On Picard Varieties of Surfaces with Equivalent Derived Categories

BY

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THESIS

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To my parents,

Vinh Pham and Le Dang,

my continual source of support and encouragement.
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LIST OF ABBREVIATIONS

Alb($X$) The Albanese variety of $X$

Aut($X$) The group of automorphisms of $X$.

Aut$^0(X)$ The connected component containing the identity of the group of automorphisms of $X$

Bir($X$) The group of birational automorphisms of $X$.

Coh($X$) The category of coherent sheaves on $X$.

$D^b(X)$ The bounded derived category of coherent sheaves on $X$.

$G_X$ The connected component containing the identity of the group of automorphisms of $X$.

Jac($C$) The Jacobian variety of a curve $C$.

Pic($X$) The Picard group of line bundles on $X$.

Pic$^0(X)$ The Picard variety of $X$ (i.e. the group of line bundles on $X$ that are algebraically equivalent to 0).

Qcoh($X$) The category of quasi-coherent sheaves on $X$.

$J^b(X)$ The Picard scheme of divisors of degree $b$ on $X$. 
SUMMARY

The study of derived categories as invariants of algebraic varieties has increasingly become an important and interesting research area. Perhaps the most famous problem related to this area of research at the moment is the Kontsevich’s Homological Mirror Symmetry Conjecture, which is of great importance to birational geometry and string theory. By the work of Bondal, Bridgeland, Maciocia, Mukai, and Orlov, many (birational) invariants of algebraic varieties are preserved by derived equivalences. Only until recently, Popa and Schnell were able to show that the irregularity of a variety, a fundamental birational invariant, is preserved under derived equivalences. They also conjectured that the Picard varieties of derived equivalent varieties are derived equivalent. This conjecture clearly holds in the case of curves. The purpose of this thesis is to investigate this conjecture in the case of surfaces. More specifically, in the case of surfaces, we provide a positive answer to the conjecture with the possible exception of the case of elliptic surfaces with constant $j$-invariant. In that case, we were able to describe the structure of the Picard variety in more details.
CHAPTER 1

INTRODUCTION

Background

Bounded derived categories of coherent sheaves on smooth projective varieties have increasingly become important and interesting objects of recent research. It has been widely accepted that it is not enough just to study coherent sheaves and their (co)homologies; instead, the right framework should be the study of complexes of coherent sheaves, and hence derived categories. The formalism of derived categories was developed by Alexander Grothendieck and his student Jean-Louis Verdier shortly after 1960. However, the study of derived categories as an invariant of algebraic varieties has not been investigated thoroughly until recently, by the work of Bondal, Mukai, and Orlov. And perhaps the most famous problem related to derived categories at the moment is the Kontsevich’s Homological Mirror Symmetry Conjecture, which is of great importance to birational geometry and string theory.

The study of the derived category as an invariant started with the work of Mukai. He constructed derived equivalences between non-isomorphic varieties. In his study, he introduced a type of transform between derived categories defined as follows. For a projective variety $X$, let us denote by $D^b(X)$ the bounded derived category of coherent sheaves on $X$. Given a pair
of smooth varieties $X$ and $Y$, and $\mathcal{E}$ an object in $D^b(X \times Y)$, one defines a *Fourier-Mukai transform*

$$D^b(X) \to D^b(Y)$$

by the formula

$$\Phi_{\mathcal{E}}(\_):= p_*(q^*(\_) \otimes \mathcal{E})$$

with $q$ and $p$ the projections from $X \times Y$ to $X$ and $Y$, respectively. Here, $p_*$ denotes the derived pushforward. Using this transform, Mukai was able to show that the Poincaré bundle of an abelian variety $A$ gives rise to an equivalence between the derived category of $A$ and that of its dual, $\hat{A}$. It turns out that the Fourier-Mukai transform is the right representation of derived equivalence due to the following remarkable theorem by Orlov. Let $X$ and $Y$ be smooth projective varieties over an algebraically closed field $k$, and let

$$F: D^b(X) \to D^b(Y)$$

be an exact equivalence of derived categories. Then there exists an object $\mathcal{E} \in D^b(X \times Y)$, unique up to isomorphism, such that

$$F \cong \Phi_{\mathcal{E}}$$

(see (31, Theorem 3.2.1)). Bondal and Orlov later gave a very convenient and simple criterion for when a Fourier-Mukai transform is an equivalence (Proposition 2.2.15 and Proposition 2.2.16).
Fourier-Mukai transforms provide important tools to study derived categories, birational invariants and moduli spaces of stable sheaves. For instance, Bridgeland and Maciocia were able to use Fourier-Mukai transforms to show that a pair of K3 (respectively, abelian) surfaces are derived equivalent if and only if one can be realized as a fine, two-dimensional moduli space of stable sheaves on the other. They also showed that a pair of elliptic surfaces of non-zero Kodaira dimension are derived equivalent if one can be realized as a fine, two-dimensional moduli space of simple stable sheaves of pure dimension one on the other. In addition, using Mukai’s study of semi-homogenous vector bundles on abelian varieties, Gulbrandsen showed that a pair of abelian varieties are derived equivalent if one is a fine moduli space of semi-homogeneous vector bundles of a fixed slope on the other.

It is in general an important problem to determine which invariants are preserved under derived equivalences. By the work of Mukai, Bondal, Orlov and Kawamata, it was known that many important birational invariants (e.g. the dimension, the (numerical) Kodaira dimension, the canonical ring, the geometric genus, certain Hodge numbers, etc) are preserved under derived equivalences. Only until recently, Popa and Schnell were able to show that the irregularity of a variety, a fundamental birational invariant, is preserved under derived equivalences. Moreover, they conjectured the following stronger result.

**Conjecture.** Let $X$ and $Y$ be smooth projective varieties with equivalent derived categories. Then

$$D^b(\text{Pic}^0(X)) \cong D^b(\text{Pic}^0(Y)).$$
Here $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ denote the Picard variety of $X$ and $Y$, respectively (i.e. the groups of line bundles algebraically equivalent to 0 on $X$ and $Y$, respectively). Derived equivalence is the strongest relation between $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ one can expect because when $X$ and $Y$ are abelian varieties it follows that $\mathcal{D}^b(\text{Pic}^0(X)) \cong \mathcal{D}^b(\text{Pic}^0(Y))$. Moreover, this conjecture would give a stronger result than Popa and Schnell’s result on the irregularity because it would imply that $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ are isogenous and, therefore, have the same dimension (i.e. the irregularity of $X$ and $Y$ are equal). We should also remark that this conjecture holds for the case of curves because two curves are derived equivalent if and only if they are isomorphic.

Popa and Schnell’s proof about the irregularity of a variety and their conjecture were inspired by a theorem of Rouquier (see Theorem 3.4.1). Let $\text{Aut}^0(X)$ denote the connected component of the identity of the group of automorphisms of a smooth projective variety $X$. The Rouquier Theorem relates the groups $\text{Aut}^0$ and $\text{Pic}^0$ of two derived equivalent varieties. In this thesis, we investigate the conjecture in the case of surfaces by analyzing Rouquier’s Theorem, the group $\text{Aut}^0$ and its action on a smooth projective surface. In addition, we also utilize theorems of Kawamata (20), Bridgeland and Maciocia (6), and (9).

**Main Results and Ideas**

The main results from analysis of the aforementioned conjecture for the case of surfaces are summarized in the following theorem (see Theorem 5.3.1).

**Theorem 1.0.1.** Let $X$ and $Y$ be smooth projective surfaces such that $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$. Then

a. $\text{Aut}^0(X) \cong \text{Aut}^0(Y)$, unless $X$ and $Y$ are abelian surfaces, in which case $\mathcal{D}^b(\text{Aut}^0(X)) \cong \mathcal{D}^b(\text{Aut}^0(Y))$, and
b. $D^b(\text{Pic}^0(X)) \cong D^b(\text{Pic}^0(Y))$ with the possible exception of the case when $X$ is a properly elliptic surface with constant $j$-invariant. In that case, there exists an isogeny between $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ with kernel isomorphic to $(\mathbb{Z}/c\mathbb{Z})^2$ for some integer $c$.

We should remark that statement (a) is in fact a byproduct of the proof of statement (b). We will explain here the main ideas together with the main difficulties for the proof of statement (b). It is clear from the statement of the theorem that the most difficult case of the theorem is the case of elliptic surfaces with constant $j$-invariant. In fact, the main difficulty in studying derived categories of surfaces usually comes from the case of elliptic surfaces with constant $j$-invariant. This can be seen from the work of Kawamata (see Theorem 2.3.1), Bridgeland and Maciocia (see Theorem 2.3.3). We will present here an alternative perspective.

Let us start with the Rouquier Theorem. Let $X$ and $Y$ be two smooth projective varieties with equivalent derived categories. Rouquier’s Theorem gives the following isomorphism of algebraic groups, which we call the Rouquier isomorphism,

$$\text{Aut}^0(X) \times \text{Pic}^0(X) \cong \text{Aut}^0(Y) \times \text{Pic}^0(Y).$$

Recall that $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ are abelian varieties. Notice that both sides of (1.1) are connected algebraic groups. Recall that, by the classical Chevalley Structure Theorem (Theorem 4.1.1), every connected algebraic group $G$ can be decomposed into a (unique) maximal affine
(linear) part, denoted by Aff\( (G) \), and a compact (abelian) part, denoted by Alb\( (G) \). In other words, there exists an exact sequence of algebraic groups

\[
0 \to \text{Aff}(G) \to G \to \text{Alb}(G) \to 0.
\]

Thus, we obtain the following isomorphisms

\[
\text{Alb}(\text{Aut}^{0}(X)) \times \text{Pic}^{0}(X) \cong \text{Alb}(\text{Aut}^{0}(Y)) \times \text{Pic}^{0}(Y),
\]

(1.1)

and

\[
\text{Aff}(\text{Aut}^{0}(X)) \cong \text{Aff}(\text{Aut}^{0}(Y)).
\]

(1.2)

It is rather straightforward to prove that the conjecture holds in the following cases,

1. \( \text{Aut}^{0}(X) \) is affine, or
2. the compact part, \( \text{Alb}(\text{Aut}^{0}(X)) \), of \( \text{Aut}^{0}(X) \) has dimension equal to \( \dim X \).

The first case is trivial due to (1.1). All smooth projective varieties of general type belong to the first case, while those in the second case are precisely abelian varieties (see Proposition 4.2.5). The main difficulties for the proof of Theorem 1.0.1(b) come from the case when \( \dim \text{Alb}(\text{Aut}^{0}(X)) = 1 \). In the case of surfaces, this case is equivalent to the case of elliptic surfaces with constant \( j \)-invariant (see Section 4.3).
We start by looking at the action of $\text{Aut}^0(X)$ on $X$ using results of Brion (11), Matsumura and Nishi (27). Let $G$ be a connected algebraic group acting faithfully on a projective variety $X$. Matsumura and Nishi gave the following estimate

$$\dim(\text{Alb}(G)) \leq \dim \text{Alb}(X) = q(X).$$

This estimate comes from the fact that the action of $G$ on $X$ induces a morphism

$$\varphi : \text{Alb}(G) \to \text{Alb}(X)$$

with finite kernel. Equivalently, the induced morphism

$$G \to \text{Alb}(G) \xrightarrow{\varphi} \text{Alb}(X)$$

is affine. In the case of the connected algebraic group $G_X = \text{Aut}^0(X)$, it was shown by Matsumura and Nishi in (27) that the image of $\varphi$ is contained in the image of the Albanese morphism $\alpha_X : X \to \text{Alb}(X)$. Hence, we obtain a more refined estimate

$$\dim \text{Alb}(G_X) \leq \dim \text{im}(\alpha_X) \leq \min\{\dim X, q(X)\}. \quad (1.3)$$

(see Lemma 4.2.1). When $X$ is a smooth projective surface, there are only 3 choices (0, 1 and 2) for the dimension of $\text{Alb}(G_X)$ . We have discussed the case $\dim \text{Alb}(G_X) = 0$ and $\dim \text{Alb}(G_X) = 2$ earlier. We are left with the main case $\dim \text{Alb}(G_X) = 1$. 
Matsumura and Nishi in (27) showed that the group of birational morphisms of a variety $X$, Bir$(X)$, contains a linear algebraic group of positive dimension if and only if $X$ is birational to $\mathbb{P}^1 \times Y$ with $Y$ another variety. Since $G_X$ acts faithfully on $X$, we have the inclusion $G_X \subseteq \text{Bir}(X)$. In the case of surfaces, this implies that the only possible minimal surfaces with Aff$(G_X)$ having positive dimension are the ruled surfaces. On the other hand, $G_X$ is an abelian group for $X$ a minimal surface of non-negative Kodaira dimension. We begin with the case of $\mathbb{P}^1$-bundles on elliptic curves, which is also the first interesting non-trivial case in the proof of Theorem 1.0.1(b).

**Theorem 1.0.2.** Let $X_1$ and $X_2$ be $\mathbb{P}^1$-bundles over elliptic curves $E_1$ and $E_2$, respectively. If $D^b(X_1) \cong D^b(X_2)$, then $E_1 \cong E_2$. See Theorem 5.1.1.

Since $X_i$’s are ruled surfaces, we have Pic$^0(X_i) \cong E_i$ with $i = 1, 2$; hence, Theorem 1.0.1(b) follows in this case. The main difficulty of proving this theorem is as follows. By applying Rouquier’s isomorphism, we obtain that

$$G_{X_1} \times \text{Pic}^0(X_1) \cong G_{X_2} \times \text{Pic}^0(X_2),$$

i.e.

$$G_{X_1} \times E_1 \cong G_{X_2} \times E_2.$$

By analyzing the group Aut$^0(X)$ for ruled surfaces $X$ (see Section 4.3.3), we can show that

$$\text{Alb}(G_{X_i}) \cong E_i.$$
for $i = 1, 2$. Hence, (1.1) gives

$$E_1 \times E_1 \cong E_2 \times E_2.$$  

Unfortunately, this is not enough to conclude that $E_1 \cong E_2$. There are examples of elliptic curves $E_1$ and $E_2$ such that $E_1 \times E_1 \cong E_2 \times E_2$ and $E_1 \not\cong E_2$ (see (32) and (38)).

The additional idea to circumvent the issue above is the following proposition (see Proposition 4.2.2).

**Lemma 1.0.3.** If $\dim \text{Alb}(G_X) > 0$, then there exists an affine subgroup (not necessarily connected) $\text{Aff}(G_X) \subset H \subset G_X$ with $H/\text{Aff}(G_X)$ finite such that

$$X \cong (G_X \times Z)/H$$

with $Z$ a connected smooth projective variety.

This result is due to Brion (11), also see (33, Lemma 2.4) for more details. Applying this lemma, we can realize the ruled surfaces $X_1$ and $X_2$ as elliptic fibrations over $\mathbb{P}^1$. Moreover, the general fibers of the fibrations $X_i \to \mathbb{P}^1$ are isomorphic to $E_i$. We then apply a theorem of Bridgeland and Maciocia about derived equivalent elliptic surfaces of non-zero Kodaira dimension to conclude that the general fibers of these two fibrations are isomorphic, i.e. $E_1 \cong E_2$. Lemma 1.0.3 also plays an important role in the investigation of the case of properly elliptic surfaces.

We start with a theorem of Bridgeland and Maciocia (see Theorem 2.3.9 and (9, Proposition 4.4)). Two properly elliptic surfaces are derived equivalent if and only if one can be realized as a
fine two-dimensional moduli space of simple stable sheaves of pure dimension one on the other. In particular, there exists an integer $b$, satisfying certain coprimality condition (see Section 2.3), such that $Y \cong J^b(X)$, the relative Picard scheme of divisors of degree $b$ of $X$ (see Section 2.3). In other words, the general point of $Y$ represents a line bundle of degree $b$ supported on a smooth fiber of the elliptic fibration of $X$. The situation is in fact symmetric. Hence, we can write $X = J^c(Y)$ for some integer $c$ satisfying the same coprimality condition for $b$. We then obtain the following result for properly elliptic surfaces.

**Theorem 1.0.4.** Let $X$ and $Y$ be properly elliptic surfaces with $D^b(X) \cong D^b(Y)$. Then either

a. $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$ or

b. there exists an isogeny $f : \text{Pic}^0(X) \to \text{Pic}^0(Y)$ with kernel $(\mathbb{Z}/c\mathbb{Z})^2$ with $c$ being the integer described above.

Moreover, the first case occurs when $\chi(\mathcal{O}_X) \neq 0$. The second case occurs when $\chi(\mathcal{O}_X) = 0$, equivalently $X$ is a properly elliptic surface with constant $j$-invariant. See Proposition 5.2.6 and Theorem 5.2.7.

The first case is Proposition 5.2.6 and is rather straightforward. Bridgeland and Maciocia’s result shows that $X$ and $Y$ are elliptic fibrations over the same curve $C$ with general fibers being isomorphic. The fact that $\chi(\mathcal{O}_X) \neq 0$ implies that $\text{Alb}(X) \cong \text{Jac}(C)$ (see Proposition 5.2.2). Similarly, $\text{Alb}(Y) \cong \text{Jac}(C)$. Since the Albanese variety is the dual of the Picard variety, we have $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$. 
In the second case of Theorem 1.0.4, we can understand \( \text{Pic}^0(X) \) and \( \text{Pic}^0(Y) \) in more detail as described in Theorem 5.2.7. The idea of the proof of this statement is as follows. First, we apply Lemma 1.0.3 to \( X \) and \( Y \) and obtain that

\[
X \cong (G_X \times Z_X)/H_X \quad \text{and} \quad Y = (G_Y \times Z_Y)/H_Y.
\] (1.4)

We then show that \( G_X \) and \( G_Y \) are isomorphic elliptic curves, \( Z_X \) and \( Z_Y \) are smooth curves, and \( Z_X/H_X \cong Z_Y/H_Y \cong C \), a curve of genus at least 2. The projections \( \pi_X : X \to C \) and \( \pi_Y : Y \to C \) are elliptic fibrations with general fibers isomorphic to \( G_X \) and \( G_Y \), respectively.

Secondly, let \( \Phi_E : \mathcal{D}^b(X) \to \mathcal{D}^b(Y) \) be the Fourier-Mukai that represents the derived equivalence. For a general point \( p \in C \), let \( X_p \) and \( Y_p \) be the fibers of \( \pi_X \) and \( \pi_Y \), respectively. We show that the restriction of the kernel \( E \) on \( X_p \times Y_p \) gives a vector bundle \( E_p \). This vector bundle then induces an equivalence

\[
\Phi_{E_p} : \mathcal{D}^b(X_p) \to \mathcal{D}^b(Y_p)
\]

(see Lemma 5.2.8). Now let \( F \) and \( H \) be the Rouquier isomorphisms associated to the derived equivalences \( \Phi_E \) and \( \Phi_{E_p} \), respectively. We obtain a commutative diagram

\[
\begin{array}{ccc}
\text{Aut}^0(X_p) \times \text{Pic}^0(X_p) & \overset{H}{\longrightarrow} & \text{Aut}^0(Y_p) \times \text{Pic}^0(Y_p) \\
\uparrow & & \uparrow \\
\text{Aut}^0(X) \times \text{Pic}^0(X) & \overset{F}{\longrightarrow} & \text{Aut}^0(Y) \times \text{Pic}^0(Y)
\end{array}
\]
(see Lemma 5.2.9). The isogeny $f$ in Theorem 1.0.5(b) is defined to be the composition

$$f : \text{Pic}^0(X) \to \text{Aut}^0(X) \times \text{Pic}^0(X) \xrightarrow{F} \text{Aut}^0(Y) \times \text{Pic}^0(Y) \xrightarrow{\text{pr}_{\text{Pic}^0(Y)}} \text{Pic}^0(Y).$$

$L \mapsto (\text{id}, L)$

Utilizing the above commutative diagram, we similarly define a new morphism $h$ as the composition

$$h : \text{Pic}^0(X_p) \to \text{Aut}^0(X_p) \times \text{Pic}^0(X_p) \xrightarrow{H} \text{Aut}^0(Y_p) \times \text{Pic}^0(Y_p) \to \text{Pic}^0(Y_p).$$

It is not hard to show that in fact ker($f$) and ker($h$) are isomorphic. It remains then to compute the kernel of $h$. The advantage we have here is that $\Phi_{\mathcal{E}_p}$ is an derived equivalence between two elliptic curves. In the case of abelian varieties, the Rouquier isomorphism induced by a derived equivalence can often be described very explicitly. And this is our next step.

As discussed earlier, we can write $Y \cong J^b(X)$ and $X \cong J^c(Y)$. Moreover, the line bundle $\mathcal{E}_p$ on $X_p \times Y_p$ is a universal bundle that parametrizes line bundles of degree $c$ on $Y_p$ and line bundles of degree $b$ on $X_p$, i.e. $X_p \cong \text{Pic}^c(Y_p)$ and $Y_p \cong \text{Pic}^b(X_p)$. Notice that this also allows us (after a choice of an origin) to realize

$$Y_p \cong \text{Pic}^0(X_p) = \hat{X}_p.$$
The universal property of $\mathcal{E}_p$ implies that

$$\mathcal{E}_p \cong \mathcal{P} \otimes q^* L \otimes p^* M$$

with $\mathcal{P}$ being the Poincaré bundle on $X_p \times \hat{X}_p$, $L \in \text{Pic}^b(X_p)$ and $M \in \text{Pic}^c(Y_p)$. This description of $\mathcal{E}_p$ allows us to explicitly describe the Rouquier isomorphism $H$ in term of the line bundles $L$ and $M$ (see Example 3.3.8). The description of $H$ then allows us to show that

$$\ker(f) = \ker(h) \cong (\mathbb{Z}/c\mathbb{Z})^2.$$
is the case of Theorem 1.0.4. Thus, we obtain the full picture for the aforementioned conjecture in the case of surfaces.

