66}$$

□

Notice, the sup-norm is given by

$$\left\| \frac{ur^2}{2v} \right\|_\infty = \frac{1-\mu}{2\mu}. \tag{4.67}$$

Lemma 4.3.2. *For $k > 0$, there exists a positive constant C such that*

$$\frac{\omega_R^2 - \omega_S^2}{\omega_R} \leq C\sqrt{k} e^{-2k}. \tag{4.68}$$

Proof. Let $\tau = \tanh k$, then from the definitions of ω_R and ω_S, a straightforward calculation shows that

$$\omega_R^2 - \omega_S^2 = \frac{k(1-\mu)^2 \tau(1-\tau)}{(\mu+\tau)(1+\mu\tau)} \tag{4.69}$$

so that

$$\begin{aligned} \frac{\omega_R^2 - \omega_S^2}{\omega_R} &= \sqrt{\frac{\mu + \tau}{k(1-\mu)\tau}} \left(\frac{k(1-\mu)^2 \tau (1-\tau)}{(\mu+\tau)(1+\mu\tau)} \right) \\ &= (1-\mu)^{3/2} \sqrt{\frac{k\tau}{\mu+\tau}} \left(\frac{1-\tau}{1+\mu\tau} \right). \end{aligned} \tag{4.70}$$

Since $\tau \in [0, 1)$, it is clear that there is a positive constant C such that

$$\sqrt{\frac{\tau}{\mu+\tau}} \left(\frac{(1-\mu)^{3/2}}{1+\mu\tau} \right) \leq C. \tag{4.71}$$

This together with $1 - \tau = e^{-k}/\cosh k \leq 2e^{-2k}$ implies

$$\frac{\omega_R^2 - \omega_S^2}{\omega_R} \leq C\sqrt{k}e^{-2k} \tag{4.72}$$

as desired. □

These lemmas along with the established ordering of the frequencies $\omega_S \leq \omega_R \leq \omega_F = \sqrt{k}$ and the relations (3.92) through (3.94) together implies

$$\begin{aligned} |\mathcal{J}(t)| \leq t \Bigg(&\left| \frac{ur^2}{2v} \right| (|\omega_F - \omega_R| + 2r|\omega_F| + 2q|\omega_R|) + \\ &\frac{1}{2} \left(|(1+q)(\omega_R - \omega_S)| + |(1-q)(\omega_R + \omega_S)| \right) \Bigg). \end{aligned} \tag{4.73}$$

From the definitions of r and q, $|\mathcal{J}(t)|$ can be estimated as

$$
\begin{aligned}
|\mathcal{J}(t)| &\leq t\left(\left|\frac{ur^2}{2v}\right|(2|\omega_F-\omega_R|+4|\omega_S|)+\left|\frac{\omega_R^2-\omega_S^2}{\omega_R}\right|\right)\\
&\leq t\left(C_1\sqrt{k}e^{-2k}+C_2\sqrt{k}e^{-2k}\right) \qquad (4.74)\\
&\leq C_3 t\sqrt{k}e^{-2k},
\end{aligned}
$$

for some positive constants C_1 and C_2 with $C_3=\max(C_1,C_2)$.

Theorem 4.3.3. *Let $s\geq 0$ and $\zeta_0=\zeta(\mathbf{x},0)$ and $\zeta_1=\partial_t\zeta(\mathbf{x},0)$ be initial data for the free-surface and rigid-lid two-fluid systems Π_f and Π_r. Let ξ be a function such that $\hat{\xi}=\hat{\zeta}_1k^{-1/2}$ and assume ζ_0 and ξ are both L^2 functions. Let $\eta(\mathbf{x},t)$ and $\zeta(\mathbf{x},t)$ be solutions for that initial data. Then there exists a positive constant C such that*

$$\|\zeta_{free}(\cdot,t)-\zeta_{rigid}(\cdot,t)\|_{H^s}^2\leq Ct^2\left(\|\zeta_0\|_2^2+\|\xi\|_2^2\right). \qquad (4.75)$$

Proof. The previous calculation for $|\mathcal{J}(t)|$ and (4.1.3) together imply

$$|\hat{\zeta}_{free}(\mathbf{k},t)-\hat{\zeta}_{rigid}(\mathbf{k},t)|\leq|\mathcal{J}(t)|(|S_+|+|S_-|)\leq C_3t\sqrt{k}e^{-2k}\left(|\hat{\zeta}_0|+\left|\frac{\hat{\zeta}_1}{\omega_S}\right|\right) \qquad (4.76)$$

so that

$$|\hat{\zeta}_{free}(\mathbf{k},t)-\hat{\zeta}_{rigid}(\mathbf{k},t)|^2\leq C_4t^2ke^{-4k}\left(|\hat{\zeta}_0|^2+\left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2\right) \qquad (4.77)$$

for $C_4 \geq 0$. Thus, the H^s-norm for the interface difference can be estimated as

$$\begin{aligned}\|\zeta_{free}(\cdot,t) - \zeta_{rigid}(\cdot,t)\|_{H^s}^2 &= \int_{\mathbb{R}^2} (1+|k|^2)^s |\hat{\zeta}_{free} - \hat{\zeta}_{rigid}|^2 \, d\mathbf{k} \\ &\leq C_4 t^2 \int_{\mathbb{R}^2} (1+|k|^2)^s |k| e^{-4k} \left(|\hat{\zeta}_0|^2 + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2 \right) \, d\mathbf{k}.\end{aligned} \tag{4.78}$$

The function

$$\Lambda_{s,n}(k) = (1+|k|^2)^s e^{-4k} k^n \tag{4.79}$$

is integrable for all s and $n > -1$ so that

$$\int_{\mathbb{R}^2} \Lambda_{s,1}(k) |\hat{\zeta}_0|^2 \, dk \leq C_5 \|\zeta_0\|_2^2. \tag{4.80}$$

Lemmas (4.1.4) and (4.1.5) together imply

$$\begin{aligned}\Lambda_{s,1}(k) \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2 &< C_6 \Lambda_{s,1} (1+|k|^2)^{1/2} \left|\frac{\hat{\zeta}_1}{k}\right|^2 \\ &= C_6 e^{-4k} (1+|k|^2)^{s+1/2} \left|\frac{\hat{\zeta}_1}{k^{1/2}}\right|^2.\end{aligned} \tag{4.81}$$

Defining ξ such that $\hat{\xi} = k^{-1/2}\hat{\zeta}_1$ implies the inequality

$$\int_{\mathbb{R}^2} (1+|k|^2)^s k e^{-4k} \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2 \, dk \leq C_7 \|\xi\|_2^2. \tag{4.82}$$

This together with (4.80) gives the desired result, namely

$$\|\zeta_{free}(\cdot,t) - \zeta_{rigid}(\cdot,t)\|_{H^s}^2 \leq Ct^2 \left(\|\zeta_0\|_2^2 + \|\xi\|_2^2\right). \tag{4.83}$$

□

4.4 Dimensionalized Estimate for the Interface

The validity of the rigid-lid assumption with respect to the interface difference will now be investigated. To this end, the estimate (4.78) will first be written in terms of physical variables by a dimensional analysis similar to that given in Section 4.2. This analysis results in the estimated interface difference

$$\begin{aligned} ||\zeta_{free} - \zeta_{rigid}||_{H^s}^2 &\leq C(\mu)t^2 \frac{g}{d_1} \int_{\mathbb{R}^2} (1 + (d_1 k)^2)^s d_1 k e^{-4d_1 k} \left(|\hat{\zeta}_0|^2 + \left| \frac{\hat{\zeta}_1}{\omega_S} \right|^2 \right) d\mathbf{k} \\ &\leq C(\mu)t^2 \frac{g}{d_1} \sup_{k\geq 0} \Theta(k,s,4) \left(\|\zeta_0\|_2^2 + \|\chi\|_2^2\right), \end{aligned} \tag{4.84}$$

where the last equality follows from (4.45) and the variables are now with dimension, i.e., no longer interpreted with tildes.

As stated in the introduction, the primary goal of this dissertation is to answer the following question: Given a wavelength λ, what must the depth d_1 be in order for the rigid-lid approximation to be good and for how long does this approximation remain valid? To answer this question, consider the case $s = 0$ (estimate the difference of the interface with respect to the L^2-norm) for initial data supported on $k \in [2\pi/\lambda, \infty)$.