Another byproduct of the proof of Theorem 1.0.1(b) is the fact that derived equivalent surfaces have the same Albanese dimensions (see Theorem 5.3.1(c)). This result is also expected to hold in higher dimensions. In particular, in his forthcoming paper (24), Lombardi shows that this result holds for varieties of non-negative Kodaira dimension by studying the Iitaka’s fibrations together with the Albanese morphism.

Organization

We begin with a chapter on Fourier-Mukai transforms. Here we review the definitions and basic properties of bounded derived categories of coherent sheaves and derived functors. We introduce the Fourier-Mukai transforms and their properties. The Bondal-Orlov criterion for derived equivalence is introduced. In addition, we enumerate the birational invariants that are known to be preserved under derived equivalence. We end the chapter with a discussion of derived equivalent surfaces. In particular, we sketch the proof of Kawamata’s theorem (Theorem 2.3.1) for non-minimal surfaces and the proof of the Bridgeland-Maciocia theorem (Theorem 2.3.7) for elliptic surfaces of non-zero Kodaira dimension. The main references for this chapter are (5), (6), (9) and (18).

Chapter 3 focuses on Fourier-Mukai for abelian varieties. We start with Mukai’s theorem which states that the Fourier-Mukai transform induced by the Poincaré bundle $\mathcal{P}$ on an abelian variety $A$ gives a derived equivalence $\Phi_{\mathcal{P}} : D^b(A) \to D^b(\tilde{A})$. We also review the Orlov-
Polishchuk criterion for two abelian varieties \( A \) and \( B \) to be derived equivalent. Associated to each equivalence \( \Phi_E : D^b(A) \to D^b(B) \) there is an isometric isomorphism

\[
f_E : A \times \hat{A} \to B \times \hat{B}.
\]

We review the construction of this isomorphism. In addition, we provide detailed descriptions of \( f_E \) for various kernels \( E \). This chapter ends with a discussion of the Rouquier isomorphism, which is a generalization of \( f_E \) to arbitrary varieties. The main references are (18), (28), (29), and (30).

Chapter 4 provides the details about the action of the group \( \Aut^0(X) \) on a variety \( X \). We start with the classical Chevalley theorem about the structure of connected algebraic groups. We then give some descriptions of the affine part and the compact part of the group \( \Aut^0(X) \) using various results of Matsumura-Nishi. We also introduce the analysis of the action of \( \Alb(\Aut^0(X)) \) on \( X \) by Brion, Popa and Schnell. The main points are Lemma 4.2.1, Proposition 4.2.2 and Proposition 4.2.3. We end the chapter with more detailed descriptions of the group \( \Aut^0(G_X) \) for minimal surfaces. The calculation of the group \( \Aut^0(X) \) for \( X \) a ruled surface is due to Maruyama (26). The main references for this chapter are (11), (27), (26), (33), and (34).

The proofs of all the aforementioned theorems are included in Chapter 5. It starts with the discussion of the conjecture in the case of ruled surfaces followed by the case of properly elliptic surfaces. Theorem 1.0.2, Theorem 1.0.4, Theorem 1.0.5 and Proposition 1.0.6 are combined
into Theorem 5.3.1 to give a complete picture of Picard varieties, the groups $\text{Aut}^0$ and the Albanese dimensions for derived equivalent surfaces.
CHAPTER 2

FOURIER-MUKAI TRANSFORMS

2.1 Derived categories of coherent sheaves

Here we recall the basic definitions and properties of derived categories and derived functors. For complete details, we refer the readers to (18). We assume that all of our schemes are of finite type over \( k \). Given a scheme \( X \), the abelian category of coherent sheaves on \( X \) is denoted by \( \text{Coh}(X) \). The derived category \( D^b(X) \) of \( X \) is by definition the derived category of the abelian category \( \text{Coh}(X) \). Recall the the derived category \( D^b(X) \) is a triangulated \( k \)-linear category whose objects are complexes of coherent sheaves, morphisms are morphisms of complexes, and isomorphisms are quasi-isomorphisms of complexes.

Definition 2.1.1. Let \( D^*(X) \) with \( * = +, - \) or \( b \) be the derived category of complexes \( \mathcal{E}^\bullet \) with \( E^i = 0 \) for \( i \ll 0, i \gg 0 \), respectively \( |i| \gg 0 \).

Notice that \( D^*(X) \) with \( * = -, - \) or \( b \) are full triangulated subcategories of \( D(X) \). There is a shift functor \( T_{D(X)} \) and we will write \([n]\) for \( T^n_{D(X)} \), the \( n \)th iteration of the shift functor \( T \). More precisely, for every object \( \mathcal{E}^\bullet \) in \( D(X) \) and for all integer \( i \), we have

\[
H^i(\mathcal{E}^\bullet[n]) = H^{i+n}(\mathcal{E}^\bullet).
\]

Definition 2.1.2. Two schemes \( X \) and \( Y \) defined over the field \( k \) are called derived equivalent (or simply D-equivalent) if there exists a \( k \)-linear exact equivalence \( D^b(X) \overset{\cong}{\longrightarrow} D^b(Y) \).
Next, we will discuss all derived functors that will be needed in the remaining sections. We begin with the following remark.

Remark 2.1.3.

a. The abelian category Coh($X$) usually contains no non-trivial injective objects. In order to compute derived functors we usually pass to the category of quasi-coherent sheaves $\text{Qcoh}(X)$ or the category of $\mathcal{O}_X$-modules $\text{Sh}_{\mathcal{O}_X}(X)$.

b. Every quasi-coherent sheaf on a noetherian scheme $X$ admits an injective resolution of $\mathcal{O}_X$-modules. Hence, we have the natural equivalence

$$D^*(\text{Qcoh}(X)) \cong D^*(\text{Sh}_{\mathcal{O}_X}(X))$$

with $* = +, -, b$.

c. For $X$ a noetherian scheme, the natural functor

$$D^h(X) \rightarrow D^h(\text{Qcoh}(X))$$

identifies $D^h(X)$ with the full subcategory $D^h_{\text{Coh}}(\text{Qcoh}(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.
Cohomology Functor

Let $X$ be a noetherian scheme over a field $k$. The global section functor $\Gamma$

$$\Gamma : \text{Qcoh}(X) \to \text{Vect}(k)$$

$$\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$$

is a left exact functor. Since $\text{Qcoh}(X)$ has enough injectives, there is an induced right derived functor

$$R\Gamma : \text{D}^b(X) \to \text{D}^b(\text{Vect}(k))$$

We denote the higher derived functor

$$H^i(X, \mathcal{F}^\bullet) := R^i\Gamma(\mathcal{F}^\bullet).$$

For a sheaf $\mathcal{F}$, these cohomology groups are just $H^i(X, \mathcal{F})$. And for a complex $\mathcal{F}^\bullet$, they are sometimes referred to as the hypercohomology groups $\mathbb{H}(X, \mathcal{F}^\bullet)$. In addition, since every complex of vector spaces splits, one has

$$R\Gamma(\mathcal{F}^\bullet) \cong \bigoplus H^i(X, \mathcal{F}^\bullet)[-i]$$

in $\text{D}^b(\text{Vect}(k))$. 
**Direct Image**

Let $f : X \to Y$ be a proper morphism of noetherian schemes. The direct image $f_*$ is a left exact functor

$$f_* : \text{Qcoh}(X) \to \text{Qcoh}(Y).$$

Again, this functor induces a right derived functor

$$Rf_* : \text{D}^b(X) \to \text{D}^b(Y),$$

and the higher derived functors are defined as

$$R^p f_*(\mathcal{F}^\bullet) := H^i(Rf_*(\mathcal{F}^\bullet)).$$

**Remark 2.1.4.**

(i) We need the properness condition in order for $Rf_*$ to map $\text{D}^b(X)$ to $\text{D}^b(Y)$. If we drop the properness condition, then we have instead a right derived functor

$$Rf_* : \text{D}^+(X) \to \text{D}^+(Y).$$

(ii) If we let $f$ be the structure morphism $X \to \text{Spec}(k)$, then the derived functor $Rf_*$ is precisely the cohomology functor $R\Gamma(X, )$. 
(iii) If $g : Y \to Z$ is another proper morphism of noetherian schemes, then there is an isomorphism of derived functors

$$R(g \circ f)_* \cong Rg_* \circ Rf_*.$$ 

We also have the Leray spectral sequence

$$E_2^{p,q} = R^p g_*(R^q f_*(\mathcal{F}^\bullet)) \Rightarrow R^{p+q}(g \circ f)_*(\mathcal{F}^\bullet). \quad (2.1)$$

In particular, we can take $g$ to be the structure morphism $Y \to \text{Spec}(k)$. Then

$$Rg_* \circ Rf_* \cong R\Gamma(X, \mathcal{F}^\bullet).$$

The spectral sequence (2.1) then becomes

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}^\bullet)) \Rightarrow H^{p+q}(X, \mathcal{F}^\bullet). \quad (2.2)$$

Global Hom

If $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ are objects in $\mathsf{D}(X)$, we define the abelian groups

$$\text{Hom}^i_X(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \text{Hom}_{\mathsf{D}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]).$$
And for $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$, these are precisely

$$\text{Hom}^i(\mathcal{E}, \mathcal{F}) \cong \text{Ext}^i(\mathcal{E}, \mathcal{F}).$$

**Local Hom**

For an element $\mathcal{F} \in \text{Qcoh}(X)$, we have a left exact functor

$$\mathcal{H}\text{om}(\mathcal{F}, \cdot) : \text{Qcoh}(X) \to \text{Qcoh}(X).$$

Recall that $\mathcal{H}\text{om}(\mathcal{F}, \mathcal{E})$ is the sheaf associated to the presheaf

$$U \mapsto \text{Hom}(\mathcal{F}(U), \mathcal{E}(U)).$$

Since $\text{Qcoh}(X)$ contains enough injectives, the derived functor

$$R\mathcal{H}\text{om}(\mathcal{F}, \cdot) : \mathcal{D}^+(X) \to \mathcal{D}^+(X)$$

exists. By definition, for any quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{E}$ on $X$, we have

$$\text{Ext}^i(\mathcal{F}, \mathcal{E}) := R^i\mathcal{H}\text{om}(\mathcal{F}, \mathcal{E}).$$
In the case of coherent sheave \( \mathcal{F} \), we have (cf. (17))

\[
\mathcal{E}xt^i(\mathcal{F}, \mathcal{E})_x \cong \text{Ext}^i(\mathcal{F}_x, \mathcal{E}_x).
\]

And for a regular scheme \( X \), we obtain a functor between the bounded derived categories

\[ R\text{Hom} : D^b(X) \to D^b(X). \]

We now generalize the above construction to the complexes \( \mathcal{F}^\bullet \in D^-(X) \). For every complex \( \mathcal{E}^\bullet \in D(X) \) and \( \mathcal{F}^\bullet \in D^-(X) \), we define \( R\text{Hom}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \) to be the complex

\[
\bigoplus_i \text{Hom}(\mathcal{E}^i, \mathcal{F}^{i+n}), \quad n \in \mathbb{Z},
\]

with differential \( d^n = \bigoplus_i (d_i^{\mathcal{E}^\bullet} + (-1)^{n+1} d_{\mathcal{F}^\bullet}^{i+n}) \). Furthermore, we can also define

\[
\mathcal{E}xt^i(\mathcal{F}^\bullet, \mathcal{E}^\bullet) := R^i\text{Hom}(\mathcal{F}^\bullet, \mathcal{E}^\bullet).
\]

In addition, for every object \( \mathcal{F}^\bullet \in D^-(X) \), we can define the \textit{derived dual} as follows

\[
\mathcal{F}^{\star \vee} := R\text{Hom}(\mathcal{F}^\bullet, \mathcal{O}_X) \in D^+(\text{Qcoh}(X)).
\]

And if \( X \) is regular, then \( \mathcal{F}^{\star \vee} \in D^b(X) \).
Tensor Product

Given a projective scheme $X$ defined over $k$. For every coherent sheaf $\mathcal{F} \in \text{Coh}(X)$, we have the right exact functor

$$\mathcal{F} \otimes_{-} : \text{Coh}(X) \to \text{Coh}(X).$$

On a projective scheme, every coherent sheaf $\mathcal{E}$ on $X$ admits a resolution by locally free sheaves. Notice also that if $\mathcal{E}^\bullet$ is an acyclic complex bounded above with all $\mathcal{E}^i$ locally free sheaf, then $\mathcal{F} \otimes \mathcal{E}^\bullet$ is still an acyclic complex for all $\mathcal{F} \in \text{Coh}(X)$. Thus, we then obtain a left derived functor

$$\mathcal{F} \otimes^L_{-} : \text{D}^{-}(X) \to \text{D}^{-}(X).$$

For every pair of objects $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in \text{D}^{-}(X)$, we define $\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet$ to be the total complex of

$$(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet)^i := \oplus_{p+q=i} \mathcal{F}^p \otimes \mathcal{E}^q$$

with the differential

$$d := d_{\mathcal{F}^\bullet} \otimes 1 + (-1)^i 1 \otimes d_{\mathcal{E}^\bullet}.$$

When $X$ is regular, every object $\mathcal{F}^\bullet \in \text{D}^b(X)$ defined a derived functor

$$\mathcal{F}^\bullet \otimes^L_{-} : \text{D}^b(X) \to \text{D}^b(X),$$
with
\[ F^* \otimes^L E^* \cong E^* \otimes^L F^* , \]

and
\[ F^* \otimes^L (E^* \otimes^L G^*) \cong (F^* \otimes^L E^*) \otimes^L G^* . \]

In addition, we define
\[ Tor_i(F, E) := H^{-i}(F \otimes E) , \]

and
\[ Tor_i(F^*, E^*) := H^{-i}(F^* \otimes^L E^*) . \]

**Pull-back**

Let \( f : X \to Y \) be a morphism of noetherian schemes. The pull-back \( f^* \) is a right exact functor
\[ f^* : \text{Sh}_\mathcal{O}_Y(Y) \to \text{Sh}_\mathcal{O}_X(X) . \]

By definition, \( f^* \) is the composition \( \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\cdot) \). We then obtain the left derived functor of \( f^* \)
\[ Lf^* := \left( \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \cdot \right) \circ f^{-1} : D^{-}(Y) \to D^{-}(X) . \]

In addition, the higher derived functors are defined as
\[ L^p f^*(F^*) := H^p(Lf_*(F^*)) . \]
Compatibilities

Let \( f : X \to Y \) be a proper morphism of projective schemes. Given and \( \mathcal{O}_X \)-modules \( \mathcal{F} \) and an \( \mathcal{O}_Y \)-module \( \mathcal{E} \), we have the projection formula

\[
f_*(\mathcal{F} \otimes f^* \mathcal{E}) \cong f_* \mathcal{F} \otimes \mathcal{E}.
\]

This formula also holds in the derived categories setting, i.e.

\[
Rf_*(\mathcal{F}^\bullet) \otimes^L \mathcal{E}^\bullet \cong Rf_*(\mathcal{F}^\bullet \otimes^L Lf^* \mathcal{E}^\bullet),
\]

for all objects \( \mathcal{F}^\bullet \in \text{D}^b(X) \) and \( \mathcal{E}^\bullet \in \text{D}^b(Y) \).

Moreover, for locally free sheaves \( \mathcal{F} \) and \( \mathcal{E} \) in \( \text{D}^b(Y) \), we have the isomorphism

\[
f^*(\mathcal{F} \otimes \mathcal{E}) \cong f^* \mathcal{F} \otimes f^* \mathcal{E}.
\]

Hence, given objects \( \mathcal{F}^\bullet, \mathcal{E}^\bullet \in \text{D}^b(Y) \), by replacing them with complexes of locally free sheaves, we obtain the formula

\[
Lf^*(\mathcal{F}^\bullet) \otimes^L Lf^*(\mathcal{E}^\bullet) \cong Lf^*(\mathcal{F}^\bullet \otimes^L \mathcal{E}^\bullet).
\]

The adjoint formula

\[
\text{Hom}_{\mathcal{O}_X}(f^* \mathcal{F}, \mathcal{E}) \cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{F}, f_* \mathcal{E})
\]
for $\mathcal{E}$ an $\mathcal{O}_X$-module and $\mathcal{F}$ and $\mathcal{O}_Y$-module also translates into the derived categories setting

$$\text{Hom}_{D^b(X)}(Lf^*\mathcal{F}^*,\mathcal{E}^*) \cong \text{Hom}_{D^b(Y)}(\mathcal{F}^*,Rf_*\mathcal{E}^*).$$

We can realize this isomorphism by replacing $\mathcal{F}^*$ by a complex of locally free sheaves and $\mathcal{E}^*$ by a complex of quasi-coherent injective sheaves, and then apply the regular adjoint formula.

**Duality**

Let $X$ be a projective scheme defined over a field $k$. Recall that for a locally free sheaf $\mathcal{E}$ on $X$, the functor

$$\mathcal{E} \otimes - : \text{Coh}(X) \to \text{Coh}(X)$$

is exact. Therefore, it induces a natural exact functor

$$\mathcal{E} \otimes - : D^\ast(X) \to D^\ast(X)$$

for $\ast = -, +, b$.

**Definition 2.1.5.** Let $X$ be a smooth projective variety of dimension $n$. The Serre functor or $S_X$, denoted by $S_X$, is defined as the composition

$$D^\ast(X) \xrightarrow{\omega_X \otimes} D^\ast(X) \xrightarrow{[n]} D^\ast(X)$$

for $\ast = -, +, b$. 
**Theorem 2.1.6 (Serre duality).** Let $X$ be a smooth projective variety of dimension $n$. Then the Serre functor

$$S_X : D^b(X) \to D^b(X)$$

is a $k$-linear equivalence such that for any $E^\bullet, F^\bullet \in D^b(X)$, there exists an isomorphism

$$\eta_{E^\bullet, F^\bullet} : \text{Hom}_{D^b(X)}(E^\bullet, F^\bullet) \xrightarrow{\sim} \text{Hom}_{D^b(X)}(F^\bullet, S_X(E^\bullet)).$$

of $k$-vector spaces which is functorial in $E^\bullet$ and $F^\bullet$.

Alternatively, since $\text{Ext}^i(E^\bullet, F^\bullet) = \text{Hom}_{D^b(X)}(E^\bullet, F^\bullet[i])$, Serre duality says that for every $i \in \mathbb{Z}$ we have an isomorphism

$$\text{Ext}^i(E^\bullet, F^\bullet) \cong \text{Ext}^{n-i}(F^\bullet, E^\bullet \otimes \omega_X)^*,$$

which is functorial in $E^\bullet$ and $F^\bullet$.

In addition, we also have the Grothendieck-Verdier duality. Let $f : X \to Y$ be a proper morphism of smooth schemes over $k$. The relative dimension of $f$ is defined as

$$\dim(f) := \dim X - \dim Y.$$
Theorem 2.1.7 (Grothendieck-Verdier duality). For every pair of objects $F^\bullet \in D^b(X)$ and $E^\bullet \in D^b(Y)$, there exists a functorial isomorphism

$$Rf_*R\text{Hom}(F^\bullet, Lf^*E^\bullet) \otimes \omega_f[\dim(f)] \cong R\text{Hom}(Rf_*F^\bullet, E^\bullet).$$

Let us define a dualizing functor

$$\mathbb{D}_X : D^b(X) \to D^b(X)$$

by

$$F^\bullet \mapsto R\text{Hom}(F^\bullet, \omega_X[\dim X] \cong F^\bullet^\vee \otimes \omega_X[\dim X].$$

Then the Grothendieck-Verdier duality can be written as

$$Rf_* \circ \mathbb{D}_X \cong \mathbb{D}_Y \circ Rf_*.$$

2.2 Fourier-Mukai transforms

2.2.1 Preliminaries

Let $X$ and $Y$ be smooth projective varieties and denote the two projections by

$$q : X \times Y \to X \text{ and } p : X \times Y \to Y.$$
Definition 2.2.1. Let $\mathcal{P}$ be an object in $\mathcal{D}^b(X \times Y)$. The Fourier-Mukai transform is defined to be the functor

$$\Phi_{\mathcal{P}} : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$$

$$\mathcal{E} \mapsto p_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P}).$$

The object $\mathcal{P}$ is called the kernel of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Here, $p_*, q^*$, and $\otimes$ denote the derived functors between derived categories. However, since the projection $q$ is flat, the functor $q^*$ is, in fact, exact; hence, $q^*$ is just the usual pullback applying to the complexes. For the rest of the paper, we will drop the $R$ and $L$ in derived functors and rely on the context to distinguish the derived functors from the usual functors.

In the literature, $\Phi_{\mathcal{P}}$ is sometimes referred to as an integral functor. And a Fourier-Mukai transform is used to denote an integral functor that is also an equivalence. In this paper, we refer to a Fourier-Mukai transform which is an equivalence as a Fourier-Mukai equivalence. We call $X$ and $Y$ Fourier-Mukai partners if there exists a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ that is an equivalence.