In order to compute $\sup_{k\geq 0}\Theta(k,0,4)$, notice the derivative of $\Theta'(k,0,4)$ with respect to k is given by $d_1e^{-4d_1k}(1-4d_1k)$ so $k_{max}=1/4d_1$ and $\Theta(k_{max},0,4)=1/4e$. Since

$$\lim_{k\to\infty}\Theta(k,0,4)=0,$$

it is clear that

$$\sup_{k\in[2\pi/\lambda,\infty)}\Theta(k,0,4)=\begin{cases}(4e)^{-1}, & \text{if } 8\pi<\epsilon;\\ \frac{2\pi}{\epsilon}e^{-8\pi/\epsilon}, & \text{if } 8\pi\geq\epsilon,\end{cases}\tag{4.85}$$

where $\epsilon=\lambda/d_1$. This observation implies

$$\frac{||\zeta_{free}(\cdot,t)-\zeta_{rigid}(\cdot,t)||_2^2}{\|\zeta_0\|_2^2+\|\chi\|_2^2}\leq\begin{cases}C_1(\mu)\frac{e^{-8\pi/\epsilon}}{\epsilon}t^2\frac{g}{d_1}, & \text{if } \epsilon\leq 8\pi;\\ C_2(\mu)t^2\frac{g}{d_1}, & \text{otherwise.}\end{cases}\tag{4.86}$$

Therefore, $\frac{||\zeta_{free}(\cdot,t)-\zeta_{rigid}(\cdot,t)||_2^2}{\|\zeta_0\|_2^2+\|\chi\|_2^2}\ll 1$ when

$$t^2\ll\begin{cases}C_3\epsilon e^{8\pi/\epsilon}\frac{d_1}{g}, & \text{if } \epsilon\leq 8\pi;\\ C_4\frac{d_1}{g}, & \text{otherwise.}\end{cases}\tag{4.87}$$

For short wave inital data, the time scale of validity of the approximation is exponentially long in ϵ, whereas for long wave intial data, the approximation is only valid for times of order $\sqrt{d_1/g}$, which is approximately the period of oscillation of a long wave.

4.5 An Estimate of the Upper Layer Energy

In this section an estimate of the free-surface energy component $E_{upper}(t)$ for $t > 0$ will be obtained given that initially, $E_{lower}(0) = E_{rigid}(0)$. To this end, the following lemma will be useful.

Lemma 4.5.1. *For $k > 0$, there exists positive constants a and b such that*

$$\left(\frac{k^{3/4}}{\sinh k}\right)^2 \le k^{-1/2}e^{-2k}(a + bk^2). \tag{4.88}$$

Proof. The definition of the hyperbolic sine function implies

$$\left(\frac{k^{3/4}}{\sinh k}\right)^2 = \frac{4k^{3/2}e^{-2k}}{(1-e^{-2k})^2} = 4k^{-1/2}e^{-2k}\left(\frac{k}{1-e^{-2k}}\right)^2. \tag{4.89}$$

To prove the lemma it suffices to show

$$\left(\frac{k}{1-e^{-2k}}\right)^2 \le a + bk^2 \tag{4.90}$$

for positive constants a and b. Since

$$\lim_{k\to 0}\frac{k}{1-e^{-2k}} = \lim_{k\to 0}\frac{1}{2e^{-2k}} = \frac{1}{2}, \tag{4.91}$$

there exists a positive constants a' and α such that $k/(1-e^{-2k}) \le a'$ for $0 < k < \alpha$.

For $\alpha \leq k$, since $(1-e^{-2k})^{-1}$ is monotonically decreasing

$$k^{-2}\left(\frac{k}{1-e^{-2k}}\right)^2 = (1-e^{-2k})^{-2} \leq (1-e^{-2\alpha})^{-2}. \tag{4.92}$$

That is, there exists a positive constant b' such that

$$\left(\frac{k}{1-e^{-2k}}\right)^2 \leq b'k^2. \tag{4.93}$$

For $k > 0$, choosing $a = (2a')^2$ and $b = 4b'$ gives the desired inequality. □

Theorem 4.5.2. *For $t \in [0, T_0]$, let ξ_j for $j = 1, \ldots, 9$ be L^2 functions depending on the initial data ζ_0 and ζ_1 defined by*

$$\begin{aligned} \hat{\xi}_1 &= k^{-1/4}\hat{\zeta}_0, \quad \hat{\xi}_2 = k^{3/4}\hat{\zeta}_0, \quad \hat{\xi}_3 = k^{-5/4}\hat{\zeta}_1, \\ \hat{\xi}_4 &= k^{-1/4}\hat{\zeta}_1, \quad \hat{\xi}_5 = k^{-3/4}\hat{\zeta}_1, \quad \hat{\xi}_6 = k^{1/4}\hat{\zeta}_1, \\ \hat{\xi}_7 &= k^{1/2}\hat{\zeta}_0, \quad \hat{\xi}_8 = k^{-1/2}\hat{\zeta}_1, \quad \hat{\xi}_9 = \hat{\zeta}_1. \end{aligned} \tag{4.94}$$

Then there exists positive constants C_1, C_2 such that

$$E_{upper}(t) \leq C_1 t \left(\sum_{j=1}^{6} \|\xi_j\|_2^2\right) + C_2 t^2 \left(\sum_{j=7}^{9} \|\xi_j\|_2^2\right). \tag{4.95}$$

Proof. The definitions of $\mathcal{A}$ and $\mathcal{B}$, the calculation given in (3.117) and the triangle inequality together imply

$$\begin{aligned}
AB^* + A^*B &\leq 2(|S_+| + |S_-|)^2 |\mathcal{A}(t)\mathcal{B}(t)| \\
&\leq 2(|S_+| + |S_-|)^2 \frac{\omega_S}{v} \left(1 + \frac{r}{2}(|1+r| + |1-r|)\right) |\mathcal{B}(t)| \\
&\leq 2(|S_+| + |S_-|)^2 \frac{\omega_S}{v} \left(1 + \frac{r}{2\omega_F}(|\omega_F + \omega_S| + |\omega_F - \omega_S|)\right) |\mathcal{B}(t)| \\
&\leq 2(|S_+| + |S_-|)^2 \frac{\omega_S}{v} \left(1 + \frac{r}{2\omega_F}(4\omega_F)\right) |\mathcal{B}(t)| \\
&\leq 2(|S_+| + |S_-|)^2 \frac{\omega_S}{v} (1 + 2r) |\mathcal{B}(t)|.
\end{aligned} \tag{4.96}$$

and

$$\begin{aligned}
|\mathcal{B}(t)| &\leq \frac{\omega_F^3 r^2}{2v} |2\omega_F t| + \frac{t\omega_S^3}{2v} (|\omega_F + \omega_S| + |\omega_F - \omega_S|) \\
&= \frac{t}{2v} \left((\omega_S\omega_F)^2 + \omega_S^3(|\omega_F + \omega_S| + |\omega_F - \omega_S|)\right) \\
&\leq \frac{t}{2v} \left(2\omega_F^4 + 4\omega_F^4\right) = \frac{3t\omega_F^4}{v}.
\end{aligned} \tag{4.97}$$

Therefore, since $\omega_S \leq \omega_F$, the inequality

$$\begin{aligned}
AB^* + A^*B &\leq Ct(|S_+| + |S_-|)^2 \frac{\omega_F^5}{v^2} \\
&= Ct(|S_+| + |S_-|)^2 \frac{k^{5/2}}{v^2}
\end{aligned} \tag{4.98}$$

is valid and this in turn implies

$$\begin{aligned} k(AB^* + A^*B) &\leq Ctk(|S_+| + |S_-|)^2 \frac{\omega_F^5}{v^2} \\ &= Ct(|S_+| + |S_-|)^2 \frac{k^{7/2}}{v^2}. \end{aligned} \tag{4.99}$$

The definition of v implies

$$v^{-1} = \frac{1}{k \sinh k} \left(\frac{1 + \mu \tanh k}{\mu(1 + \tanh k)} \right). \tag{4.100}$$

The definition of the hyperbolic tangent implies

$$\frac{1 + \mu \tanh k}{\mu(1 + \tanh k)} = \frac{1}{2\mu}(1 + \mu + (1 - \mu)e^{-2k}) \tag{4.101}$$

so that

$$\sup_{k \geq 0} \frac{1 + \mu \tanh k}{\mu(1 + \tanh k)} = \mu^{-1}. \tag{4.102}$$