Notice that $\mathcal{P} \in \mathcal{D}^b(X \times Y)$ also gives rise to another Fourier-Mukai transform

$$\Psi_{\mathcal{P}} : \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(X)$$

$$\mathcal{E}^\bullet \mapsto q_*(p^*\mathcal{E}^\bullet \otimes \mathcal{P})$$

For an object $\mathcal{P} \in \mathcal{D}^b(X \times Y)$, we will always denote by $\Phi_{\mathcal{P}}$ the functor $\mathcal{D}^b(X) \rightarrow \mathcal{D}^b(Y)$ and $\Psi_{\mathcal{P}}$ the function $\mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(X)$. 
Remark 2.2.2. Since $\Phi_P$ is a composition of three exact functors $p_*, q^*$, and $\otimes$, $\Phi_P$ itself is an exact functor.

Example 2.2.3. (18, Example 5.4) We will represent some of the common derived functors in the form of Fourier-Mukai transform

(a) The identity $\text{id} : D^b(X) \to D^b(X)$ is naturally isomorphic to the Fourier-Mukai transform $\Phi_{O_\Delta}$, where $O_\Delta$ denote the structure sheaf of the diagonal image $\Delta \subset X \times X$ of the diagonal embedding $\iota : X \to \Delta \subset X \times X$. Indeed,

$$
\Phi_{O_\Delta}(E^\bullet) = p_*(q^*E^\bullet \otimes O_\Delta) = p_*(q^*E^\bullet \otimes \iota_*O_X) \\
\cong p_*(\iota_*(q^*E^\bullet \otimes O_X)) \text{ (projection formula)} \\
\cong (p \circ \iota)_*((q \circ \iota)^*E^\bullet) \cong E^\bullet.
$$

(b) Let $f : X \to Y$. Let $\Gamma_f \subset X \times Y$ be the graph of $f$. We then have

$$
\Phi_{\Gamma_f}(E^\bullet) = p_*(q^*E^\bullet \otimes \Gamma_f) = p_*(q^*E^\bullet \otimes f_*O_X) \\
\cong p_*(f_*(f^*q^*E^\bullet \otimes O_X)) \text{ (projection formula)} \\
\cong (p \circ f)_*((q \circ f)^*E^\bullet) \cong (p \circ f)_*E^\bullet \cong f_*E^\bullet,
$$

i.e., $f_* \cong \Phi_{\Gamma_f}$. Similarly, we have that $f^* \cong \Psi_{\Gamma_f}$.

(c) Let $L$ be a line bundle on $X$. The derived functor $E^\bullet \mapsto E^\bullet \otimes L$ defines an equivalence $D^b(X) \to D^b(X)$. Let $\iota : X \to \Delta \subset X \times X$ be the diagonal embedding. Then by following
similar argument as the previous cases, the derived functor $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes L$ is isomorphic to the Fourier-Mukai transforms $\Phi_{\epsilon_* L}$.

(d) The shift functor $T : D(X) \to D(X)$ is clearly isomorphic to the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[1]$.

(e) Consider again the diagonal embedding $\iota : X \to \Delta \subset X \times X$. Let $S_X$ be the Serre functor $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes \omega_X[n]$ with $n = \dim X$. Applying (c) and (d), we obtain that

$$S^k_X[-nk] \cong \Phi_{\epsilon_* \omega}^k$$

(f) Let $\mathcal{P}$ be a coherent sheaf on $X \times Y$ flat over $X$. Consider the Fourier-Mukai transform $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$. If $x \in X$ is a closed point with $k(x) \cong k$, then

$$\Phi_{\mathcal{P}}(k(x)) \cong \mathcal{P}_x,$$

where $\mathcal{P}_x := \mathcal{P}|_{\{x\} \times Y}$ is considered as a sheaf on $Y$ via the second projection $\{x\} \times Y \to Y$.

Every exact equivalence has a left and right adjoint. In particular, the left and right adjoint of a Fourier-Mukai transforms are again Fourier-Mukai transforms whose kernels can be described explicitly.

**Definition 2.2.4.** For any object $\mathcal{P} \in D^b(X \times Y)$ we define

$$\mathcal{P}_L := \mathcal{P}^\vee \otimes p^* \omega_Y[\dim Y] \quad \text{and} \quad \mathcal{P}_R := \mathcal{P}^\vee \otimes q^* \omega_X[\dim X],$$
both objects in $D^b(X \times Y)$.

Remark 2.2.5. The above kernels induce the Fourier-Mukai transforms $\Psi_R : D^b(Y) \to D^b(X)$ and $\Psi_L : D^b(Y) \to D^b(X)$. In fact, those transforms can be equivalently realized as

\[ \Psi_R \cong \Psi^\vee \circ S_Y \quad \text{and} \quad \Psi_L \cong S_X \circ \Psi^\vee. \]

Proposition 2.2.6 (Mukai). Let $F = \Phi_P : D^b(X) \to D^b(Y)$ be the Fourier-Mukai transform with kernel $\mathcal{P}$. Then

\[ G := \Psi_L : D^b(Y) \to D^b(X) \quad \text{and} \quad H := \Psi_R : D^b(Y) \to D^b(X) \]

are left, respectively right adjoint to $F$.

Proposition 2.2.7. The composition

\[ D^b(X) \xrightarrow{\Phi_P} D^b(Y) \xrightarrow{\Phi_Q} D^b(Z) \]

is isomorphic to the Fourier-Mukai transform

\[ \Phi_R : D^b(X) \to D^b(Z), \]

where $\mathcal{R} = \pi_{XZ*}(\pi_{XY*} \mathcal{P} \otimes \pi_{YZ*} \mathcal{Q})$ with $\pi_{XZ}, \pi_{XY}$ and $\pi_{YZ}$ being the projections from $X \times Y \times Z$ to $X \times Z$, $X \times Y$, and $Y \times Z$, respectively. See (29, Section 1).
Proposition 2.2.8. (18, Exercise 5.13) Consider two kernels $\mathcal{P}_i \in D^b(X_i \times Y_i)$ with induced Fourier-Mukai transforms $\Phi_{\mathcal{P}_i} : D^b(X_i) \to D^b(Y_i)$, $i = 1, 2$, and their exterior tensor product $\mathcal{P}_1 \boxtimes \mathcal{P}_2 \in D^b((X_1 \times X_2) \times (Y_1 \times Y_2)$ with induced Fourier-Mukai transform $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2} : D^b(X_1 \times X_2) \to D^b(Y_1 \times Y_2)$.

1. There exists isomorphisms

$$\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet) \cong \Phi_{\mathcal{P}_1}(\mathcal{F}_1^\bullet) \boxtimes \Phi_{\mathcal{P}_2}(\mathcal{F}_2^\bullet)$$

which are functorial in $\mathcal{F}_i \in D^b(X_i)$, $i = 1, 2$.

2. For every object $\mathcal{R} \in D^b(X_1 \times X_2)$ and its image $\mathcal{S} := \Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{R})$, we have the following commutative diagram

$$\begin{array}{ccc}
D^b(X_1) & \xleftarrow{\Psi_p} & D^b(Y_1) \\
\Phi_{\mathcal{R}} \downarrow & & \downarrow \Phi_{\mathcal{S}} \\
D^b(X_2) & \xrightarrow{\Phi_{\mathcal{P}_2}} & D^b(Y_2)
\end{array}$$

The notation $\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}$ is sometimes used instead of $\Phi_{\mathcal{P}_1} \times \Phi_{\mathcal{P}_2}$. See (31, Section 2.1).

The following celebrated theorem of Orlov clarifies the relationship between arbitrary functors and those of Fourier-Mukai type.

Theorem 2.2.9 (Orlov). Let $X$ and $Y$ be two smooth projective varieties and let $F : D^b(X) \to D^b(Y)$
be a fully faithful exact functor. If $F$ admits right and left adjoint functors, then there exists an object $\mathcal{P} \in D^b(X \times Y)$ unique up to isomorphism such that $F$ is isomorphic to $\Phi_{\mathcal{P}}$. See (31, Theorem 3.2.1).

Orlov’s theorem has many important implications in the case that $F$ is an equivalence.

**Corollary 2.2.10.** Let $F : D^b(X) \to D^b(Y)$ be an equivalence between the derived categories of two smooth projective varieties. Then $F$ is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to an object $\mathcal{P} \in D^b(X \times Y)$, which is unique up to isomorphism.

**Proposition 2.2.11.** (18, Exercise 5.19) Let $\mathcal{P}_i \in D^b(X_i \times Y_i), i = 1, 2,$ be objects such that

$$
\Phi_{\mathcal{P}_i} : D^b(X_i) \to D^b(Y_i)
$$

are equivalences. Then the box product $\mathcal{P}_1 \boxtimes \mathcal{P}_2 \in D^b((X_1 \times X_2) \times (Y_1 \times Y_2))$ defines an equivalence

$$
\Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2} : D^b(X_1 \times X_2) \to D^b(Y_1 \times Y_2).
$$

**Corollary 2.2.12.** Let $X$ and $Y$ be smooth projective varieties with equivalent derived categories $D^b(X)$ and $D^b(Y)$. Then $\dim X = \dim Y$. See (18).

**Corollary 2.2.13.** Suppose that $\Phi : D^b(X) \to D^b(Y)$ is an equivalence such that for any closed point $x \in X$ there exists a closed point $f(x) \in Y$ with

$$
\Phi(k(x)) \cong k(f(x)).
$$
Then $f : X \to Y$ defines an isomorphism and $\Phi$ is the composition of $f_*$ with the twist by some line bundle $M \in \text{Pic}(Y)$, i.e.
\[
\Phi \cong (M \otimes (\ )) \circ f_* .
\]

See (18)

In fact, we have a more refined version of the above corollary.

**Proposition 2.2.14.** Suppose there exists a closed point $x_0 \in X$ such that
\[
\Phi_P(k(x_0)) \cong k(y_0)
\]
for some closed point $y_0 \in Y$. Then there exists an open neighborhood $U \subset X$ of $x_0$ and a morphism $f : U \to Y_0$ with $f(x_0) = y_0$ and such that
\[
\Phi_P(k(x)) \cong k(f(x))
\]
for all closed points $x \in U$. In particular, $f$ is a birational map. See (18).

### 2.2.2 Criteria for derived equivalence

Consider the Fourier-Mukai transform $\Phi_P : \text{D}^b(X) \to \text{D}^b(Y)$ between derived categories of smooth projective varieties $X$ and $Y$ given by an object $P \in \text{D}^b(X \times Y)$. 
Proposition 2.2.15 (Bondal, Orlov). The functor $\Phi_P$ is fully faithful if and only if for any two closed points $x, y \in X$ one has

$$\text{Hom}(\Phi_P(k(x)), \Phi_P(k(y)[i])) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \text{dim } X. \end{cases}$$

See (18), (3), (7).

Proposition 2.2.16. Suppose $\Phi_P : D^b(X) \to D^b(Y)$ is a fully faithful Fourier-Mukai transform between smooth projective varieties. Then $\Phi_P$ is an equivalence if and only if

$$\text{dim } X = \text{dim } Y \text{ and } P \otimes q^* \omega_X \cong P \otimes p^* \omega_Y.$$  

An immediate consequence of this proposition is every fully faithful exact functor $D^b(X) \to D^b(Y)$ between smooth projective varieties of the same dimension and with trivial canonical bundle $\omega_X$ and $\omega_Y$ is an equivalence.

Proposition 2.2.17 (Bridgeland). Suppose $\Phi_P : D^b(X) \to D^b(Y)$ is fully faithful. Then $\Phi_P$ is an equivalence if and only if

$$\Phi_P(k(x)) \otimes \omega_Y \cong \Phi_P(k(x))$$

for all closed point $x \in X$. See (7).

Corollary 2.2.18. Let $P$ be a sheaf on $X \times Y$ flat over $X$. Assume that $\Phi_P$ is fully faithful. Then $\Phi_P$ is an equivalence if and only if $P_x \cong P_x \otimes \omega_Y$ for all $x \in X$. 
Corollary 2.2.19. Let $X$ be a smooth projective surface with a fixed polarization, and let $Y$ be a smooth, fine, complete, two-dimensional moduli space of special, stable sheaves on $X$. Then there is a universal sheaf $P$ on $Y \times X$ and the function $\Phi_P : D^b(Y) \to D^b(X)$ is a Fourier-Mukai transform. See (9, Corollary 2.8).

Lemma 2.2.20. Let $\Phi : D^b(Y) \to D^b(X)$ be a Fourier-Mukai transform and take a point $y \in Y$. Then there is an inequality

$$\sum_i \dim Ext^i_X(\Phi^i(O_y), \Phi^i(O_y)) \leq 2.$$  

and moreover, each of the sheaves $\Phi^i(O_y)$ is special. See (9, Lemma 2.9).

2.2.3 Birational invariants and derived equivalence

We have seen that derived equivalence preserve the dimension (Corollary 2.2.14). In addition, many important birational invariants are preserved by derived equivalence as well. Given a smooth projective variety $X$, we denote by $R(X)$ the canonical ring of $X$, i.e.

$$R(X) := \bigoplus_{k>0} H^0(X, \omega^k_X).$$

The following theorem is due to Kawamata and Orlov.
Theorem 2.2.21 (Kawamata, Orlov). Suppose that \( X \) and \( Y \) are smooth projective varieties with equivalent derived categories \( D^b(X) \cong D^b(Y) \). Then there exists a ring isomorphism

\[
R(X) \cong R(Y),
\]

and, in particular, the Kodaira dimensions of \( X \) and \( Y \) are equal, i.e. \( \text{kod}(X) = \text{kod}(Y) \). See (31, Theorem 2.1.8) and (20, Theorem 2.3).

Corollary 2.2.22. Suppose that \( X \) and \( Y \) are smooth projective varieties with the canonical being ample and that \( D^b(X) \cong D^b(Y) \). Then \( X \cong Y \).

Remark 2.2.23. Both Theorem 2.2.22 and Corollary 2.2.23 hold for the anti-canonical bundles \( \omega^*_X \) and \( \omega^*_Y \) as well. The proofs remain the same in this case.

Proposition 2.2.24 (Kawamata). Let \( X \) and \( Y \) be smooth projective varieties with equivalent derived categories. Then the (anti-)canonical bundles of \( X \) is nef if and only if the (anti-)canonical of \( Y \) is nef. See (20).

Corollary 2.2.25. If \( X \) and \( Y \) are smooth projective varieties with equivalent derived categories, then \( \omega_X \) is numerically trivial if and only if \( \omega_Y \) is. See (20).

Proposition 2.2.26. Let \( X \) and \( Y \) be smooth projective varieties with equivalent derived categories. Then the canonical bundles \( \omega_X \) and \( \omega_Y \) have the same order, including the case when \( \omega_X \) and \( \omega_Y \) are not torsion. In particular, \( \omega_X \) is trivial if and only if \( \omega_Y \) is. See (18, Proposition 4.1)
Proposition 2.2.27 (Kawamata). Let $X$ and $Y$ be smooth projective varieties with equivalent derived categories. Then $X$ and $Y$ have the same numerical Kodaira dimension. See (31).

We include here a few notions, principles and techniques that are used to study other finer birational invariants as well as to study the Fourier-Mukai transforms of surfaces. We refer the reader to Chapter 6 of (18), (20) and (31) for the details. Given a Fourier-Mukai equivalence

$$
\Phi = \Phi_P : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)
$$

with kernel $\mathcal{P}$. We denote by $\Gamma$ the support of $\mathcal{P}$ (or, equivalently, of $\mathcal{P}^\vee$), i.e.

$$
\Gamma := \bigcup \text{supp}(\mathcal{H}^i)
$$

with $\mathcal{H}^i$ the cohomology sheaves of $\mathcal{P}$ (respectively $\mathcal{P}^\vee$). Abusing the notation, let $q : \Gamma \to X$ and $p : \Gamma \to Y$ be the projections. The support of $\mathcal{P}$ contains many geometric information. It is known that the fibers of the projection $\Gamma \to X$ are connected (18, Lemma 6.11). There exists an irreducible component $Z \subset \Gamma$ that dominates $X$ (18, Corollary 6.5). This component may or may not dominates $Y$. Moreover, when $\dim \Gamma = \dim X$, the projection $p_1 : Z \to X$ is in fact a birational morphism (18, Corollary 6.12). And if such a component exists, then there no other component of $\Gamma$ dominates $X$. In addition, we have the following property.

**Lemma 2.2.28.** Let $Z \subset \text{supp}(\mathcal{P})$ be an irreducible component that dominates $X$. Assume that $\dim Z = \dim X$. Then $Z \to Y$ is also a birational morphism. Hence, $X$ and $Y$ are birational.
Proof. If \( \dim Z = 2 \), then \( Z \to X \) is necessarily generically finite and hence by Corollary 6.12 in (18) birational. Moreover, we may find an open dense subset \( U \subset X \) such that \( \Gamma_U = Z_U \cong U \) (see Remark 6.13 in (18)).

If for this component the second projection \( Z \to Y \) is not dominant then one finds distinct points \( x_1 \neq x_2 \in U \) with \( y := p(x_1) = p(x_2) \in Y \). Hence, \( \Phi(k(x_1)) \) and \( \Phi(k(x_2)) \) would both be complexes concentrated in \( y \). In particular, for some \( i \in \mathbb{Z} \) one has

\[
\text{Ext}^i(\Phi(k(x_1)), \Phi(k(x_2))) \neq 0.
\]

Indeed, if \( m_1 \) and \( m_2 \) are maximal, respectively minimal with \( \mathcal{H}^{m_1}(\Phi(k(x_i))) \neq 0 \), then there exists a non-trivial morphism \( \mathcal{H}^{m_1}(\Phi(k(x_1))) \to \mathcal{H}^{m_2}(\Phi(k(x_2))) \) (both are sheaves concentrated in \( y \in Y \)), which gives rise to a non-trivial morphism \( \Phi(k(x_1))[m_1] \to \Phi(k(x_2))[m_2] \). This then would contradict the fact that \( \text{Ext}^i(k(x_1), k(x_2)) = 0 \) for all \( i \).

Therefore, the projection \( Z \to Y \) is also dominant and hence generically finite. As before, this suffices to conclude that \( Z \to Y \) is birational as well. So under the assumption of the existence of an irreducible component \( Z \subset \Gamma \) of dimension 2 dominating one of the two factor, we can find a birational correspondence \( X \leftarrow Z \to Y \). By Lemma 6.9 in (18), we can conclude that \( X \) and \( Y \) are isomorphic. 

\[\square\]

**Corollary 2.2.29.** Let \( \Phi := \Phi_P : D^b(X) \to D^b(Y) \) be a Fourier-Mukai equivalence between smooth projective surfaces. If there exists an irreducible \( Z \subset \text{supp}(P) \) dominating \( X \) with \( \dim Z = \dim X \), then \( X \) and \( Y \) are isomorphic.
Lemma 2.2.30. Let $C$ be a complete reduced curve and $\varphi : C \to X \times Y$ be a morphism with image in $\text{supp}(\mathcal{P})$. Then

$$\text{deg}(\varphi^*q^*\omega_X) = \text{deg}(\varphi^*p^*\omega_Y).$$

In other words, the pull-backs $q^*\omega_X|_{\text{supp}(\mathcal{P})}$ and $p^*\omega_Y|_{\text{supp}(\mathcal{P})}$ are numerically equivalent.

Corollary 2.2.31. Suppose $Z \subset \text{supp}(\mathcal{P})$ is a closed subvariety such that the restriction of $\omega_X$ (or its dual $\omega_X^*$) to the image of $q : Z \to X$ is ample. Then $p : Z \to Y$ is a finite morphism.

Lemma 2.2.32. Let $Z \subset \text{supp}(\mathcal{P})$ be a closed irreducible subvariety with a normalization $\mu : \tilde{Z} \to Z$. Then there exists an integer $r > 0$ such that

$$\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r,$$

where $\pi_X = q \circ \mu$ and $\pi_Y = p \circ \mu$.

We end this section with a discussion of another notion of equivalence.

Definition 2.2.33. Two varieties $X$ and $Y$ are called K-equivalent if there exists birational correspondence

$$X \leftarrow \overset{\pi_X}{Z} \overset{\pi_Y}{\rightarrow} Y$$

with $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$. 
Proposition 2.2.34 (Kawamata). Let $X$ and $Y$ be smooth projective varieties over an algebraically closed field. Suppose that there exists an exact equivalence

$$D^b(X) \cong D^b(Y).$$

If $\text{kod}(X) = \dim X$ or $\text{kod}(X, \omega_X^*) = \dim X$, then $X$ and $Y$ are $K$-equivalent. See (20).

2.3 Fourier-Mukai transforms for surfaces

Fourier-Mukai transforms of surfaces are well-understood by the work of Kawamata (20) and Bridgeland-Maciocia (9). In particular, Kawamata in (20) showed that smooth projective surfaces only have finitely many Fourier-Mukai partners. Moreover, Bridgeland and Maciocia, in (9), give a detailed descriptions of Fourier-Mukai partners of a given minimal surface. In this section, we will give a summary of the main results. We begin with smooth projective surfaces that are non-minimal.

Theorem 2.3.1 (Kawamata). Let $X$ be a smooth projective surface containing a $(-1)$-curve. Suppose there is a smooth projective variety $Y$ with $D^b(X) \cong D^b(Y)$. Then

(i) $X \cong Y$ or

(ii) $X$ is a relatively minimal elliptic rational surface.

See (20), (18, Chapter 12).

Proof. We already knew that $\dim X = \dim Y$. Let $\Phi := \Phi_P : D^b(X) \rightarrow D^b(Y)$ be a Fourier-Mukai equivalence and $\Gamma = \text{supp}(P)$. If $\dim \Gamma = 2$ then we can find an irreducible component $Z$ of
dimension 2 that dominates $X$. By Corollary 2.2.30, $X$ and $Y$ are isomorphic. Thus, we can assume that $\dim \Gamma \geq 3$. Let $C$ be a $(-1)$-curve on $X$, and let $\Gamma_C = q^{-1}(C) = \Gamma \times_X C$. Consider the following diagram

$$
\begin{array}{ccc}
\Gamma_C := q^{-1}(C) & \longrightarrow & \Gamma \\
\downarrow & & \downarrow^p \\
C & \longrightarrow & X
\end{array}
$$

Let $q_C$ and $p_C$ be the projections from $\Gamma_C$ to $X$ and $Y$, respectively. First, we claim that $p_C : \Gamma_C \rightarrow Y$ is a finite and dominant morphism, and $\dim \Gamma_C = 2$. Since $C$ is a $(-1)$-curve, $\omega_X|_C$ is ample. Corollary 2.2.33 implies that $p_C$ is a finite morphism. Thus, $\dim \Gamma_C \leq 2$. On the other hand, $q : \Gamma \rightarrow X$ is a dominant morphism with $\dim \Gamma \geq 3$; hence, $q : \Gamma \rightarrow X$ and $q_C : \Gamma_C \rightarrow X$ have fibers of dimension at least 1, i.e. $\dim \Gamma_C > 1$. Thus, $\dim \Gamma_C = 2$ and $p_C$ is finite and dominant.

Next, we will show that $\omega_Y^*$ is nef and $\nu(Y, \omega_Y^*) = 1$. Since $\omega_X^*|_C$ is ample, $q_C^*(\omega_X^*) = q^*(\omega_X^*|_C)$ is nef. Since $p_C^*(\omega_Y^*)$ and $q_C^*(\omega_X^*)$ are numerically equivalent on $\Gamma_C \subset \Gamma$ (by Lemma 2.5.3), it follows that $p_C^*(\omega_Y^*)$ is also nef. Since $p_C : \Gamma_C \rightarrow Y$ is surjective, $p_C^*(\omega_Y^*)$ being nef is equivalent to $\omega_Y^*$ being nef. The fact that $\nu(Y, \omega_Y^*) = 1$ follows from the following equalities

$$\nu(Y, \omega_Y^*) = \nu(\Gamma_C, p_C^*(\omega_Y^*)) = \nu(\Gamma_C, q_C^*(\omega_Y^*)) = \nu(\Gamma_C, q_C^*(\omega_X^*|_C)) = \nu(C, \omega_X^*|_C) = 1.$$  

The first and the fourth equalities follow from the fact that $p_C$ and $q_C : \Gamma_C \rightarrow C$ are surjective. The second equality follows from Lemma 2.2.31. And the last equality follows from the fact that $\omega_X^*|_C$ is ample. We then can conclude that $\omega_X^*$ is nef and $\nu(X, \omega_X^*) = 1$. These two facts imply that $\kod(X) = \kod(Y) = -\infty$ as follows. Assume that $H^0(X, \omega_X^k) \neq 0$ for some $k \in \mathbb{Z}^+$. 
Let $s \in H^0(X, \omega_X^k)$ and consider $Z(s)$, the zeros of $s$. Then either $Z(s)$ is empty (i.e. $\omega_X^k$ is trivial) or $\dim Z(s) = 1$. Notice that since $\nu(X, \omega_X^*) = 1$, $Z(s) \neq \emptyset$. And $\dim Z(s) = 1$ contradicts the fact that $\omega_X^*$ is nef. This shows that $\text{kod}(X) = \text{kod}(Y) = -\infty$.

By the classification of surfaces, the minimal model of $X$ is either a rational or a ruled surface over a curve of genus $g \geq 1$. Since $\omega_X^*$ is nef, $c_1^2(X) = K_X^2$ is non-negative. Recall that for a minimal surface $X_{\min}$ ruled over a curve of genus $g$, $c_1^2(X_{\min}) = 8(1-g)$. Moreover $c_1^2$ decreases under blow-up. Therefore, $X$ is either a rational surface or a minimal surface ruled over an elliptic curve. Since we assume that $X$ is minimal, it remains that $X$ is a rational surface. And since $q(X) = q(Y)$ and $h^0(X, \omega_X^2) = h^0(Y, \omega_Y^2)$, Castelnuovo’s criterion for rational surfaces implies that $Y$ is also a rational surface.

Then for purely numerical reasons $X$ is either a nine-fold blow-up of $\mathbb{P}^2$ or an eight-fold blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$. In either case, one can find a family of elliptic curves that cover $X$. The details are left to the readers.

We are now left with minimal surfaces. We begin with the following Lemma.

**Lemma 2.3.2.** Let $X$ be a smooth projective surface and $Y$ be a smooth projective variety with $D^b(X) \cong D^b(Y)$. Then $X$ and $Y$ are surfaces of the same type in the Kodaira’s classification.

We have the following theorem whose proof can be found in (9).
Theorem 2.3.3 (Bridgeland, Maciocia). Let \( X \) and \( Y \) be smooth projective minimal surfaces with equivalent derived categories \( D^b(X) \cong D^b(Y) \). Then \( X \cong Y \) unless \( X \) (and, hence, \( Y \)) belongs to one of the following classes of surfaces

a. abelian surfaces,

b. \( K3 \)-surfaces, or

c. elliptic surfaces of non-zero Kodaira dimension.

Remark 2.3.4.

a. The case of abelian surfaces are treated in the general context of abelian varieties in the next chapter.

b. For the case of surfaces of general type, it follows from Proposition 2.2.34 that \( X \) and \( Y \) are birational. Since \( X \) and \( Y \) are both minimal, they are isomorphic.

Elliptic Surfaces of non-zero Kodaira dimension

Since the case of elliptic surfaces with non-zero Kodaira dimension are of particular interest in Chapter 5, we include a brief treatment here. The details of the proofs can be found in (6), (9), and (18, Chapter 12). For the rest of the section, let us fix a surface \( X \) and a relatively minimal elliptic fibration \( \pi : X \to C \). Given an object \( E \) of \( D^b(X) \), one can define the fiber degree of \( E \) by

\[
d(E) = c_1(E) \cdot f,
\]
where \( f \) denotes a general fiber of \( \pi \). Let \( \lambda_{X/C} \) denote the greatest common factor of the fiber degrees of objects of \( D^b(X) \). Equivalently, \( \lambda_{X/C} \) is the smallest integer \( d \) such that there exists a holomorphic \( d \)-section of \( \pi \).

For integers \( a > 0 \) and \( b \) with \( \gcd(b, a\lambda_{X/C}) = 1 \), by (6, Section 4), there exists a smooth, 2-dimensional component
\[
Y = J_{X/C}(a, b)
\]
of the moduli space of pure dimension one stable sheaves on \( X \), whose general point represents rank \( a \), degree \( b \) stable vector bundle supported on a smooth fiber of \( \pi \). In addition, there is a natural morphism \( Y \to C \), taking a point representing a sheaf supported on the fiber \( \pi^{-1}(p) \) of \( X \) to the point \( p \), and this morphism is a relatively minimal elliptic fibration.

Remark 2.3.5. We should note here that by using a different polarization of \( X \), we can define different elliptic surfaces \( J_{X/C}(a, b) \) and \( J'_{X/C}(a, b) \). Since the stability of sheaves on a smooth curve does not depend on the choice of the polarization, the two spaces are isomorphic over an open set \( U \). By the relatively minimal condition over \( C \), the two spaces are in fact isomorphic as elliptic surfaces over \( C \).

When \( a = 1 \), the elliptic surface \( J_{X/C}(a, b) \) is the relative Picard scheme described in Friedman (12), which we will denote by \( J^b(X) \). For every integer \( d \) there is an elliptic surface \( J^d(X) \to C \) whose generic fiber over \( \eta \in C \) is the Picard scheme of divisors of degree \( d \) on the elliptic curve \( X_\eta \). Clearly, \( J^1(X) \cong X \). In addition, \( J^0(X) \cong J(X) \), the Jacobian surface associated to \( X \) (see (13) and (1)). More specifically, for every elliptic surface \( X \to C \) there
associates a unique (up to fiber-preserving isomorphism) elliptic surface $J(X) \rightarrow C$ with a section. And every smooth fiber of $J(X) \rightarrow C$ is in fact the Jacobian of the corresponding smooth fiber of $X \rightarrow C$. The important property here is that: given any pair of integers $a$ and $b$ with $b$ coprime to $a\lambda_{X/C}$, there is an isomorphism

$$J_{X/C}(a,b) \cong J^b(X).$$

Moreover, there is a natural isomorphism

$$J^i(X) \cong J^{i+\lambda_{X/C}}(X).$$

This natural isomorphism is given by intersecting with the $\lambda_{X/C}$-section of $\pi$. One should also notice that the relationship between $X$ and $Y = J_{X/C}(a,b)$ is entirely symmetric in the following sense

**Proposition 2.3.6.** There exists an integer $c$ such that $X \cong J_{Y/C}(a,c)$. In particular, $\lambda_{X/C} = \lambda_{Y/C}$.

Moreover, there exists a universal sheaf $\mathcal{P}$ on $Y \times X$, supported on $Y \times_C X$, and the corresponding Fourier-Mukai functor $\Phi_{\mathcal{P}} : \mathcal{D}^b(Y) \rightarrow \mathcal{D}^b(X)$ is a Fourier-Mukai equivalence.

**Theorem 2.3.7** (Bridgeland, Maciocia). Let $\pi : X \rightarrow C$ be a relatively minimal elliptic surface and take an element

$$\begin{pmatrix} c & a \\ d & b \end{pmatrix} \in SL_2(\mathbb{Z}),$$

$$\lambda_{X/C} = \lambda_{Y/C}.$$
such that \( \lambda_X \) divides \( d \) and \( a > 0 \). Let \( Y \) be the elliptic surface \( J_X(a,b) \) over \( C \). Then there exists sheaves \( \mathcal{P} \) on \( X \times Y \), flat and strongly simple over both factors such that for any point \((x,y) \in X \times Y\), \( \mathcal{P}_y \) has Chern class \((0,af,-b)\) on \( X \) and \( \mathcal{P}_x \) has Chern class \((0,af,-c)\) on \( Y \).

For any such sheaf \( \mathcal{P} \), the result functor \( \Phi = \Phi_{\mathcal{P}} : D^b(Y) \to D^b(X) \) is an equivalence and satisfies

\[
\begin{pmatrix}
    r(\Phi(E)) \\
    d(\Phi(E))
\end{pmatrix} =
\begin{pmatrix}
    c & a \\
    d & b
\end{pmatrix}
\begin{pmatrix}
    r(E) \\
    d(E)
\end{pmatrix},
\]

for all objects \( E \in D^b(Y) \). See (6, Theorem 5.3).

**Remark 2.3.8.**

a. The integers \( c \) and \( d \) above are not uniquely determined. The existence of the \( \lambda_{X/C} \)-section allows us to replace them by \( c + n\lambda_{X/C}a \) and \( d + n\lambda_{X/C}b \) by twisting \( \mathcal{P} \) by the pull-back of a line bundle of fibre degree \( n\lambda_{X/C} \) on \( Y \).

b. We also notice that \( X \cong J_{Y/C}(a,c) \).

c. Theorem 2.2.21 implies that \( X \) and \( Y \) have the same Kodaira dimension.

d. The coprimality condition \( \gcd(b, a\lambda_{X/C}) = 1 \) implies that the general fibers of \( X \to C \) and \( Y \to C \) are in fact isomorphic by Atiyah’s Theorem.

**Theorem 2.3.9.** Let \( X \) be a relatively minimal surface of non-zero Kodaira dimension, with an elliptic fibration \( \pi : X \to C \). If \( Y \) is a Fourier-Mukai partner of \( X \), then \( Y \) is isomorphic to the relative Picard scheme \( J^b(X) \), for some integer \( b \) coprime to \( \lambda \).
Proof. Let $\Phi : D^b(Y) \to D^b(X)$ be a Fourier-Mukai equivalence between $Y$ and $X$. For every closed point $y \in Y$ let $E = \Phi(O_y)$. The object $E$ has the following properties.

**Claim 1:**

i. $E$ is simple, i.e. $\text{Hom}_{D^b(X)}(E, E) \cong k$, and, hence, $\text{supp}(E)$ is connected.

ii. $E$ is special, i.e $E \otimes \omega_X \cong E$.

iii. $\text{supp}(E)$ is contained in a fiber of $\pi : X \to C$.

iv. $E$ is a sheaf after some shift.

Let us consider the proof of this claim at the end and proceed with the proof of the theorem.

By choosing $y$ to be a general point, we may assume that $\text{supp}(E)$ is contained in a smooth fiber $F_x$ of $\pi$. By composing $\Phi$ with a shift, we may also assume that $E$ is a sheaf. Then the support of $E$ is either a point $z$ or the whole smooth fiber $F_x$.

If $\text{supp}(E) = \{z\}$, then $E_z \cong (O_z)^{\oplus m}$ for some positive integer $m$. Since $\text{Hom}_{D^b(X)}(E, E) \cong k$, we have $E_z \cong O_z$; hence, $\Phi(O_y) = O_z$. Proposition 2.2.14 then implies that there exists a birational morphism $g : Y \to X$. In this case, the argument shown in Claim 2 below shows that $X$ and $Y$ are in fact isomorphic.

On the other hand, assume that $\text{supp}(E) = F_x$. Let $f$ be a general fiber of $\pi$. The Chern class of $E$ is of the form $(0, rf, b)$ with $r, b \in \mathbb{Z}$. We have the equality

$$\chi(E, \Phi(O_Y)) = \chi(\Phi(O_y), \Phi(O_Y)) = \chi(O_y, O_Y) = 1.$$
If we write the Chern class of $\Phi(\mathcal{O}_Y)$ as $(a, cf, d)$ with $a, b, c \in \mathbb{Z}$, then

$$1 = \chi(E, \Phi(\mathcal{O}_Y)) = rc + ab$$

(see Section 2.2 in (9)). This shows that $r$ and $d$ are coprime. By Atiyah’s Theorem ((18, Theorem 12.23)), $E$ is a stable simple sheaf.

Therefore, as discussed above, we can compare $Y$ with $Y^+ = J_{X/C}(b)$, a smooth, two-dimensional component of the moduli space of pure dimension one stable sheaves on $X$. Also recall that $J_{X/C}(b)$ comes with a relatively minimal elliptic fibration $\pi^+: Y^+ \to C$. In addition, there is an equivalence

$$\Psi: D^b(Y^+) \to D^b(X),$$

which takes the structure sheaf of some point $p \in Y^+$ to $E$. The composition $\Psi^{-1} \circ \Phi$ is an equivalence $D^b(Y) \to D^b(Y^+)$. Moreover, by construction, we have

$$(\Psi^{-1} \circ \Phi)(\mathcal{O}_y) = \mathcal{O}_p.$$ 

Thus, by Proposition 2.2.14, there exists a birational morphism $f: Y \dashrightarrow Y^+$. Notice that this birational $f$ and the earlier birational morphism $g$ enjoy the same properties.

**Claim 2:** $f$ and $g$ extend to an isomorphism.

We just need to show this claim for the morphism $f$. If $X$ has Kodaira dimension 1, then so do $Y$ and $Y^+$. In this case, $Y$ and $Y^+$ are minimal surfaces of non-negative Kodaira dimension
that are birational; hence, they are isomorphic. The only other possibility is that $X$ is a
minimal surface ruled over an elliptic base. For numerical reasons, $Y$ and $Y^+$ are also minimal
surfaces ruled over an elliptic base. By Theorem 2.3.3, we can assume that $Y$ admits an elliptic
fibration (otherwise, $X$ and $Y$ are isomorphic). Since elliptic fibration of a non-zero Kodaira
dimension is unique. It follows that $Y$ and $Y^+$ admit elliptic fibrations over $C$. We can show
that $f: Y \to Y^+$ is in fact a birational morphism of relatively minimal elliptic surfaces.
Therefore, by (1, Proposition III.8.4), $f$ extends to an isomorphism.

It remains to prove Claim 1. We have $\text{Hom}_{\mathcal{D}^b(X)}(\Phi(O_Y), \Phi(O_Y)) = \text{Hom}_{\mathcal{D}^b(Y)}(O_Y, O_Y) = k$; thus, (i) follows. Since $\Phi$ is an equivalence, (ii) follows from Proposition 2.2.17. Let $\mathcal{F}$ be
$\mathcal{H}^i(E)$ for an arbitrary integer $i$. Since $X$ is a relatively minimal elliptic of non-zero Kodaira
dimension, $\Omega_X$ is a line bundle associated to a linear combination of fibers (see (1)) and is
non-trivial. This fact together with $E \otimes \Omega_X \cong$ suffice to show that $\text{supp}(E)$ is contained in a
fiber. Next, recall Lemma 2.2.20,

$$\sum_i \dim \text{Ext}^1(\mathcal{H}^i(E), \mathcal{H}^i(E)) \leq 2.$$  \hspace{1cm} (2.3)

Since $\mathcal{H}^i(E)$ is contained a sheaf on a smooth elliptic curve ($\text{supp}(E)$ is contained in a smooth
fiber), we have

$$\text{Ext}^i(\mathcal{H}^i(E), \mathcal{H}^i(E)) \neq 0 \text{ if } \mathcal{H}^i(E) \neq 0.$$
Moreover, the Chern class of $\mathcal{H}^i(E)$ is of the form $(0, rf, d)$. Then

$$\chi(\mathcal{H}^i(E), \mathcal{H}^i(E)) = 0.$$ 

On the other hand,

$$\chi(\mathcal{H}^i(E), \mathcal{H}^i(E)) = \dim \text{Ext}^0(\mathcal{H}^i(E), \mathcal{H}^i(E)) - \dim \text{Ext}^1(\mathcal{H}^i(E), \mathcal{H}^i(E)) + \dim \text{Ext}^2(\mathcal{H}^i(E), \mathcal{H}^i(E)).$$

Serre’s duality gives $\dim \text{Ext}^0(\mathcal{H}^i(E), \mathcal{H}^i(E)) = \dim \text{Ext}^2(\mathcal{H}^i(E), \mathcal{H}^i(E))$. This shows that $\dim \text{Ext}^1(\mathcal{H}^i(E), \mathcal{H}^i(E))$ is either 0 or even. Combining this with inequality (2.3), we obtain that $E$ is a sheaf after some shift. \qed
CHAPTER 3

DERIVED EQUIVALENCE OF ABELIAN VARIETIES

3.1 Equivalence induced by the Poincaré bundle

In this section, we will give some specific properties of Fourier-Mukai transforms on abelian varieties, which is the starting point of the theory of Fourier-Mukai transforms. An abelian variety $A$ is a projective connected algebraic group over $k$. In particular, $A$ is equipped with morphisms

$$m : A \times A \to A, \quad \iota : A \to A, \quad \text{and} \quad e : \text{Spec}(k) \to A.$$ 

satisfying the usual axioms of a group. We often write $a + b$ for $m(a, b)$, $-a$ for $\iota(a)$ and $0 \in A$ for $e \in A$. Every closed point $a \in A$ gives rise to a translation morphism

$$t_a : A \to A, \quad b \mapsto a + b.$$ 

In addition, we have the Picard group of $A$

$$\text{Pic}^0(A) := \{L \in \text{Pic}(A) \mid c_1(L) = 0\}.$$

We can also characterize $\text{Pic}^0(A)$ as

$$\text{Pic}^0(A) = \{L \in \text{Pic}(A) \mid t_a^*L \cong L \text{ for all } a \in A\}.$$
Moreover, there is a fine moduli space \( \hat{A} \), called the dual of \( A \), that parametrizes line bundles on \( A \) that belong to \( \text{Pic}^0(A) \). As an abstract group, \( \hat{A} \) is isomorphic to \( \text{Pic}^0(A) \). In addition, there is a universal Poincaré bundle \( P \) on \( A \times \hat{A} \) with the following properties

1. If \( \alpha \in \hat{A} \) corresponds to a line bundle \( L \in \text{Pic}^0(A) \) on \( A \), then \( P|_{A \times \{ \alpha \}} \) is isomorphic to \( L \).

2. The restriction \( P|_{\{a\} \times \hat{A}} \) is trivial.

To shorten the notation, for every \( a \in A \) and \( \alpha \in \hat{A} \), we write \( P_a \) for \( P|_{\{a\} \times \hat{A}} \) and \( P_\alpha \) for \( P|_{A \times \{\alpha\}} \). The starting point of the theory of Fourier-Mukai transforms is the following theorem due to Mukai

**Theorem 3.1.1 (Mukai).** Let \( A \) be an abelian variety and \( P \) the Poincaré bundle on \( A \times \hat{A} \).

Then

\[
\Phi_P : D^b(\hat{A}) \rightarrow D^b(A)
\]

is an equivalence. Moreover, the composition

\[
D^b(\hat{A}) \xrightarrow{\Phi_P} D^b(A) \xrightarrow{\Psi_P} D^b(\hat{A})
\]

is isomorphic to \( i^* \circ [-g] \), where \( g = \dim A \) and \( i : \hat{A} \rightarrow \hat{A} \) being the inverse morphism of \( \hat{A} \).