Using this result in the calculation for v^{-1} implies $v^{-1} \leq (\mu k \sinh k)^{-1}$. Therefore,

$$\begin{aligned} k(AB^* + A^*B) &\leq Ct(|S_+| + |S_-|)^2 \frac{k^{7/2}}{k^2 \sinh^2 k} \\ &= Ct(|S_+| + |S_-|)^2 \left(\frac{k^{3/4}}{\sinh k} \right)^2. \end{aligned} \tag{4.103}$$

This inequality together with Lemmas 4.1.4 and 4.3.1 imply

$$\begin{aligned} k(AB^* + A^*B) &\leq C_3 t(C_1 k^{-1/2} + C_2)\left(|\hat{\zeta}_0|^2 + \alpha|\hat{\zeta}k^{-1}|^2(1+\frac{\beta}{\alpha}k)\right) \\ &= C_3 t e^{-2k}(C_1 k^{-1/2} + C_2 k^{3/2})\left(|\hat{\zeta}_0|^2 + \alpha|\hat{\zeta}_1 k^{-1}|^2 + \beta|\hat{\zeta}_1 k^{-1/2}|^2\right). \end{aligned} \tag{4.104}$$

Expanding the right-hand side, results in the relation

$$\begin{aligned} k(AB^* + A^*B) \leq C_4 t e^{-2k}(&|k^{-1/4}\hat{\zeta}_0|^2 + |k^{3/4}\hat{\zeta}_0|^2 + |k^{-5/4}\hat{\zeta}_1|^2 \\ &+ |k^{-1/4}\hat{\zeta}_1|^2 + |k^{-3/4}\hat{\zeta}_1|^2 + |k^{1/4}\hat{\zeta}_1|^2) \end{aligned} \tag{4.105}$$

for some positive constant C_4.

For $i = 1, \ldots 6$, let ξ_i be functions defined by

$$\begin{aligned} &\hat{\xi}_1 = k^{-1/4}\hat{\zeta}_0, \quad \hat{\xi}_2 = k^{3/4}\hat{\zeta}_0, \quad \hat{\xi}_3 = k^{-5/4}\hat{\zeta}_1, \\ &\hat{\xi}_4 = k^{-1/4}\hat{\zeta}_1, \quad \hat{\xi}_5 = k^{-3/4}\hat{\zeta}_1, \quad \hat{\xi}_6 = k^{1/4}\hat{\zeta}_1. \end{aligned} \tag{4.106}$$

Then the previous inequality implies

$$\int_{\mathbb{R}^2} k(AB^* + A^*B)\, d\mathbf{k} \leq C_4 t \int_{\mathbb{R}^2} \left(e^{-2k}\sum_{j=1}^{6}|\hat{\xi}_j|^2\right)\, d\mathbf{k}. \tag{4.107}$$

The calculation made for $|\hat{\eta}|^2$ in Theorem 4.1.6 along with Lemma (4.1.4) implies

$$
\begin{aligned}
|\hat{\eta}|^2 &\leq C_5 t^2 k e^{-2k} \left(|\hat{\zeta}_0|^2 + |\omega_S^{-1} \hat{\zeta}_1|^2 \right) && (4.108) \\
&\leq C_5 t^2 e^{-2k} \left(|k^{-1/2} \hat{\zeta}_0|^2 + |k^{-1/2} \hat{\zeta}_1|^2 + |\hat{\zeta}_1|^2 \right) && (4.109)
\end{aligned}
$$

for some positive constant C_5.

For for $i = 7, 8, 9$, define functions ξ_i such that

$$\hat{\xi}_7 = k^{1/2} \hat{\zeta}_0, \quad \hat{\xi}_8 = k^{-1/2} \hat{\zeta}_1, \quad \hat{\xi}_9 = \hat{\zeta}_1. \tag{4.110}$$

Then the last inequality above implies

$$\|\eta\|_2^2 \leq C_5 t^2 \int_{\mathbb{R}^2} \left(e^{-2k} \sum_{j=7}^{9} |\hat{\xi}_j|^2 \right) \, d\mathbf{k}. \tag{4.111}$$