See (29, Theorem 2.2).

**Proposition 3.1.2.** For any \( a \in A \) and \( \alpha \in \hat{A} \), one has

1. \( (P_\alpha^* \otimes ( )) \circ \Phi_P \cong \Phi_P \circ i_\alpha^* : D^b(\hat{A}) \rightarrow D^b(A) \)
2. $\Psi_P \circ (P_\alpha \otimes ( )) \cong t_\alpha^* \circ \Psi_P : D^b(A) \to D^b(\hat{A})$.

See (29, Section 3).

3.2 Criteria for derived equivalence of abelian varieties

We now study derived equivalences between two different abelian varieties. For all abelian varieties $A$, we have the following automorphism

$$\mu : A \times A \to A \times A$$

$$(a_1, a_2) \mapsto (a_1 + a_2, a_2).$$

In addition, the Poincaré bundle $P$ on $A \times \hat{A}$ induces the following equivalence, by Proposition 2.2.13,

$$\text{id} \times \Phi_P : D^b(A \times \hat{A}) \to D^b(A \times A).$$

We then have the following composition

$$D^b(A \times \hat{A}) \xrightarrow{\text{id} \times \Phi_P} D^b(A \times A) \xrightarrow{\mu^*} D^b(A \times A).$$

For every closed point $a \in A$ and $\alpha \in \hat{A}$, let $k(a)$ and $k(\alpha)$ denote the function field of $a$ and $\alpha$, respectively. It is clear that

$$\Phi_P(k(a)) = P_\alpha.$$

In addition, we have the following calculation
Lemma 3.2.1. (18, Example 9.33) Let \( \mathcal{P}_\alpha \) be the line bundle corresponding to \( \alpha \) on \( A \). Then

\[
\mu_* \circ (id \times \Phi_\mathcal{P}) (k(a) \boxtimes k(\alpha)) = (\mathcal{O}_A \boxtimes \mathcal{P}_\alpha) \otimes \mathcal{O}_{\Gamma_{-a}},
\]

where \( \Gamma_{-a} \) denotes the graph of the translation \( t_{-a} \).

Proof. By Proposition 2.2.8a, we have

\[
(id \times \Phi_\mathcal{P})(k(a) \boxtimes k(\alpha)) = k(a) \boxtimes \Phi_\mathcal{P}(k(\alpha)) = k(a) \boxtimes \mathcal{P}_\alpha.
\]

Moreover, let \( i_a : A \to A \times A \) be the morphism \( a' \mapsto (a, a') \). Then

\[
\mu_* (k(a) \boxtimes \mathcal{P}_\alpha) = \mu_* (i_a)_* \mathcal{P}_\alpha.
\]

Notice also that

\[
\mu \circ i_a : A \to A \times A \to A \times A
\]

\[
a' \mapsto (a, a') \mapsto (a' + a, a)
\]

is the graph of the morphism \( t_{-a} : A \to A \). Therefore, \((\mu \circ i_a)_* = \Phi_{\Gamma_{-a}}\), and

\[
(\mu \circ i_a)_* (\mathcal{P}_\alpha) = (\mathcal{O}_A \boxtimes \mathcal{P}_\alpha) \otimes \mathcal{O}_{\Gamma_{-a}}.
\]
Remark 3.2.2. The object \((O_A \boxtimes \mathcal{P}_a) \otimes O_{\Gamma-a} \in D^b(A \times A)\) gives rise to a Fourier-Mukai transform, which is clearly the composition \((\mathcal{P}_a \otimes (\ )) \circ t_{-a^*}\).

Now let \(\Phi_E : D^b(A) \to D^b(B)\) be a derived equivalence between two abelian varieties \(A\) and \(B\). Since the canonical bundle of an abelian variety is trivial, we have

\[
\mathcal{E}_R = \mathcal{E}^\vee[g]
\]

with \(g = \dim A = \dim B\). The Fourier-Mukai transform

\[
\Psi_{\mathcal{E}_R} : D^b(B) \to D^b(A)
\]

is a quasi-inverse of \(\Phi_E\). Moreover, we have the following induced equivalences

\[
\Psi_E : D^b(B) \to D^b(A)
\]

\[
\Phi_{\mathcal{E}_R} : D^b(A) \to D^b(B).
\]

Applying Proposition 2.2.13 gives the product equivalence

\[
\Phi_E \times \Phi_{\mathcal{E}_R} : D^b(A \times A) \to D^b(B \times B)
\]

Notice also that the composition

\[
D^b(A) \xrightarrow{\Phi_{\mathcal{E}_R}} D^b(B) \xrightarrow{\Psi_E} D^b(A)
\]
is the identity.

**Definition 3.2.3.** For each equivalence \( \Phi : D^b(A) \to D^b(B) \) we associate the equivalence

\[
F_\varepsilon : D^b(A \times \hat{A}) \to D^b(B \times \hat{B})
\]

defined by the following diagram

\[
\begin{array}{ccc}
D^b(A \times \hat{A}) & \xrightarrow{F_\varepsilon} & D^b(B \times \hat{B}) \\
id \times \Psi_{\Phi A} \downarrow & & \uparrow (id \times \Psi_{\Phi B})^{-1} \\
D^b(A \times A) & \xrightarrow{\Phi} & D^b(B \times B) \\
\mu_A \downarrow & & \mu_B \\
D^b(A \times A) & \xrightarrow{\Phi \times \Phi B} & D^b(B \times B)
\end{array}
\]

**Lemma 3.2.4.** The above construction \( \Phi_\varepsilon \mapsto F_\varepsilon \) is compatible with composition. More specifically, given a composition of equivalences \( \Phi : D^b(A) \xrightarrow{\Phi} D^b(B) \xrightarrow{\Phi F} D^b(C) \) of abelian varieties, then the isomorphism induced by \( \Phi \) is the composition \( F_\varepsilon \circ F_{\Phi} \) with \( F_\varepsilon \) and \( F_{\Phi} \) the isomorphisms induced by \( \Phi_\varepsilon \) and \( \Phi_\Phi \), respectively.

An equivalence \( D^b(A) \xrightarrow{\xi} D^b(A) \) is referred to as an autoequivalence. Let \( \text{Aut}(D^b(A)) \) denotes the group of autoequivalences. Applying the above association to the case \( B = A \), we obtain

**Corollary 3.2.5.** The map

\[
\Phi_\varepsilon \mapsto F_\varepsilon
\]

\[
\text{Aut}(D^b(A)) \to \text{Aut}(D^b(A \times \hat{A}))
\]
is a group homomorphism.

Orlov and Polishchuk noticed that passing through $\Phi_E \times \Phi_{E_R}$ makes the situation more geometric. We will illustrate this by the following computations in which we give an explicit formula for $F_E$ in special cases

**Example 3.2.6.** (18, Example 9.38)

(a) Suppose that $\Phi_E$ is given by $M \otimes ( ) : D^b(A) \to D^b(A)$. First of all, $E$ is given by $\Delta_* M$ with $\Delta : A \to A \times A$ the diagonal embedding. The inverse functor, $\Psi_{E_R}$, of $\Phi_E$ is then $M^{-1} \otimes ( )$. Hence, $E_R = \Delta_* (M^{-1})$ and $\Phi_E \times \Phi_{E_R} : D^b(A \times A) \to D^b(A \times A)$ is just $(M \boxtimes M^{-1}) \otimes ( )$.

In order to study the associated equivalence $F_E$, we will compute $F_E(k(a) \boxtimes k(\alpha))$ for all closed points $(a, \alpha) \in A \times \hat{A}$. Let $\mathcal{P}$ denote the Poincaré bundle of $A$. Lemma 3.2.1 gives

$$
\mu_*(id \times \Phi_{\mathcal{P}})(k(a) \boxtimes k(\alpha)) = (\mathcal{O}_A \boxtimes \mathcal{P}_\alpha) \otimes \mathcal{O}_{\Gamma-a}.
$$

Hence,

$$
\mathcal{G} := (\Phi_E \times \Phi_{E_R})(\mu_*(id \times \Phi_{\mathcal{P}})(k(a) \boxtimes k(\alpha))) = (M \boxtimes M^{-1}) \otimes ((\mathcal{O}_A \boxtimes \mathcal{P}_\alpha) \otimes \mathcal{O}_{\Gamma-a})
$$

$$
= (M \boxtimes (M^{-1} \otimes \mathcal{P}_\alpha)) \otimes \mathcal{O}_{\Gamma-a}.
$$
Recall from Remark 3.2.2 that $\mu_*(id \times \Phi_*)(k(a) \boxtimes k(\alpha))$ is the kernel of the transform $\mathcal{P}_\alpha \otimes t_{-a*}(\ )$. Applying Proposition 2.2.8b, it follows that $\Phi_\mathcal{G}$ is the composition

$$D^b(A) \xrightarrow{\Phi_\mathcal{G}=M\otimes(\ )} D^b(A) \xrightarrow{\mathcal{P}_\alpha \otimes t_{-a*}(\ )} D^b(A) \xrightarrow{\Psi_{\mathcal{G}R}=M^{-1}\otimes(\ )} D^b(A),$$

i.e. $\Phi_\mathcal{G} = M^{-1} \otimes \mathcal{P}_\alpha \otimes t_{-a*}(M \otimes ( ))$. We then have two cases

**Case 1:** $M \in \text{Pic}^0(A)$. Then $t_{-a*}M \cong M$. Hence

$$\Phi_\mathcal{G} = M^{-1} \otimes \mathcal{P}_\alpha \otimes t_{-a*}(t_{-a}^*M \otimes ( ))$$

$$= M^{-1} \otimes \mathcal{P}_\alpha \otimes M \otimes t_{-a*}(\ )$$

$$= \mathcal{P}_\alpha \otimes t_{-a*}(\ ).$$

Since the functor $\mathcal{P}_\alpha \otimes t_{-a*}(\ )$ is represented by $(\mathcal{O}_{\hat{A}} \boxtimes \mathcal{P}_\alpha) \otimes \mathcal{O}_{\Gamma-a}$, we have

$$\mathcal{G} \cong (\mathcal{O}_{\hat{A}} \boxtimes \mathcal{P}_\alpha) \otimes \mathcal{O}_{\Gamma-a}.$$  

Therefore, $F_{\mathcal{E}}(k(a) \boxtimes k(\alpha)) = k(a) \boxtimes k(\alpha)$ for all closed point $(a, \alpha) \in A \times \hat{A})$. Applying Corollary 2.2.14, we know that $F_{\mathcal{E}}$ is of the form $(N \otimes ( )) \circ f_*$. The above calculation shows that $f$ is the identity. It remains to give a description of $N$. Notice also that $N = F_{\mathcal{E}}(\mathcal{O}_{A \times \hat{A}})$. Following the same line of calculation as above, we can show that

$$F_{\mathcal{E}}(\mathcal{O}_{A \times \hat{A}}) \cong M \boxtimes \mathcal{O}_{\hat{A}}.$$
This implies $F_E$ is isomorphic to $(M \boxtimes \mathcal{O}_A) \otimes (\ )$.

**Case 2:** $M \not\in \text{Pic}^0(A)$. Recall the homomorphism

$$\varphi_M : A \to \hat{A}$$

$$a \mapsto t_a^*M \otimes M^{-1}.$$

We then can write

$$\Phi_G = M^{-1} \otimes \mathcal{P}_a \otimes t_{-a*}(M \otimes t_{-a}^*M^{-1} \otimes t_{-a}^*M \otimes (\ ))$$

$$= M^{-1} \otimes \mathcal{P}_a \otimes t_{-a*}(t_{-a}^*(M \otimes t_{-a}^*M^{-1}) \otimes t_{-a}^*M \otimes (\ ))$$

$$= M^{-1} \otimes \mathcal{P}_a \otimes M \otimes t_{-a}^*M^{-1} \otimes M \otimes t_{-a*}(\ )$$

$$= \mathcal{P}_a \otimes \varphi_{M^{-1}}(-a) \otimes t_{-a*}(\ ).$$

Notice that $\mathcal{P}_a \otimes \varphi_{M^{-1}}(-a) \otimes t_{-a*}(\ )$ is represented by the kernel

$$(\mathcal{O}_A \boxtimes (\mathcal{P}_a \otimes \varphi_{M^{-1}}(-a)) \otimes \mathcal{O}_{\Gamma-a}.)$$

Hence, $F_E(k(a) \boxtimes k(\alpha)) = (k(a) \boxtimes k(\alpha + \varphi_{M^{-1}}(-a)))$. The associated isomorphism $f : A \times \hat{A} \to A \times \hat{A}$ is $(a, \alpha) \mapsto (a, \varphi_{M^{-1}}(-a) + \alpha)$.

(b) Suppose that $\Phi_E = t_{a_0*} : \text{D}^h(A) \to \text{D}^h(A)$ for a closed point $a_0 \in A$. Recall that $\mathcal{E} = \mathcal{O}_{\Gamma_{a_0}}$ and $\mathcal{E}_R = \mathcal{O}_{\Gamma_{a_0}}$; hence, $\Phi_{\mathcal{E}_R} \cong t_{a_0*}$. We follow the same strategy as above and try to compute $F_E(k(a) \boxtimes k(\alpha))$ with $(a, \alpha) \in A \times \hat{A}$ an arbitrary closed point. Applying Lemma 3.2.1 and the fact that $t_{a_0*}\mathcal{P}_a \cong \mathcal{P}_a$ for $\mathcal{P}_a \in \text{Pic}^0(A)$, we have
\[(\Phi_E \times \Phi_{E_R})(\mu_*(id \times \Phi_P)(k(a) \boxtimes k(\alpha))) = (t_{a_0*} \times t_{a_0*})(O_{\hat{A}} \boxtimes \mathcal{P}_\alpha) \otimes O_{\Gamma-a}) \]

\[= (O_{\hat{A}} \boxtimes \mathcal{P}_\alpha) \otimes O_{\Gamma-a}.\]

Again $F_E$ sends $k(a) \boxtimes k(\alpha)$ to itself. Then $F_E$ is of the form $N \otimes ( )$ for some line bundle $N$ on $A \times \hat{A}$. Again, $N$ is given by $F_E(O_{A \times \hat{A}})$, and we have

\[(t_{a_0*} \times t_{a_0*})(\mu_*(id \times \Phi_P)(O_{A \times \hat{A}})) = (t_{a_0*} \times t_{a_0*})(O_{\hat{A}} \boxtimes k(e)[-g]) \]

\[= O_{\hat{A}} \boxtimes k(a_0)[-g].\]

Now notice that $\mu^*(O_{\hat{A}} \boxtimes k(a_0)[-g]) \cong O_{\hat{A}} \boxtimes k(a_0)[-g]$ and $\Phi_P(\mathcal{P}_\alpha^*) \cong k(a_0)[-g]$. Hence, $N = F_E(O_{A \times \hat{A}}) = O_{\hat{A}} \boxtimes \mathcal{P}_\alpha^*$. In other words, the equivalence $F_E : \text{D}^b(A \times \hat{A}) \to \text{D}^b(A \times \hat{A})$ induced by $t_{a_0*} : \text{D}^b(A) \to \text{D}^b(A)$ is given by the tensor product with $O_{\hat{A}} \boxtimes \mathcal{P}_\alpha^*$.

(c) Let $(a, \alpha) \in A \times \hat{A}$ be a closed point and consider

\[\Phi_{(a,\alpha)} := (\mathcal{P}_\alpha \otimes ( )) \circ t_{a*} : \text{D}^b(A) \to \text{D}^b(A).\]

From the calculation in (a) and (b), the induced equivalences for $\mathcal{P}_\alpha \otimes ( )$ and $t_{a*}( )$ are $(\mathcal{P}_\alpha \boxtimes O_A) \otimes ( )$ and $(O_{\hat{A}} \boxtimes \mathcal{P}_\alpha^*) \otimes ( )$, respectively. Applying Corollary 3.2.5, we have

\[F_{(a,\alpha)} = (\mathcal{P}_\alpha \boxtimes O_A) \otimes (O_{\hat{A}} \boxtimes \mathcal{P}_\alpha^*) \otimes ( ) \]

\[= (\mathcal{P}_\alpha \boxtimes \mathcal{P}_\alpha^*) \otimes ( ).\]
(d) Let $\Phi_E$ be the shift functor. Then $F_E : D^b(A \times \hat{A}) \to D^b(A \times \hat{A})$ is in fact the identity. This is because $(\mathcal{O}_\Delta[n])_R \cong \mathcal{O}_\Delta[-n]$.

(e) For completeness, we consider $\Phi_P : D^b(A) \to D^b(\hat{A})$ given by the Poincaré bundle $P$ on $A \times \hat{A}$. Then $F_P : D^b(A \times \hat{A}) \to D^b(\hat{A} \times A)$ is given as

$$(P \otimes ( )) \circ f_P^*$$

with $f_P : A \times \hat{A} \to \hat{A} \times A$ given by

$$(a, \alpha) \mapsto (-\alpha, a).$$

In all of the examples above, the associated equivalence $F_E$ sends closed points to closed points; hence, by Corollary 2.2.14, $F_E$ is given, up to a twist by a line bundle, by an automorphism. This phenomenon in fact holds in general.

**Proposition 3.2.7** (Orlov). Let $\Phi_E : D^b(A) \to D^b(B)$ be an equivalence. The associated equivalence

$$F_E : D^b(A \times \hat{A}) \to D^b(B \times \hat{B})$$

is of the form

$$F_E = (N_E \otimes ( )) \circ f_E^*$$

with $N_E \in \text{Pic}(B \times \hat{B})$ and $f_E : A \times \hat{A} \to B \times \hat{B}$ an isomorphism of abelian varieties.
The automorphism in the Example 3.2.6 (a) - (d), except for Case 2 of (a), is in fact the identity. We give another example when the induced automorphism is slightly more complicated and it will be used later.

**Example 3.2.8.** Let $A$ be an abelian variety, $L$ a line bundle on $A$ and $M$ a line bundle on $\hat{A}$. Let $\Psi : D^b(A) \to D^b(\hat{A})$ be an equivalence defined by

$$\Psi(L) := \mathcal{P} \otimes q^* L \otimes p^* M$$

with $q : A \times \hat{A} \to A$ and $p : A \times \hat{A} \to \hat{A}$. Then the induced automorphism $f_\mathcal{E} : A \times \hat{A} \to \hat{A} \times A$ is given by the matrix

$$\begin{pmatrix}
-\varphi_L & -\mathrm{id}_{\hat{A}} \\
\mathrm{id}_A - \hat{\varphi}_M \circ \varphi_L & -\hat{\varphi}_M
\end{pmatrix}$$

with $\varphi_L : A \to \hat{A}$ and $\hat{\varphi}_M : \hat{A} \to A$ the morphisms induced by $L$ and $M$, respectively. To see this, first notice that $\Phi_{\mathcal{E}}$ is the composition

$$D^b(A) \xrightarrow{L \otimes (\_)} D^b(A) \xrightarrow{\Psi_p} D^b(\hat{A}) \xrightarrow{M \otimes (\_)} D^b(\hat{A}).$$

Applying the calculation in Example 3.2.6 and Corollary 3.2.5, we have that $f_\mathcal{E}$ is given by

$$\begin{pmatrix}
\mathrm{id}_{\hat{A}} & 0 \\
\varphi_{M^{-1}} \circ -\mathrm{id}_A & \mathrm{id}_{\hat{A}}
\end{pmatrix} \begin{pmatrix}
0 & -\mathrm{id}_{\hat{A}} \\
\mathrm{id}_A & 0
\end{pmatrix} \begin{pmatrix}
\mathrm{id}_A & 0 \\
\varphi_{L^{-1}} \circ -\mathrm{id}_A & \mathrm{id}_A
\end{pmatrix} = \begin{pmatrix}
\varphi_{L^{-1}} & -\mathrm{id}_{\hat{A}} \\
-\hat{\varphi}_{M^{-1}} \circ \varphi_{L^{-1}} & \hat{\varphi}_{M^{-1}}
\end{pmatrix}$$

In addition, we can characterize the induced isomorphism $f_\mathcal{E}$ as follows
\textbf{Corollary 3.2.9.} Suppose that $\Phi_E : \mathcal{D}^b(A) \to \mathcal{D}^b(B)$ is an equivalence and the induced isomorphism

$$f_E : A \times \hat{A} \to B \times \hat{B}.$$ 

Then $f_E(a, \alpha) = (b, \beta)$ if and only if

$$\Phi_{(b, \beta)} \circ \Phi_E \cong \Phi_E \circ \Phi_{(a, \alpha)} : \mathcal{D}^b(A) \to \mathcal{D}^b(B).$$

\textbf{Proof.} By Orlov’s Theorem (Proposition 3.2.7), we have $f_E(a, \alpha) = (b, \beta)$ if and only if $F_E(k(a) \boxtimes k(\alpha)) = k(b) \boxtimes k(\beta)$. By Lemma 3.2.1, we know that $\mu_{A*} \circ (\text{id} \times \Phi_{P\hat{A}})$ sends $k(a) \boxtimes k(\alpha)$ to $(\mathcal{O}_{\hat{A}} \boxtimes \mathcal{P}_a) \otimes \mathcal{O}_{\Gamma_{-a}}$. Similarly, $\mu_{B*} \circ (\text{id} \times \Phi_{P\hat{B}})$ sends $k(b) \boxtimes k(\beta)$ to $(\mathcal{O}_{\hat{B}} \boxtimes \mathcal{P}_b) \otimes \mathcal{O}_{\Gamma_{-b}}$. Hence

$$F_E(k(a) \boxtimes k(\alpha)) = k(b) \boxtimes k(\beta)$$

if and only if

$$\Phi_{(b, \beta)} \times \Phi_{E_R}((\mathcal{O}_{\hat{A}} \boxtimes \mathcal{P}_a) \otimes \mathcal{O}_{\Gamma_{-a}}) = (\mathcal{O}_{\hat{B}} \boxtimes \mathcal{P}_b) \otimes \mathcal{O}_{\Gamma_{-b}}.$$ 

Moreover, $(\mathcal{O}_{\hat{A}} \boxtimes \mathcal{P}_a) \otimes \mathcal{O}_{\Gamma_{-a}}$ is the kernel of the equivalence $\Phi_{(a, \alpha)}$ and $(\mathcal{O}_{\hat{B}} \boxtimes \mathcal{P}_b) \otimes \mathcal{O}_{\Gamma_{-b}}$ is the kernel of the equivalence $\Phi_{(b, \beta)}$. Now applying Proposition 2.2.9 gives

$$\Phi_{(b, \beta)} = \Phi_E \circ \Phi_{(a, \alpha)} \circ \Psi_E = \Phi_E \circ \Phi_{(a, \alpha)} \circ \Phi_{E}^{-1},$$

i.e. $\Phi_E \circ \phi_{(a, \alpha)} = \Phi_{(b, \beta)} \circ \Phi_E$. 

$\square$
We can furthermore characterize the isomorphism $f_E$ in the following way. Every isomorphism $f : A \times \hat{A} \to B \times \hat{B}$ can be written as

$$f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

We can associate to $f$ an isomorphism $\tilde{f} : B \times \hat{B} \to A \times \hat{A}$

$$\tilde{f} = \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix}$$

**Definition 3.2.10.** We define the set

$$U(A \times \hat{A}, B \times \hat{B}) := \{ f : A \times \hat{A} \to B \times \hat{B} \text{ isomorphism} \mid f^{-1} = \tilde{f} \}.$$ 

An isomorphism in $U(A \times \hat{A}, B \times \hat{B})$ is called an *isometric isomorphism*.

**Corollary 3.2.11.** The isomorphism $f_E : A \times \hat{A} \to B \times \hat{B}$ associated to an equivalence $\Phi_E : D^b(A) \to D^b(B)$ is contained in $U(A \times \hat{A}, B \times \hat{B})$.

**Proof.** First of all, recall that for every homomorphism $f : A \to B$ of abelian varieties, the dual homomorphism $\hat{f} : \hat{B} \to \hat{A}$ is defined by $\hat{f}(\beta) = \alpha$ if and only if $f^* P_\beta \cong P_\alpha$ on $A$. By Corollary 3.2.5, the identity $\Phi_E \circ \Phi_{(a,\alpha)} = \Phi_{(b,\beta)} \circ \Phi_E$ implies that

$$F_E \circ F_{(a,\alpha)} = F_{(b,\beta)} \circ F_E.$$
Using Proposition 3.2.7, we can write $F_E = f_{E^*} (\ ) \otimes N_E$ with $N_E$ being a line bundle on $B \times \hat{B}$.

Example 3.2.6(c) gives

$$(f_{E^*} (\ ) \otimes N_E) \circ ((\mathcal{P}_a \boxtimes \mathcal{P}_a^{-1}) \otimes (\ )) = ((\mathcal{P}_{\beta \boxtimes \mathcal{P}_b^{-1}}) \otimes (\ )) \circ (f_{E^*} (\ ) \otimes N_E).$$

Applying both sides to $\mathcal{O}_{A \times \hat{A}}$, we obtain

$$f_{E^*} (\mathcal{P}_a \boxtimes \mathcal{P}_a^{-1}) \otimes N_E = (\mathcal{P}_{\beta \boxtimes \mathcal{P}_b^{-1}}) \otimes N_E.$$

This implies

$$f_{E^*} (\mathcal{P}_{\beta \boxtimes \mathcal{P}_b^{-1}}) = (\mathcal{P}_a \boxtimes \mathcal{P}_a^{-1}),$$

i.e. $\hat{f}(-b, \beta) = (-a, \alpha)$. Therefore, $\hat{f}(b, \beta) = (a, \alpha)$, i.e. $\hat{f} = f^{-1}$. \hfill \qed

The converse to the above corollary also holds. The following result is originally due to Polishchuk. An alternative proof was given by Orlov.

**Proposition 3.2.12** (Orlov, Polishchuk). Consider two abelian varieties $A$ and $B$. Any $f \in U(A \times \hat{A}, B \times \hat{B})$ is of the form $f = f_E$ for some equivalence $\Phi_E : D^b(A) \to D^b(B)$. See (30) and (31)

**Corollary 3.2.13.** Two abelian varieties $A$ and $B$ are derived equivalent if and only if there exists an isomorphism of abelian varieties $f : A \times \hat{A} \to B \times \hat{B}$ such that $f^{-1} = \tilde{f}$, i.e. $U(A \times \hat{A}, B \times \hat{B}) \neq \emptyset$. 
3.3 **Objects representing derived equivalences of abelian varieties**

In this section, we will provide more details about the sheaf $\mathcal{E}$ that represents the equivalence $\Phi_{\mathcal{E}} : \text{D}^b(A) \to \text{D}^b(B)$ between two abelian varieties. First of all Orlov showed that

**Proposition 3.3.1.** Let $\Phi_{\mathcal{E}} : \text{D}^b(A) \to \text{D}^b(B)$ be a Fourier-Mukai equivalence between two abelian varieties. Then $\mathcal{E}$ has only one non-trivial cohomology, i.e. $\mathcal{E}$ is a sheaf on $A \times B$ up to a shift. See (30).

Moreover, $\mathcal{E}$ is in fact a semi-homogenous sheaf. The following definition and a complete description of simple semi-homogenous vector bundles can be found in (28).

**Definition 3.3.2.** A sheaf $E$ on $X$ is semi-homogenous if for every $x \in X$, there exists a line bundle $L$ on $X$ such that

$$T_x^*(E) \cong E \otimes L,$$

where $T_x$ is the translate of $A$ by $x$.

We have the following results from Orlov (30).

**Proposition 3.3.3.** Given an isometric isomorphism $f : A \times \hat{A} \to B \times \hat{B}$. One can construct a semi-homogenous vector bundle $\mathcal{E}$ on $A \times B$ such that the functor $\Phi_{\mathcal{E}} : \text{D}^b(A) \to \text{D}^b(B)$ is an equivalence. Moreover, the isometric isomorphism $f_{\mathcal{E}}$ induced by $\Phi_{\mathcal{E}}$ is precisely $f$. See (30, Proposition 4.11 and 4.12).

In general, we have the following result by Gulbrandsen (16).
Proposition 3.3.4. Let $A$ and $B$ be abelian varieties and $E$ a vector bundle on $A \times B$. Then the following are equivalent

a. The Fourier-Mukai transform $\Phi_E : D^b(A) \to D^b(B)$ is an equivalence.

b. The variety $Y$, equipped with the family $E$ is a fine moduli space of simple semi-homogenous vector bundles on $X$.

3.4 Rouquier’s isomorphism

We end this chapter with a general principle behind of Proposition 3.2.7, which is due to Rouquier. Suppose that $F : D^b(X) \to D^b(Y)$ is an equivalence. We then obtain an induced isomorphism between the groups of auto-equivalences

$$\text{Aut}(D^b(X)) \xrightarrow{\sim} \text{Aut}(D^b(Y))$$

$$\Phi \mapsto F^* \Phi := F \circ \Phi \circ F^{-1}.$$  

Equivalently, this isomorphism is given by the commutative diagram

$$
\begin{array}{ccc}
D^b(X) & \xleftarrow{F^{-1}} & D^b(Y) \\
\Phi \downarrow & & \downarrow F^* \Phi \\
D^b(X) & \xrightarrow{F} & D^b(Y).
\end{array}
$$
In addition, if we represent $F$ by a Fourier-Mukai equivalence $\Phi_E$. Then on the level of Fourier-Mukai kernels, the isomorphism $\Phi_R \mapsto \Phi_E \circ \Phi_R \circ \Phi_E^{-1}$ can be described as

$$\mathcal{R} \mapsto F^*\mathcal{R} := (\Phi_E \times \Phi_E)(\mathcal{R})$$

(See Proposition 2.2.9). It is an interesting question to study the image of the identity $\text{id} : D^b(X) \to D^b(X)$. The general idea used to do this is as follows (see (18, Section 9.4)). We consider the semi-direct product $\text{Aut}(X) \ltimes \text{Pic}(X)$ as a subgroup of $\text{Aut}(D^b(X))$ by associating to $(\varphi, L) \in \text{Aut}(X) \ltimes \text{Pic}(X)$ the equivalence $\Phi_{(\varphi, L)} := (L \otimes (\_)) \circ \varphi_*$. On can show that $F^*\Phi_{(\text{id}_X, \mathcal{O}_X)} \cong \Phi_{(\text{id}_Y, \mathcal{O}_Y)}$. And since any open neighborhood of the identity $(\text{id}_X, \mathcal{O}_X) \in \text{Aut}(X) \ltimes \text{Pic}(X)$ generates the connected component $\text{Aut}^0(X) \ltimes \text{Pic}^0(X)$, the isomorphism induces a morphism

$$\text{Aut}^0(X) \ltimes \text{Pic}^0(X) \to \text{Aut}^0(Y) \ltimes \text{Pic}^0(Y).$$

Using the same argument for the inverse $F^{-1}$, one can show that it is in fact an isomorphism.

These are the main ideas for the following theorem.

**Theorem 3.4.1** (Rouquier). *Let $X$ and $Y$ be smooth projective varieties. Any equivalence $F : D^b(X) \to D^b(Y)$ induces an isomorphism of algebraic groups

$$f : \text{Aut}^0(X) \times \text{Pic}^0(X) \xrightarrow{\cong} \text{Aut}^0(Y) \times \text{Pic}^0(Y).$$*
See (34).

In the original paper by Rouquier, the result is stated as an isomorphism of semi-direct products, i.e.
\[ f : \text{Aut}^0 \rtimes \text{Pic}^0(X) \to \text{Aut}^0(Y) \rtimes \text{Pic}^0(Y) \]
is an isomorphism. However, in further analysis as follows we can show that \( \text{Aut}^0(X) \) acts trivially on \( \text{Pic}^0(X) \). Hence, the semi-direct product in the theorem is in fact a direct product.

**Lemma 3.4.2.** Let \( X \) be a smooth projective variety. Then \( \text{Aut}^0(X) \) acts trivially on \( \text{Pic}^0(X) \).

**Proof.** First of all, \( \text{Aut}(X) \) acts on \( \text{Pic}(X) \) as follows: let \( \varphi \in \text{Aut}(X) \), then \( \varphi \) acts on \( \text{Pic}(X) \) by \( \varphi^* : \text{Pic}(X) \to \text{Pic}(X) \). Then for every \( \varphi \in \text{Aut}^0(X) \), \( \varphi^* \in \text{Aut}^0(\text{Pic}^0(X)) \). On the other hand, \( \text{Pic}^0(X) \) is an abelian variety; hence, \( \text{Aut}^0(\text{Pic}^0(X)) \cong \text{Pic}^0(X) \) and elements of \( \text{Aut}^0(\text{Pic}^0(X)) \) are translations. Thus, \( \varphi^* \) corresponds to a translation on \( \text{Pic}^0(X) \). Since \( \varphi^* \mathcal{O}_X = \mathcal{O}_X \) and translations do not fix the origin, it means that \( \varphi^* \) acts trivially on \( \text{Pic}^0(X) \). \( \square \)

**Remark 3.4.3.** For abelian varieties \( A \) and \( B \) considered in the previous section, this is exactly what is expressed by Proposition 3.2.7 after identifying \( A \cong \text{Aut}^0(A) \) via \( a \mapsto t_a \) and \( \hat{A} \cong \text{Pic}^0(A) \).

Finally, we also have a characterization of the Rouquier isomorphism which is similar to that of the induced isomorphism \( f_E \) in the case of abelian varieties.

**Proposition 3.4.4.** Let \( F \) be represented by \( \Phi_E \). Then \( f(\varphi, L) = (\psi, M) \) if and only if
\[ (\varphi \times \psi)^* \mathcal{E} \cong (L^{-1} \boxtimes M) \otimes \mathcal{E}. \]
See (33, Lemma 3.1).
4.1 Preliminaries on actions of algebraic groups

We start with a classical structure theorem about connected algebraic groups by Chevalley.

**Theorem 4.1.1 (Chevalley).** Let $G$ be a connected algebraic group. Then there exists a unique maximal connected affine subgroup, denoted $\text{Aff}(G)$, of $G$ such that the quotient of $G$ by $\text{Aff}(G)$, is an abelian variety, i.e. we have a short exact sequence of connected algebraic groups

$$0 \to \text{Aff}(G) \to G \xrightarrow{\alpha_G} \text{Alb}(G) \to 0.$$  \hfill (4.1)

*See (11, Section 0).*

**Remark 4.1.2.** The morphism $\alpha_G : G \to \text{Alb}(G)$ has the universal property of the usual Albanese morphisms for projective varieties. We will refer to $\text{Alb}(G)$ as the Albanese variety of $G$ or the compact part of $G$.

Given a variety $X$ and a connected algebraic group $G$. We say that $G$ acts on $X$ if there is a morphism satisfying the usual properties of group actions (see (11, Section 0))

$$\lambda : G \times X \to X.$$
Many properties of $G$ can be studied through its action on $X$. In particular, the affine part of $G$ is directly related to the group of birational morphisms of $X$. On the other hand, the compact part of $G$ is studied through its action on the Albanese of $X$, $\text{Alb}(X)$. The following analysis is mainly due to Matsumura and Nishi (15).

Let us denote by $\text{Bir}(X)$ the group of birational morphisms of $X$. Notice that $\text{Bir}(X)$ is the group of automorphisms of the function field of $X$, $k(X)$. Clearly, there is a homomorphism $G \to \text{Bir}(X)$. Moreover, when $G$ acts on $X$ faithfully (i.e. if $g \cdot x = x$ for all $x \in X$, then $g$ is the identity), this homomorphism is injective. Moreover, if two connected algebraic groups $G$ and $G'$ act on $X$ faithfully and have the same image in $\text{Bir}(X)$, then they are necessarily isomorphic. Thus, we can speak of algebraic subgroups of $\text{Bir}(X)$ without ambiguity. The following propositions taken from (27) give a condition for the existence of an affine subgroup of the group of birational morphisms of $X$.

**Proposition 4.1.3.** The group $\text{Bir}(X)$ contains a linear algebraic group of positive dimension if and only if $X$ is birationally equivalent to the product of $\mathbb{P}^1$ and another variety. In particular, if a complete non-singular variety $X$ has a multicanonical system $|nK|$ for some $n > 0$, then $\text{Bir}(X)$ cannot contain any linear algebraic group of positive dimension. See (27, Corollary 1).

**Proposition 4.1.4.** If $X$ is complete and non-singular, and if the rational mapping of $X$ determined by $|nK|$ is birational for some $n > 0$, then $\text{Bir}(X)$ is a finite group. See (27, Corollary 2).

The above propositions allow us to give a description of the linear part of a connected algebraic group $G$ acting faithfully on $X$. In particular, we will consider the case when $G =$
\text{Aut}^0(X)$, the connected component of Aut$(X)$ containing the identity. Clearly, Aut$(X)$ acts faithfully on $X$. We have an inclusion Aut$(X) \hookrightarrow $ Bir$(X)$.

**Corollary 4.1.5.** Let $X$ be a smooth projective variety of general type. Then Bir$(X)$ and Aut$(X)$ are finite.

*Proof.* Clearly, this follows directly from Proposition 4.1.4. □

**Corollary 4.1.6.** Let $X$ be a smooth projective variety of non-negative Kodaira dimension then Aut$^0(X)$ is an abelian variety.

*Proof.* If Aut$^0(X)$ has a linear part of positive dimension, then Proposition 4.1.3 implies that $X \sim_{\text{bir}} V \times \mathbb{P}^1$ for some smooth variety $V$. This contradicts the non-negative Kodaira dimension condition. Hence, dim Aff(Aut$^0(X)) = 0$. □

We now turn to the abelian part of a connected algebraic group. Again, let $G$ be a connected algebraic group acting faithfully on a variety $X$. Matsumura and Nishi, in (27), showed the following estimate

**Proposition 4.1.7.**

$$\dim \text{Alb}(G) \leq q(X) = \dim \text{Alb}(X)$$

Let $\alpha_X : X \rightarrow \text{Alb}(X)$ be the Albanese morphism of $X$. The action of $G$ on $X$ induces a morphism

$$G \times X \rightarrow \text{Alb}(G \times X).$$
Since \( \text{Alb}(G \times X) = \text{Alb}(G) \times \text{Alb}(X) \) and, by the universal property of Albanese morphisms, we have the following commutative diagram

\[
\begin{array}{ccc}
G \times X & \xrightarrow{\lambda} & X \\
\downarrow{\alpha_G \times \alpha_X} & & \downarrow{\alpha_X} \\
\text{Alb}(G) \times \text{Alb}(X) & \xrightarrow{\beta} & \text{Alb}(X),
\end{array}
\]

where \( \beta \) is uniquely defined and \( \alpha_G : G \to \text{Alb}(G) \) is the homomorphism defined in Chevalley’s Theorem. Moreover, as a morphism between abelian varieties, \( \beta \) is a composition of a homomorphism and a translation. Observe that

\[
\beta(0,z) = z, \quad \forall x \in \text{im}(\alpha_X).
\]

Since \( \text{im}(\alpha_X) \) is not contained in any translation of a smaller abelian variety, it follows that

\[
\beta(0,z) = z, \quad \forall z \in \text{Alb}(X).
\]

Thus, we can write

\[
\beta(y,z) = \alpha_\lambda(y) + z
\]

where \( \alpha_\lambda : \text{Alb}(G) \to \text{Alb}(X) \) is a homomorphism. In other words, the action of \( G \) on \( X \) induces an action of \( \text{Alb}(G) \) on \( \text{Alb}(X) \) by translations via \( \alpha_\lambda \).

**Theorem 4.1.8** (Matsumura-Nishi). *Let \( G \) be a connected algebraic group acting faithfully on a variety \( X \). Then, using the notation as above, the induced morphism \( \alpha_\lambda : \text{Alb}(G) \to \text{Alb}(X) \)
has finite kernel. And equivalently, the induced morphism $G \xrightarrow{\alpha_G} \text{Alb}(G) \to \text{Alb}(X)$ is an affine morphism. See (27, Theorem 2).

Remark 4.1.9.

- Since $\alpha_\lambda$ is finite, $\dim \text{Alb}(G) \leq \dim \text{Alb}(X) = q(X)$. Hence, we obtain Proposition 4.1.7.
- Let $G$ act on itself via left multiplication. It follows that the Albanese morphism

$$\alpha_G : G \to \text{Alb}(G)$$

is a surjective group homomorphism with affine kernel. It can be shown that this kernel is smooth and connected. Thus, we obtain the Chevalley’s Structure Theorem.

In a more general context, Brion proved, in (11), the following analogue of the Matsumura-Nishi Theorem.

**Theorem 4.1.10 (Brion).** Let $X$ be a normal, quasi-projective variety on which $G$ acts faithfully. Then there exists a $G$-morphism

$$\psi : X \to A$$

where $A$ is a quotient of $\text{Alb}(G)$ by a finite subgroup scheme. Moreover, $X$ admits a $G$-equivariant embedding into the projectivization of a $G$-homogeneous vector bundle over $A$. See (11, Section 3).

Remark 4.1.11.
a. Notice that $\psi$ is not unique as we can replace $A$ with any finite quotient.

b. In characteristic 0, we can assume that the fibers of $\psi$ are normal varieties by using Stein factorization.

c. The existence of a $G$-morphism $\psi$ as in Theorem 4.1.8 is equivalent to the kernel of the morphism $G \to \text{Alb}(X)$ being affine, and also to the kernel of $\alpha_\lambda$ being finite.

d. When $X$ is normal and almost homogeneous, the morphism $\psi$ has the universal property, namely, $\psi$ is the Albanese morphism.

4.2 Actions of automorphism groups.

In this section, we specialize to the group of automorphisms of a smooth projective variety. More precisely, let $X$ be a smooth projective variety, and let $G_X$ denote $\text{Aut}^0(X)$, the connected component of $\text{Aut}(X)$ containing the identity. From the previous section, the action of $G_X$ on $X$ induces a homomorphism

$$\alpha : \text{Alb}(G_X) \to \text{Alb}(X)$$

with finite kernel. Moreover, we have the commutative diagram

$$\begin{array}{ccc}
\text{Alb}(G_X) \times \text{Alb}(X) & \xrightarrow{\beta} & \text{Alb}(X) \\
\downarrow & & \downarrow \\
\text{Alb}(G_X) & \xrightarrow{\alpha} & \text{Alb}(X) \leftarrow \alpha_X \quad X
\end{array}$$

From the discussion in Section 3 of (27) (also see (33, Lemma 2.2)) it follows that

$$\text{im}(\beta) \subseteq \alpha_X(X);$$
hence,

\[ \alpha(\text{Alb}(G_X)) \subseteq \alpha_X(X). \]

Since \( \alpha \) has finite kernel, we obtain the following Lemma

**Lemma 4.2.1.**

\[ \dim \text{Alb}(G_X) \leq \dim \text{im}(\alpha_X) \leq \min\{\dim X, q(X)\}. \]

Moreover, the analysis of Matsumura-Nishi and Brion can be carried out in further details. In particular, we have the following result.