□

4.6 Dimensionalized Energy Estimate

The dimensionless energy is presented in (2.50) as

$$
\begin{aligned}
\|\Psi\|_E = \frac{\mu}{2} \int_{\mathbb{R}^2} \int_{-1}^{0} |\nabla \phi_1|^2 \, d\tilde{z} d\tilde{\mathbf{x}} + \frac{\mu}{2} \int_{\mathbb{R}^2} \tilde{\eta}^2 \, d\tilde{\mathbf{x}} \\
+ \frac{1}{2} \int_{\mathbb{R}^2} \int_{-(1+\Gamma)}^{-1} |\tilde{\nabla} \tilde{\phi}_2|^2 \, d\tilde{z} d\tilde{\mathbf{x}} + \frac{(1-\mu)}{2} \int_{\mathbb{R}^2} \tilde{\zeta}^2 \, d\tilde{\mathbf{x}},
\end{aligned} \tag{4.112}
$$

where $\Gamma \to \infty$. Defining the dimensionalized energy by

$$E = \|\Psi\|_E \, d_1^4 g \rho_2 \tag{4.113}$$

implies

$$\begin{aligned} E =& d_1^4 g \rho_2 \left[\frac{\mu}{2} \int_{\mathbb{R}^2} \int_{-d_1}^{0} \left| (L\nabla) \frac{\phi_1}{L^2/T} \right|^2 \frac{d\tilde{z} d\tilde{\mathbf{x}}}{L^3} + \frac{\mu}{2L^4} \int_{\mathbb{R}^2} \tilde{\eta}^2 \, d\tilde{\mathbf{x}} \right. \\ & \left. + \frac{1}{2} \int_{\mathbb{R}^2} \int_{-(1+\Gamma)}^{-d_1} \left| (L\tilde{\nabla}) \frac{\tilde{\phi}_2}{L^2/T} \right|^2 \frac{d\tilde{z} d\tilde{\mathbf{x}}}{L^3} + \frac{(1-\mu)}{2L^4} \int_{\mathbb{R}^2} \tilde{\zeta}^2 \, d\tilde{\mathbf{x}} \right] \end{aligned} \tag{4.114}$$

where $\tilde{\nabla} = L\nabla$ is the non-dimensional gradient.

By recalling

$$L = d_1, \quad T^2 = d_1/g, \quad \mu = \rho_1/\rho_2, \tag{4.115}$$

the dimensionalized energy, after clearing denominators, becomes

$$\begin{aligned} E =& \frac{\rho_1}{2} \int_{\mathbb{R}^2} \int_{-d_1}^{0} |\nabla \phi_1|^2 \, dz d\mathbf{x} + \frac{\rho_1 g}{2} \int_{\mathbb{R}^2} \eta^2 \, d\mathbf{x} + \\ & \frac{\rho_2}{2} \int_{\mathbb{R}^2} \int_{-(1+\Gamma)}^{-d_1} |\nabla \phi_2|^2 \, dz d\mathbf{x} + \frac{g(\rho_2 - \rho_1)}{2} \int_{\mathbb{R}^2} \zeta^2 \, d\mathbf{x}. \end{aligned} \tag{4.116}$$

A dimensionalized estimate of the upper energy (3.116), which is written in terms of tilde variables as

$$\tilde{E}_{upper}(t) = \frac{\mu}{2} \left(\int_{\mathbb{R}^2} |\hat{\tilde{\eta}}|^2 \, d\tilde{\mathbf{k}} + \frac{1}{2} \int_{\mathbb{R}^2} \tilde{k} (AB^* \mp A^* B) \, d\tilde{\mathbf{k}} \right), \tag{4.117}$$

can be obtained by dimensionalizing (4.30) and (4.99) along with (4.113). After some straightforward computations, the dimensionalized upper energy estimate becomes

$$E_{upper} \leq \frac{B_1\rho_1 g^2 t^2}{2d_1}\int_{\mathbb{R}^2} d_1 k e^{-2d_1 k}\left(|\hat{\zeta}_0|^2 + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2\right) d\mathbf{k} + \frac{B_2\rho_2 g^{3/2} t}{2\sqrt{d_1}}\int_{\mathbb{R}^2}\left(\frac{(d_1k)^{3/4}}{\sinh d_1 k}\right)^2\left(|\hat{\zeta}_0|^2 + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2\right) d\mathbf{k}. \quad (4.118)$$

where $B_1(\mu)$ and $B_2(\mu)$ are positive constants.