**Proposition 4.2.2.** There is an affine subgroup \( \text{Aff}(G_X) \subset H \subset G_X \) with \( H/\text{Aff}(G_X) \) finite, such that \( X \) admits a \( G_X \)-equivariant map \( \psi : X \to G_X/H \). Consequently, \( X \) is isomorphic to the equivariant fiber bundle \( G_X \times^H Z \) with fiber \( Z = \psi^{-1}(0) \) being connected. See (11, Section 3) and (33, Lemma 2.4).

**Proposition 4.2.3.** With the same notation in the previous proposition, we have

a. \( G_X \to G_X/H \) is a principal \( H \)-bundle.

b. \( X \to G_X/H \) and \( G \to G/H \) are isotrivial. In particular, all the smooth fibers of \( X \to G_X/H \) are isomorphic to \( Z \).

c. If \( \dim \text{Alb}(G_X) > 0 \) (i.e. \( G_X \) is not affine), then \( \chi(\mathcal{O}_X) = 0 \).

See (33).
Remark 4.2.4. In the same setting as the previous two propositions, there is a surjective $G_X$-morphism

$$\text{Alb}(X) \to G_X/H,$$

and the induced $G_X$-equivariant map $\psi$ is the composition $X \xrightarrow{\alpha_X} \text{Alb}(X) \to G_X/H$. In addition, one has in mind the following commutative diagram

$$
\begin{array}{cccccc}
0 & \to & H/\text{Aff}(G_X) & \to & G_X/\text{Aff}(G_X) & \to & G/H & \to & 0 \\
& & \downarrow= & & \downarrow= & & \downarrow= & & \\
0 & \to & F & \to & \text{Alb}(G_X) & \to & \text{Alb}(G_X)/F & \to & 0.
\end{array}
$$

Applying these results we can analyze the extreme cases of Lemma 4.2.1 as follows. We first consider the case of $\dim \text{Alb}(G_X) = 0$. By Corollary 4.1.5, varieties of general type are examples of this case, and in this case, $\dim \text{Aff}(G_X) = 0$. If $\text{Aff}(G_X)$ is of positive dimension, then $\text{Bir}(X)$ contains a linear algebraic group of positive dimension and, by Proposition 4.1.3, $X$ is birational to $\mathbb{P}^1 \times V$ with $V$ a smooth projective variety. In the case of $\dim \text{Alb}(G_X) = \dim X$, we have the following result.

**Proposition 4.2.5.** Let $X$ be a smooth projective variety with $\dim \text{Alb}(G_X) = \dim X$. Then $X$ is an abelian variety.

**Proof.** The equality $\dim \text{Alb}(G_X) = \dim \text{im}(\alpha_X) = \dim X = n$ implies that $X$ is of maximal Albanese dimension and $\alpha(\text{Alb}(G_X)) = \alpha_X(X)$. Since $\alpha_X(X)$ generates $\text{Alb}(X)$, the homomorphism $\alpha : \text{Alb}(G_X) \to \text{Alb}(X)$ is surjective (with finite kernel). Thus, $q(X) = n$. By Proposition 4.2.2, $X$ admits a $G$-equivariant map $\psi : X \to G/H$ with connected fiber $Z$. No-
Notice that $G/H \cong \text{Alb}(G_X)/F$ with $F = H/\text{Aff}(G_X)$ being finite. Hence, $\psi$ must be a finite morphism. The fact that $Z$ is connected implies that $\psi$ is an isomorphism, i.e. $X$ is an abelian variety.

Another consequence of Proposition 4.2.2 and Proposition 4.2.3 is the following theorem about the automorphism groups of derived equivalent surfaces. We leave the proof of this theorem to the next chapter.

**Theorem 4.2.6.** Let $X$ and $Y$ be smooth projective surfaces with $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$. Then

$$\text{Aut}^0(X) \cong \text{Aut}^0(Y),$$

unless $X$ is an abelian surface, in which case, we have $\mathcal{D}^b(\text{Aut}^0(X)) \cong \mathcal{D}^b(\text{Aut}^0(Y))$. See Theorem 5.3.1a.

We conclude this chapter with some calculations of the automorphism groups of minimal surfaces. These results will be used in the proof of the above theorem and in the next chapter.

### 4.3 Automorphism groups of minimal surfaces

Throughout this section, let $X$ be minimal smooth projective surface over an algebraically closed field $k$. We will give a description of the group $G_X = \text{Aut}^0(X)$ for minimal surfaces. By Corollary 4.1.5, for $X$ a surface of general type, we know that $\text{Aut}^0(X) = \{e\}$. Hence, we are left with the cases of Kodaira dimension $-\infty$, 0, and 1. We begin with surfaces of Kodaira dimension 0.
4.3.1 **Kodaira dimension 0**

By the Kodaira classification of minimal surfaces, we know that $X$ is an abelian surface, a K3 surface, an Enriques surface or a bi-elliptic surface. Moreover, Corollary 4.1.6 states that $G_X$ in this case is an abelian variety, i.e. $\dim \text{Aff}(G_X) = 0$. We proceed in a case-by-case basis as follows.

(a) If $X$ is an abelian surface, then it is clear that $G_X \cong X$.

(b) For $X$ a K3 surface or an Enriques surface, we have $q(X) = 0$. Proposition 4.1.7 implies that $\dim \text{Alb}(G_X) = 0$, i.e. $G_X = \{e\}$.

(c) Assume that $X$ is a bi-elliptic surface. Then $q(X) = 1$. By the definition of bi-elliptic surface (see (1), (2)), we can write $X \cong (E \times F)/G$ with $E$ and $F$ elliptic curves and $G$ a finite group acting on $E$ by translations and on $F$ as a subgroup of $\text{Aut}(F)$ such that $F/G \cong \mathbb{P}^1$. It is clear that $E$ acts on $X$; hence, $E \subset \text{Aut}^0(X)$. The fact that $\dim \text{Alb}(G_X) \leq q(X) = 1$ implies that $\text{Aut}^0(X) \cong E$.

4.3.2 **Kodaira dimension 1**

This is the case of properly elliptic surface, i.e. $X$ admits an elliptic fibration $\pi : X \to C$ with $g(C) \geq 2$. Recall that an elliptic fibration $\pi : X \to C$ is a flat surjective morphism with connected fibers and the general fiber is an elliptic curve. By Proposition 4.1.6, we know that $\dim \text{Aff}(G_X) = 0$ and $G_X = \text{Alb}(G_X)$ is an abelian variety. By Lemma 4.2.1 and Proposition
4.2.5, it follows that \( \dim G_X \) is either 0 or 1. We start with the case \( \dim G_X = 1 \). It follows from Proposition 4.2.2 that \( X \) admits a \( G_X \)-equivariant morphism

\[ \psi : X \to G_X/H \]

with \( H \) a finite subgroup of \( G_X \) (since \( \dim \text{Aff}(G_X) = 0 \)). Since \( \dim G_X/H = 1 \), the fiber \( Z = \psi^{-1}(0) \) is a smooth curve. Moreover, we can write

\[ X \cong G_X \times^H Z = (G_X \times Z)/H. \]

This implies that \( X \) admits a morphism \( X = (G_X \times Z)/H \to Z/H \) whose general fiber is \( G_X \).

Since elliptic surface of non-zero Kodaira dimension admits a unique elliptic fibration (see (37)), it follows then that \( G_X \) is a general fiber of the fibration \( \pi : X \to C \) and \( Z/H \cong C \). Moreover, Corollary 4.2.3 c) gives that \( \chi(O_X) = 0 \). This condition is equivalent to the fact that \( X \to C \) is a properly elliptic surface with constant \( j \)-invariant (see Lemma 5.2.1 and Remark 5.2.3). It follows then that the case \( \dim G_X = 0 \) (i.e. \( G_X = \{e\} \)) corresponds to the case of properly elliptic surfaces with non-constant \( j \)-invariant.

The above argument gives the following proposition.

**Proposition 4.3.1.** Let \( \pi : X \to C \) be a properly elliptic surface. If \( X \) has non-constant \( j \)-invariant, then \( G_X = \{id\} \). If \( X \) has constant \( j \)-invariant, then \( G_X \) is isomorphic to the
general fiber of $\pi$. Moreover, in that case, there exists a finite subgroup $H$ of $G_X$ and a smooth curve $Z$ of genus at least 2 such that the elliptic fibration $\pi$ is in fact

$$\pi : X \cong (G_X \times Z)/H \to Z/H$$

and $Z/H \cong C$.

4.3.3 Kodaira dimension $-\infty$

This is the case of ruled surfaces and $\mathbb{P}^2$. The classification of ruled surfaces and their automorphism groups are studied extensively by Maruyama in (25) and (26). Let $\pi : X \to C$ be a ruled surface over a curve $C$. Then we can write $X$ as a projective bundle $X = \mathbb{P}(\mathcal{E}) \to C$ with $\mathcal{E}$ being a rank 2 vector bundle on $C$. In short, the automorphism group of $X$ is computed by, first, relating $\text{Aut}(X)$ with $\text{Aut}_C(X)$, the group of automorphisms of $X$ that fix the base curve $C$, (see (26, Lemma 4)) and, second, relating $\text{Aut}_C(X)$ with $\text{Aut}(\mathcal{E})$ (see (26, Lemma 1)). We have the following results taken from (25) and (26).

**Proposition 4.3.2.** If $g(C) \geq 2$ and $X \neq \mathbb{P}^1 \times C$, then $\text{Aut}(X)$ is a finite group and, thus, $G_X = \{e\}$. In the case $X \cong \mathbb{P}^1 \times C$, we have $\text{Aut}(X) \cong \text{Aut}(\mathbb{P}^1) \times \text{Aut}(C) = \text{PGL}(1) \times \text{Aut}(C)$ and $G_X = \text{PGL}(1)$, with $\text{PGL}(1)$ being the projective general linear group of $2 \times 2$ matrices over $k$. See (25, Lemma 7).
Proposition 4.3.3. If $C$ is rational, i.e. $C = \mathbb{P}^1$, then $X \cong F_n$ with $n \geq 0$. Recall that $F_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. In this case, the $\text{Aut}(X)$ is described by the following exact sequence of algebraic groups

$$e \to H_{n+1} \to \text{Aut}(S) \to \text{PGL}(1) \to e,$$

with

$$H_{n+1} = \left\{ \left( \begin{array}{cc} \alpha & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} \alpha & t_1 \\ 0 & 1 \end{array} \right), \ldots, \left( \begin{array}{cc} \alpha & t_n \\ 0 & 1 \end{array} \right) \right\} | t_1, \ldots, t_n \in k \text{ and } \alpha \in \mathbb{G}_m \right\}.$$

See (26, Theorem 3(1)).

Proposition 4.3.4. Assume that $g(C) = 1$, i.e. $C$ is an elliptic curve. We have the following cases

a. Case 1: $X \cong \mathbb{P}^1 \times C$. Similar to the case in Proposition 4.3.1, we have $\text{Aut}(X) \cong \text{PGL}(1) \times \text{Aut}(C)$ and $G_X \cong \text{PGL}(1) \times \text{Aut}^0(C) \cong \text{PGL}(1) \times C$.

b. Case 2: $\mathcal{E}$ is decomposable and $X \neq \mathbb{P}^1 \times C$. The description of $\text{Aut}(X)$ in this case is rather involved. And since it will not be used in any other part of this paper, we refer the reader to (26, Theorem 3(2)).

c. Case 3: $\mathcal{E}$ is given by the non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0.$$
Then $\text{Aut}(X)$ is described by the following exact sequence of algebraic groups

\[ e \to G_a \to \text{Aut}(X) \to \text{Aut}(C) \to e \]

Moreover, $\text{Aut}^0(X)$ is commutative group and it is a non-trivial extension of $\text{Aut}^0(C)$ by $G_a$.

d. Case 4: $E$ is given by the non-split extension

\[ 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(p) \to 0, \]

with $p$ a point on $C$. Then there is an exact sequence of algebraic groups

\[ e \to \Delta \to \text{Aut}(X) \to \text{Aut}(C) \to e, \]

where $\Delta \cong (\mathbb{Z}/2\mathbb{Z})^2$ is the group of $2$-torsion points of $C$. Moreover, $\Delta \cap \text{Aut}^0(X) = \Delta$.

Hence, we have

\[ e \to \Delta \to G_X \to \text{Aut}^0(C) \to e. \]

Since $C$ is an elliptic curve, $\text{Aut}^0(C) = C$; hence, the morphism $G_X \to C$ is with kernel $\Delta = (\mathbb{Z}/2\mathbb{Z})^2$ is the multiplication by 2. Thus, $G_X \cong C$.

**Lemma 4.3.5.** Let $X \to E$ be a ruled surface that admits an elliptic fibration. Then $E$ is an elliptic curve and $G_X$ is either $\text{PGL}(1) \times E$ or $E$. In particular, $\text{Alb}(G_X) \cong E$. 
Proof. We are working over a field of characteristic 0. By Theorem 4 in (26), $X$ is an elliptic surface if and only if $E$ is an elliptic curve and $X$ is a projective bundle $\mathbb{P}(\mathcal{E}) \to E$ with $\mathcal{E}$ is given either by

i. $\mathcal{E} = \mathcal{O}_E \oplus \mathcal{O}_E$ or

ii. the non-split sequence

$$0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{E}_E(p) \to 0,$$

where $p$ is a point on $E$.

In case (i), we have $G_X \cong \text{PGL}(1) \times E$ (by Proposition 4.3.4(a)). In case (ii), we have $G_X \cong E$ (by Proposition 4.3.4(d)). Hence, $\text{Alb}(G_X) \cong E$. \qed
5.1 Ruled surfaces

We start with the following theorem.

**Theorem 5.1.1.** Let $X_1$ and $X_2$ be $\mathbb{P}^1$-bundles over elliptic curves $E_1$ and $E_2$, respectively. If $D^b(X_1) \cong D^b(X_2)$, then $E_1 \cong E_2$.

**Proof.** If $X_1$ is a ruled surface that does not admit an elliptic fibration, then by Theorem 2.3.3, $X_1 \cong X_2$; hence, $E_1 \cong E_2$. The remaining case is the case when $X_1$ and $X_2$ are ruled surfaces that admit an elliptic fibration. Standard results about ruled surfaces show that $X_1$ and $X_2$ admit an elliptic fibration over $\mathbb{P}^1$. By Lemma 4.3.5, we have $\text{Alb}(G_{X_i}) \cong E_i$. Thus, applying Proposition 4.2.2 and Remark 4.2.4, we can write

$$X_i \cong (\text{Alb}(G_{X_i}) \times Z_i)/F_i = (E_i \times Z_i)/F_i$$

with $Z_i$ a smooth curve and $F_i$ a finite subgroup of $E_i$, $i = 1, 2$. Notice that the elliptic fibrations of $X_1$ and $X_2$ are the projections $X_i = (\text{Alb}(G_{X_i}) \times Z_i)/F_i \to Z_i/F_i$ with generic fiber $E_i$. Since an elliptic surface of non-zero Kodaira dimension admits a unique elliptic fibration (see (37)), it
means $Z_i/F_i \cong \mathbb{P}^1$. By Theorem 2.3.9, the fact that $D^b(X_1) \cong D^b(X_2)$ implies that the general fibers of $X_1 \to \mathbb{P}^1$ and $X_2 \to \mathbb{P}^1$ are isomorphic; i.e. $E_1 \cong E_2$. \hfill \Box

\textbf{Remark 5.1.2.}

\begin{enumerate}[a.]
\item A similar argument using Proposition 4.2.2 and Remark 4.2.4 as above can also be applied to properly elliptic surfaces. This is done in Proposition 5.2.5.
\item It is noteworthy to point out that applying Rouquier’s Theorem in this situation gives

$$\text{Pic}^0(X_1) \times G_{X_1} \cong \text{Pic}^0(X_2) \times G_{X_2}.$$ 

It is a standard result that $\text{Pic}^0(X_i) = \text{Jac}(E_i) = E_i$; hence,

$$E_1 \times G_{X_1} \cong E_2 \times G_{X_2}.$$ 

By Chevalley’s Theorem, we have that $E_1 \times \text{Alb}(\text{Aut}^0(X_1)) \cong E_2 \times \text{Alb}(\text{Aut}^0(X_2))$, i.e. $E_1 \times E_1 \cong E_2 \times E_2$. Nevertheless, this fact does not suffice to prove that $E_1$ and $E_2$ are isomorphic. There are examples of elliptic curves $E$ and $F$ (with large endomorphism rings) such that $E \times E \cong F \times F$ but $E \neq F$ (see (38), (32)).

An immediate corollary is

\textbf{Corollary 5.1.3.} \textit{Let $X_1$ and $X_2$ be ruled surfaces with $D^b(X_1) \cong D^b(X_2)$ then}

\begin{enumerate}[a.]
\item $\text{Pic}^0(X_1) \cong \text{Pic}^0(X_2)$,
\end{enumerate}
b. \(G_{X_1} \cong G_{X_2}\).

**Proof.** As above, if \(X_1\) does not admit an elliptic fibration then Theorem 2.3.3 implies that \(X_1\) and \(X_2\) are isomorphic. Now assume that \(X_1\) and \(X_2\) are ruled surfaces that admit an elliptic fibration. Then by (26, Theorem 4) \(X_i\) is ruled over an elliptic curve \(E_i\) for \(i = 1, 2\). The above proposition then implies \(E_1 \cong E_2\); hence, \(\text{Pic}^0(X_1) \cong \text{Pic}^0(X_2)\). The fact that \(q(X_1) = q(X_2) = 1\) implies that the Albanese dimensions of \(X_1\) and \(X_2\) are both 1. By Lemma 4.3.5, either \(G_{X_i} = E_i\) or \(G_{X_i} = \text{PGL}(1) \times E_i\). Rouquier’s Theorem together with Chevalley’s Theorem imply that the affine part of \(G_{X_1}\) and \(G_{X_2}\) are isomorphic. Therefore, \(G_{X_1} \cong G_{X_2}\). \(\square\)

### 5.2 Properly elliptic surfaces

We begin with some standard facts about elliptic surfaces. Let \(\pi : X \to C\) be an elliptic fibration, i.e. \(\pi\) is a flat surjective morphism from a smooth projective surface to a smooth curve with connected fibers and the general fiber is a smooth elliptic curve. Hence,

\[
\pi_* \mathcal{O}_X = \mathcal{O}_C.
\]

Moreover, the sheaf \(L := (R^1\pi_* \mathcal{O}_X)^\vee\) is a line bundle. And

**Lemma 5.2.1.** \(\chi(\mathcal{O}_X) = \deg L \geq 0\). Moreover, \(L \cong \mathcal{O}_C\) if and only if \(q(X) = g(C) + 1\), and \(L \not\cong \mathcal{O}_C\) if and only if \(q(X) = g(C)\).

**Proof.** The results follow from the Leray spectral sequence

\[
E_2^{p,q} = H^p(C, R^q\pi_* \mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X),
\]
and the induced equality

\[ h^1(X, \mathcal{O}_X) = h^0(C, R^1\pi_*\mathcal{O}_X) + h^1(C, \pi_*\mathcal{O}_X) = h^0(C, R^1\pi_*\mathcal{O}_X) + h^1(C, \mathcal{O}_C). \]

It is clear from the spectral sequence above that \( \chi(\mathcal{O}_X) = \deg L \). We claim that \( \deg R^1\pi_*\mathcal{O}_X \leq 0 \). Hence, \( h^0(C, R^1\pi_*\mathcal{O}_X) = 1 \) if and only if \( R^1\pi_*\mathcal{O}_X \cong \mathcal{O}_C \). See (1, V.12), (13, Section 1.3.5).

The Albanese variety of \( \pi : X \to C \) is characterized as follows.

**Proposition 5.2.2.** Let \( X \) be a properly elliptic surface with an elliptic fibration \( \pi : X \to C \).

Then

a. either \( \text{Alb}(X) \cong \text{Jac}(C) \), or

b. there exists an exact sequence of abelian varieties

\[ 0 \to F \to \text{Alb}(X) \xrightarrow{\tilde{\pi}} \text{Jac}(C) \to 0, \]

with \( F := \ker(\tilde{\pi}) \) being an elliptic curve isogenous to the general fiber of \( \pi \), and \( \tilde{\pi} : \text{Alb}(X) \to \text{Jac}(C) \) the morphism induced by the universal property of Albanese varieties.

Case a) is equivalent to \( q(X) = g(C) \), and case b) is equivalent to \( q(X) = g(C) + 1 \). See (2).
Proof. Consider the following commutative diagram induced by the universal property of the Albanese morphism

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & C \\
\downarrow{\alpha} & & \downarrow{j} \\
\text{Alb}(X) & \xrightarrow{\bar{\pi}} & \text{Jac}(C)
\end{array}
\]

with \(\alpha\) and \(j\) being the Albanese morphisms of \(X\) and \(C\), respectively. Since \(C\) is a smooth curve, \(j\) is an injective morphism. Since \(\bar{\pi} \circ \alpha = j \circ \pi\) and \(\text{im}(j)\) generates \(\text{Jac}(C)\), it is clear that \(\bar{\pi}\) is a surjective homomorphism. Lemma 5.2.1 states that either \(g(X) = g(C)\) (iff \(R^1 \pi_* \mathcal{O}_X \ncong \mathcal{O}_C\)) or \(g(X) = g(C) + 1\) (iff \(R^1 \pi_* \mathcal{O}_X \cong \mathcal{O}_C\)). Before proceeding, let us observe the following.