In the example of a class of initial data with k supported by $[2\pi\lambda, \infty)$, elementary calculus arguments show that $((d_1k)^{3/4}/\sinh d_1k)^2$ is monotonically decreasing, so that

$$\begin{aligned}\varepsilon_2 := \sup_{k\in[2\pi\lambda,\infty)}\left(\frac{(d_1k)^{3/4}}{\sinh d_1 k}\right)^2 &= \left(\frac{(2\pi/\epsilon)^{3/4}}{\sinh(2\pi/\epsilon)}\right)^2 \\ &\leq (2\pi/\epsilon)^{-1/2}e^{-4\pi/\epsilon}(a + b(2\pi/\epsilon)^2)\end{aligned} \quad (4.119)$$

and

$$\varepsilon_1 := \sup_{k\in[2\pi\lambda,\infty)} d_1 k e^{-2d_1 k} = \begin{cases}\frac{2\pi}{\epsilon}e^{-4\pi/\epsilon}, & \text{if } \epsilon < 4\pi; \\ (2e)^{-1}, & \text{otherwise,}\end{cases} \quad (4.120)$$

where the last inequality follows from Lemma 4.4.1. An estimate for the dimensionalized upper layer energy becomes

$$E_{upper}(t) \leq \frac{\rho_1 g t}{d_1}\left[\varepsilon_1 B_3(\mu, \|\zeta_0\|_2^2, \|\chi\|_2^2) g t + \varepsilon_2 B_4(\mu, \|\zeta_0\|_2^2, \|\chi\|_2^2)\sqrt{g d_1}\right], \quad (4.121)$$

where B_3 and B_4 are positive constants depending on the initial data.

In the example of a class of initial data with $k \ll 1$, the approximations

$$d_1 k e^{-2d_1 k} \approx d_1 k \qquad \text{and} \tag{4.122}$$

$$\left(\frac{(d_1 k)^{3/4}}{\sinh d_1 k}\right)^2 \approx B_5 (d_1 k)^{-1/2} \tag{4.123}$$

are valid, where the last approximation follows from the calculation presented in Lemma 4.4.1 and B_5 is a positive constant with $[B_5] = L^{-1}$. An estimate of the dimensionalized upper layer energy is thus given by

$$E_{upper} \leq \frac{B_1 \rho_1 g^2 t^2}{2d_1} \int d_1 k \left(|\hat{\zeta}_0|^2 + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2 \right) d\mathbf{k} + \frac{B_2 \rho_2 g^{3/2} t}{2\sqrt{d_1}} \int B_3 k^{-1/2} \left(|\hat{\zeta}_0|^2 + \left|\frac{\hat{\zeta}_1}{\omega_S}\right|^2 \right) d\mathbf{k}. \tag{4.124}$$

CHAPTER 5

DISCUSSION

The oceanographic rigid-lid approximation was mathematically investigated by comparing two linear initial-boundary-value problems (IBVP) for a fluid system in which two immiscible layers were separated by a sharp free interface with featureless, deep bottom. In particular, when in hydrostatic equilibrium, the fluid system was composed of a top homogeneous fluid layer with depth d_1 and density ρ_1 and a bottom homogeneous fluid layer of depth d_2 and density $\rho_2 > \rho_1$. The fluid system was modeled using the linearized Euler's equations and the case of small amplitude surface and internal wave propagation was considered.

Regarding the initial-boundary-value problems, the first free-surface IBVP had both free surface and interfacial boundaries. The second rigid-lid IBVP was obtained by neglecting surface variations. By utilizing standard Fourier transform methods used in the analysis of linear PDEs, it was shown that these IBVPs are equivalent to two linear dynamical systems (Chapter 3, Theorem 3.1.1). The time evolution of these systems were compared by choosing the same internal wave initial data (Sections 3.3-3.5 and Chapter 4). In particular, given data supported for wave numbers $k \in [2\pi/\lambda, \infty)$, the L^2-based Sobolev norms of the difference of the solutions were computed in order to find depths d_1 and time scales in which the rigid-lid approximation is reasonable (Theorems 4.1.6, 4.3.3 along with Sections 4.2 and 4.4). Moreover, the rigid-lid approximation does not significantly change the internal mode of vibration for sufficiently small and large wavelengths (Section 3.2) and, in the case of the free-surface system, when the mechanical energy is initially concentrated in the lower layer, it is transported to the upper layer linearly in time (Theorem 4.5.2 and Section 4.6).

In general, the two-layer fluid system conserves mechanical energy (Theorem 2.3.1). The positive eigenvalues of the free-surface system are given by a fast mode $\omega_F = \sqrt{gk}$, corresponding to surface wave motion, and a slow mode $\omega_S = \omega_F \sqrt{\frac{(\rho_2-\rho_1)\tanh d_1 k}{\rho_2+\rho_1 \tanh d_1 k}}$, corresponding to internal wave motion. On the other hand, the rigid-lid system has a positive eigenvalue given by $\omega_R = \omega_F \sqrt{\frac{(\rho_2-\rho_1)\tanh d_1 k}{\rho_1+\rho_2 \tanh d_1 k}}$. These elementary modes of vibration are ordered as $\omega_S \leq \omega_R \leq \omega_F$ and ω_R and ω_S are of the same order of magnitude for wavelengths both large and small compared to the depth d_1 (Theorem 3.1.1 and Lemmas 3.1.2, 3.2.1 and 3.2.2). Thus internal waves propagate slower than surface waves. Finally, sharp L^2 estimates with respect to the a small parameter $\epsilon = \lambda/d_1$ where obtained for of the free surface and interface difference providing depths d_1 and time scales over which the rigid-lid approximation remains good (Sections 4.2 and 4.4).

The analysis of the present dissertation could be extended with difficulty to the nonlinear case (see [2] and [14] for asymptotic models of this situation) or by comparing the free-surface and rigid-lid systems with underwater topography as suggested in [1].

Notation

$\mathbf{x} = (x, y)$

Π_r, Π_f represent rigid-lid and free-surface system, respectively.

$\zeta(\mathbf{x}, t)$ represents the interface.

$\eta(\mathbf{x}, t)$ represents the free-surface.

Ω_t is a time-dependent domain in $\mathbb{R}^3$.

$\mathbf{u} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$ fluid velocity field.

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$

$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

For $i = 1, 2$, $\phi_i = \phi_i(x, y, z, t)$ are velocity potential for the upper and lower fluid domains, respectively.

$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

$\phi_x = \frac{\partial \phi}{\partial x}$

$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$

$|\nabla\phi|^2 = \nabla\phi \cdot \nabla\phi = \phi_x^2 + \phi_y^2 + \phi_z^2$

$\mu = \rho_1/\rho_2$

$\Gamma = d_2/d_1$

$\Psi = (\zeta, \phi_1, \eta, \phi_2)$

$k = \sqrt{\mathbf{k} \cdot \mathbf{k}} = |\mathbf{k}|$ with $k = (k_1, k_2)$

$L^2 = \{\text{measurable functions}, f : \int_{R^2} |f|^2 < \infty\}$

$\omega_F = \sqrt{k}$

$\omega_S = \sqrt{\frac{k(1-\mu)\tanh k}{1+\mu\tanh k}}$

$\omega_R = \sqrt{\frac{k(1-\mu)\tanh k}{\mu+\tanh k}}$

$u(k) = \cosh k(k\tanh k - \omega_F^2)$

$v(k) = \cosh k(k\tanh k - \omega_S^2)$

$r = \omega_S/\omega_F$

$q = \omega_R/\omega_F$

$\hat{f} = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\mathbf{x}\cdot\mathbf{k}} f \; d\mathbf{x}$

$||f||_{H^s}^2 = \int_{\mathbb{R}^2} \left(1+|\mathbf{k}|^2\right)^s |\hat{f}(\mathbf{k})|^2 \; d\mathbf{k}$

$\Upsilon_{s,n}(k) = (1+|k|^2)^s e^{-2k} k^n$

$\Lambda_{s,n}(k) = e^{-2k}\Upsilon_{s,n}(k)$

$\Theta(k,s,n) = (1+d_1^2 k^2)^s d_1 k e^{-nd_1 k}$

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