(i) Since \(\pi\) has connected fibers and \(j\) is injective, the composition \(\bar{\pi} \circ \alpha\) has connected fibers.

(ii) We can fix a general point \(b_0 \in C\) such that \(j(b_0) = 0\) and the fibre \(F_0\) of \(\pi\) over \(b_0\) is a smooth fibre. And the commutativity of the above diagram implies that \(\alpha(F_0) \subset \ker(\bar{\pi})\).

(iii) For every general fiber \(F_b\) of \(\pi\) over a point \(b \in C\), the Albanese morphism \(F_b \to \text{Alb}(F_b)\) is surjective. Hence, \(\alpha(F_b)\) is a translate of a fixed sub-abelian variety, \(K\), of \(\ker(\bar{\pi})\), i.e.

for every general \(F_b\) there exists a unique \(p_b \in \text{Alb}(X)\) such that \(F_b = p_b + K\).

Notice that (iii) gives a rational map \(f : C \to \text{Alb}(X)/K\). Since \(\text{Alb}(X)/K\) is an abelian variety, this map extends to a morphism. In addition, we have the induced homomorphism \(\bar{f} : \text{Jac}(C) \to \text{Alb}(X)/K\) by the universal property of Albanese varieties. By construction, the image of \(f\) generates \(\text{Alb}(X)/K\); hence, the homomorphism \(\bar{f}\) is surjective.
Assume that \( q(X) = g(C) + 1 \). The surjectivity of \( \bar{f} \) implies that

\[
q(X) - \dim K \leq g(C),
\]
equivalently, \( \dim K \geq 1 \). Since \( K \subset \ker(\bar{\pi}) \), it follows that \( K = \ker(\bar{\pi}) \). Therefore, \( \alpha : F_0 \to \ker(\bar{\pi}) \) is an isogeny.

Assume that \( q(X) = g(C) \). In this case, \( K \) is a point, i.e. \( \alpha \) contracts all the fibers \( F_b \). By construction, it follows that \( \alpha = \pi \circ f \). We then obtain the following

\[
\alpha = f \circ \pi = \bar{f} \circ f \circ \pi = \bar{f} \circ \bar{\pi} \circ \alpha,
\]
i.e. \( \bar{f} \circ \bar{\pi} \) is the identity on \( \text{im}(\alpha) \). Since \( \text{im}(\alpha) \) generates \( \text{Alb}(X) \), it means that \( \bar{f} \circ \bar{\pi} = \text{id} \), i.e. \( \text{Alb}(X) \cong \text{Jac}(C) \).

**Remark 5.2.3.** By applying Lemma 5.2.1, it is clear that \( q(X) = g(C) + 1 \) if and only if \( \chi(\mathcal{O}_X) = \deg L = \deg \mathcal{O}_C = 0 \). By results from (13), the case \( q(X) = g(C) \) corresponds to the case of elliptic surfaces with non-constant \( j \)-invariant. On the other hand, the case \( q(X) = g(C) + 1 \) corresponds to the case of elliptic surfaces with constant \( j \)-invariant; equivalently, all the generic fibers of \( \pi \) are isomorphic and the singular fibers are multiples of smooth elliptic curves.

**Corollary 5.2.4.** We continue to use the same notation as the proposition above. When \( q(X) = g(C) \), \( X \) has Albanese dimension 1, i.e. \( \dim \alpha(X) = 1 \). And when \( q(X) = g(C) + 1 \), \( X \) is of maximal Albanese dimension, i.e. \( \dim \alpha(X) = 2 \).
Proof. If \( q(X) = g(C) \), then \( \bar{\pi} \) is an isomorphism. In this case, we have

\[
\dim \alpha(X) = \dim \bar{\pi}(\alpha(X)) = \dim j(\pi(X)) = \dim j(C) = 1.
\]

Assume that \( q(X) = g(C) + 1 \). From the proof above, \( \alpha \) is generically finite; i.e. \( X \) is of maximal Albanese dimension.

Before considering the group \( \text{Pic}^0 \) of properly elliptic surfaces that have equivalent derived categories, we show the following relation between their automorphism groups. The idea of the proof is combining Theorem 2.3.9, Remark 2.3.8 and the analysis of the group \( \text{Aut}^0 \) of properly elliptic surface in Section 4.3.2.

**Proposition 5.2.5.** Let \( X \) and \( Y \) be properly elliptic surfaces with equivalent derived categories. Then \( G_X \cong G_Y \).

*Proof.* By Proposition 4.3.1, we can write \( X = (G_X \times Z_X)/F_X \) with \( F_X \) a finite subgroup of \( G_X \) and \( Z_X \) a smooth curve. Similarly, \( Y = (G_Y \times Z_Y)/F_Y \). Since \( D^b(X) \cong D^b(Y) \), \( X \) and \( Y \) are elliptic fibered over the same curve \( C \) with the generic fibers being isomorphic (Remark 2.3.10). Since \( X \) and \( Y \) are of Kodaira dimension 1, their elliptic fibrations are unique. Therefore \( Z_X/F_X \cong Z_Y/F_Y \) and \( G_X \cong G_Y \). 

We turn our attention now to the Picard varieties. We consider the two cases of Proposition 5.2.2. The first case is rather straightforward.
**Proposition 5.2.6.** Let $X$ and $Y$ be properly elliptic surfaces with equivalent derived categories, i.e. $D^b(X) \cong D^b(Y)$. Assume that $\chi(\mathcal{O}_X) \neq 0$. Then $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$.

**Proof.** Notice that $X$ and $Y$ are elliptic surfaces of non-zero Kodaira dimension. Hence, Theorem 2.3.8 applies and gives that $X$ and $Y$ are elliptically fibered over the same curve $C$ and their general fibers are isomorphic. Proposition 5.2.2 then gives that $\text{Alb}(X) \cong \text{Jac}(C) \cong \text{Alb}(Y)$. Since the Picard variety is the dual of the Albanese variety, $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$. \hfill $\Box$

There remains the case when $\chi(\mathcal{O}_X) = 0$ (i.e. the case of properly elliptic surface with constant $j$-invariant), in which we have:

**Theorem 5.2.7.** Let $X$ and $Y$ be properly elliptic surfaces with equivalent derived categories. Assume that $\chi(\mathcal{O}_X) = 0$. Then there exists the following commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \ker(i_X^*)_X & \rightarrow & \text{Pic}^0(X) & \rightarrow & \text{Pic}^0(G_X) & \rightarrow & 0 \\
\cong & & & f & & h & & & \\
0 & \rightarrow & \ker(i_Y^*)_Y & \rightarrow & \text{Pic}^0(Y) & \rightarrow & \text{Pic}^0(G_Y) & \rightarrow & 0
\end{array}
$$

where $f$ and $h$ are both isogenies having kernel $(\mathbb{Z}/c\mathbb{Z})^2$, $c$ is a positive integer such that $Y$ can be realized as $J^c(X)$, and $i_X, i_Y$ are the inclusion of generic fibers into $X$ and $Y$, respectively.

The proof of the above theorem follows from the following discussion and lemmas. First of all, Theorem 2.3.8 shows that $Y$ can be realized as $J^b(X)$ for some positive integer $b$. By symmetry, $X$ can be realized as $J^c(Y)$ for some positive integer $c$. Moreover, the derived equivalence $D^b(X) \rightarrow D^b(Y)$ can be represented by $\Phi_{\mathcal{E}}$ with $\mathcal{E}$ a sheaf supported on $X \times_C Y$. Theorem 2.3.6 gives that the sheaf $\mathcal{E}$ is flat over both factors, and for any point $(x, y) \in X \times Y$,
\( \mathcal{P}_y \) has Chern class \((0, f, -b)\) and \( \mathcal{P}_y \) has Chern class \((0, f, -c)\) on \( Y \). In other words, let \( p \in C \) be a general point. Let \( \mathcal{E}_p \) be the restriction of \( \mathcal{E} \) on \( X_p \times Y_p \) with \( X_p \) and \( Y_p \) the fibers of \( \pi_X \) and \( \pi_Y \) over \( p \), respectively. Then \( \mathcal{E}_p \) is a vector bundle that parametrizes line bundles of degree \( c \) on \( Y_p \) and line bundles of degree \( b \) on \( X_p \). Therefore, we have

**Lemma 5.2.8.** \( \Phi_{\mathcal{E}_p} : D^b(X_p) \to D^b(Y_p) \) is an equivalence.

Now, by the discussion in Section 4.3.2, by choosing points \( x \in X_p \) and \( y \in Y_p \), we have the identifications

\[
\text{Aut}^0(X_p) \cong X_p \cong G_X \quad \text{and} \quad \text{Aut}^0(Y_p) \cong Y_p \cong G_Y.
\]

More precisely, the action of \((\varphi, \psi) \in \text{Aut}^0(X_p) \times \text{Aut}^0(Y_p)\) on \( X_p \times Y_p \) is compatible with the action of \((\varphi, \psi)\) on \( X \times Y \), i.e. we have the following commutative diagram

\[
\begin{array}{ccc}
X_p \times Y_p & \xrightarrow{\varphi \times \psi} & X_p \times Y_p \\
\downarrow_{i_X \times i_Y} & & \downarrow_{i_X \times i_Y} \\
X \times Y & \xrightarrow{\varphi \times \psi} & X \times Y
\end{array}
\ \ (5.1)
\]

with \( i_X : X_p \hookrightarrow X \) and \( i_Y : Y_p \hookrightarrow Y \) being the inclusions.

By the discussion in Section 3.4, the equivalences \( \Phi_{\mathcal{E}} : D^b(X) \to D^b(Y) \) and \( \Phi_{\mathcal{E}_p} : D^b(X_p) \to D^b(Y_p) \) induce the Rouquier isomorphisms. We claim that those two isomorphisms are compatible in the following sense.
**Lemma 5.2.9.** There exists a commutative diagram

\[
\begin{array}{ccc}
\text{Aut}^0(X_p) \times \text{Pic}^0(X_p) & \xrightarrow{H} & \text{Aut}^0(Y_p) \times \text{Pic}^0(Y_p) \\
(id,i_X^\ast) & & (id,i_Y^\ast) \\
\text{Aut}^0(X) \times \text{Pic}^0(X) & \xrightarrow{F} & \text{Aut}^0(Y) \times \text{Pic}^0(Y) \\
\end{array}
\]

with $F$ and $H$ being the Rouquier isomorphisms induced by $\Phi_E$ and $\Phi_{E_p}$, respectively.

**Proof.** By Proposition 3.4.4, we have that $F(\varphi, L) = (\psi, M)$ if and only if

\[(\varphi \times \psi)^\ast E \cong (L^{-1} \boxtimes M) \otimes E\]

for all $(\varphi, L) \in G_X \times \text{Pic}^0(X)$ and $(\psi, M) \in G_Y \times \text{Pic}^0(Y)$. By the commutativity of the diagram (5.1) we have

\[(i_X \times i_Y)^\ast(\varphi \times \psi)^\ast E \cong (i_X \times i_Y)^\ast(L^{-1} \boxtimes M) \otimes (i_X \times i_Y)^\ast E\]

\[\iff (\varphi \times \psi)^\ast(i_X \times i_Y)^\ast E \cong (i_X^\ast(L^{-1}) \boxtimes i_Y^\ast M) \otimes (i_X \times i_Y)^\ast E.\]

\[\iff (\varphi \times \psi)^\ast E_p \cong (i_X^\ast(L^{-1}) \boxtimes i_Y^\ast M) \otimes E_p,\]

i.e $H(\varphi, i_X^\ast L) = (\psi, i_Y^\ast M)$. \qed
Notice that the restrictions \( i^*_X : \text{Pic}^0(X) \to \text{Pic}^0(X_p) \) and \( i^*_Y : \text{Pic}^0(Y) \to \text{Pic}^0(Y_p) \) are surjective. We will now consider the diagram

\[
\begin{array}{ccccccccc}
0 & \to & \ker(i^*_X) & \to & \text{Pic}^0(X) & \to & \text{Pic}^0(X_p) & \to & 0 \\
& & \downarrow g & & \downarrow f & & \downarrow h & & \\
0 & \to & \ker(i^*_Y) & \to & \text{Pic}^0(Y) & \to & \text{Pic}^0(Y_p) & \to & 0.
\end{array}
\]

(5.3)

where \( f \) is the composition

\[ f = \text{pr}_{\text{Pic}^0(Y)} \circ F(id, \_). \]

and \( h \) is the composition

\[ h = \text{pr}_{\text{Pic}^0(Y_p)} \circ H(id, \_). \]

**Lemma 5.2.10.** The morphism \( f \) maps \( \ker(i^*_X) \) isomorphically onto \( \ker(i^*_Y) \).

**Proof.** Let \( L \in \text{Pic}^0(X) \), and let \( F(id, L) \cong (\psi, M) \in G_Y \times \text{Pic}^0(Y) \). We claim that \( L \in \ker(i^*_X) \) if and only if \( \psi = \text{id} \) and \( M \in \ker(i^*_Y) \). Assume that \( L \in \ker(i^*_X) \). Then

\[
H(id, i^*_X L) = (\psi, i^*_Y M)
\]

\[
\implies H(id, \mathcal{O}_{X_p}) = (\psi, i^*_Y M).
\]
Since $H$ is an isomorphism of algebraic groups, $(\psi, i_Y^* M) = (\text{id}, \mathcal{O}_{Y_p})$. Now assume that $F(\text{id}, L) = (\text{id}, M)$ with $i_Y^* M = \mathcal{O}_{Y_p}$. Then

$$H(\text{id}, i_X^* L) = (\text{id}, i_Y^* M)$$

$$\implies \mathcal{E}_p \cong (q^* i_X^* L^{-1} \otimes p^* i_Y^* M) \otimes \mathcal{E}_p \ (\text{by Proposition 3.4.4})$$

$$\implies q^* i_X^* L \cong p^* i_Y^* M \ (\text{since } \mathcal{E}_p \text{ is a line bundle})$$

$$\implies i_X^* L = \mathcal{O}_{X_p}.$$ 

From this, it follows that $f$ maps $\ker(i_X^*)$ isomorphically onto $\ker(i_Y^*)$. \hfill \qed

Lemma 5.2.10 shows that the morphism $g$ is just the restriction of $f$ onto $\ker(i_X^*)$; therefore, the diagram (5.3) commutes. Furthermore, by making the identification $X_p \cong G_X$ and $Y_p \cong G_Y$, diagram (5.3) is precisely the diagram in Theorem 5.2.7. We now need only to prove the second part of Theorem 5.2.7.

First let us recall the equivalence $D^b(X_p) \cong D^b(Y_p)$ given by $\Phi_{\mathcal{E}_p}$. Moreover, $\mathcal{E}_p$ parametrizes line bundles of degree $c$ on $Y_p$ and line bundles of degree $b$ on $X_p$. Once we fix a point $x \in X_p$ and $y \in Y_p$, we have the following identifications

$$Y_p \cong \text{Pic}^b(X_p),$$

and by symmetry,

$$X_p \cong \text{Pic}^c(Y_p).$$
The choice of $x$ and $y$ also allows us to identify $Y_p \cong \text{Pic}^0(X_p) = \widehat{X}_p$. The universal property of the vector bundle $E_p$ implies that we can choose $E_p$ to have the form

$$E_p \cong \mathcal{P} \otimes q^*L \otimes p^*M$$

with $\mathcal{P}$ the Poincaré bundle on $X_p \times \widehat{X}_p$, $L \in \text{Pic}^h(X_p)$ and $M \in \text{Pic}^c(Y_p)$.

Proof of Theorem 5.2.7. The fact that $g$ is an isomorphism and the commutativity of diagram (5.3) imply that

$$\ker(f) \cong \ker(h),$$

and

$$\coker(f) \cong \coker(h).$$

Since $h$ is a morphism between two elliptic curves, it is either an isogeny or a trivial morphism. When $h$ is trivial then $\ker(h) = \text{Pic}^0(G_X) \cong \ker(f)$ and $\coker(h) = \text{Pic}^0(G_Y) \cong \coker(f)$. In that case, horizontal exact sequence splits and we obtain

$$\text{Pic}^0(X) = \ker(i_X^* \chi) \times \text{Pic}^0(G_X),$$

and

$$\text{Pic}^0(Y) = \ker(i_Y^* \gamma) \times \text{Pic}^0(G_Y).$$
By Proposition 5.2.5, $G_X \cong G_Y$; hence, $\text{Pic}^0(X) \cong \text{Pic}^0(Y)$. From now on, assume that $h$ is surjective. It remains to compute $\ker(h)$. By the commutative diagram (5.2) and the definition of $h$, it is clear that $h$ is the composition

$$
\begin{array}{ccc}
\text{Pic}^0(X_p) & \longrightarrow & \text{Aut}^0(X_p) \times \text{Pic}^0(X_p) \\
L & \longrightarrow & (\text{id}, L)
\end{array}
\text{Aut}^0(Y_p) \times \text{Pic}^0(Y_p) \xrightarrow{pr_{\text{Pic}^0(Y_p)}} \text{Pic}^0(Y_p)
$$

Since the equivalence $D^b(X_p) \cong D^b(Y_p)$ can be seen as the equivalence between elliptic curves $D^b(X_p) \cong D^b(\hat{X}_p)$ given by the transform $\Phi_{E_p}$ with $\mathcal{E}_p \cong \mathcal{P} \otimes q^*L \otimes p^*M$, we can apply the computation in Example 3.2.8 and obtain that $H$ is given by the matrix

$$
H = \begin{pmatrix}
\varphi_{L^{-1}} & -\text{id}_{X_p} \\
-\hat{\varphi}_{M^{-1}} \circ \varphi_{L^{-1}} & \hat{\varphi}_{M^{-1}}
\end{pmatrix}.
$$

It follows that $h(L) = H(\text{id}, L) = (\psi, \hat{\varphi}_{M^{-1}}(L))$ for some $\psi \in \text{Aut}^0(Y_p)$; thus, $L \in \ker(h)$ if and only if $L \in \ker(\hat{\varphi}_{M^{-1}})$. Since $M \in \text{Pic}^c(Y_p)$, we have

$$
\ker(h) = \ker(\hat{\varphi}_{M^{-1}}) \cong (\mathbb{Z}/c\mathbb{Z})^2.
$$

We then conclude that $\ker(f) \cong \ker(h) \cong (\mathbb{Z}/c\mathbb{Z})^2$. 

5.3 Conclusions

In this section, we will prove the following theorem

**Theorem 5.3.1.** Let $X$ and $Y$ be smooth projective surfaces such that $D^b(X) \cong D^b(Y)$. Then
a. \( G_X \cong G_Y \), unless \( X \) and \( Y \) are abelian surfaces, in which case \( D^b(G_X) \cong D^b(G_Y) \),

b. \( D^b(\text{Pic}^0(X)) \cong D^b(\text{Pic}^0(Y)) \) with the possible exception of the case when \( X \) is a properly elliptic surface with constant \( j \)-invariant, which is the case of Theorem 5.2.7, and

c. \( X \) and \( Y \) have the same Albanese dimension.

Remark 5.3.2. We should notice here that the condition of constant \( j \)-invariant is equivalent to the fact that the all the smooth fibers of the elliptic fibration of \( X \) are isomorphic and the only singular fibers are multiples of smooth elliptic curves.

Proof of Theorem 5.3.1. We start with the case of \( X \) being non-minimal surface. Applying Theorem 2.3.1, we have that either

- \( X \cong Y \), or

- \( X \) and \( Y \) are relatively minimal elliptic rational surfaces.

Theorem 5.3.1 clearly holds in the first case. In the second case, we have \( q(X) = q(Y) = 0 \); hence, (b) and (c) are trivial. By Proposition 4.1.6, we have \( \dim \text{Alb}(G_X) \leq q(X) = 0 \). Hence, \( G_X \) and \( G_Y \) are affine algebraic groups. Since the isomorphism (5.5) below also apply in the case of non-minimal surfaces, it follows that \( G_X \cong G_Y \). Thus, we can assume that \( X \) and \( Y \) are minimal surfaces.

We start with the Rouquier isomorphism induced by the equivalence \( D^b(X) \cong D^b(Y) \)

\[ G_X \times \text{Pic}^0(X) \cong G_Y \times \text{Pic}^0(Y). \]
Applying Chevalley’s Structure Theorem (Theorem 4.1.1), we have

\[ \text{Alb}(G_X) \times \text{Pic}^0(X) \cong \text{Alb}(G_Y) \times \text{Pic}^0(Y), \quad (5.4) \]

and

\[ \text{Aff}(G_X) \cong \text{Aff}(G_Y). \quad (5.5) \]

By Theorem A of (33), we know that \( q(X) = q(Y) \); hence, \( \dim \text{Alb}(G_X) = \dim \text{Alb}(G_Y) \).

Next, recall from Chapter 4 that the action of \( G_X \) on \( X \) induces a morphism with finite kernel

\[ \varphi : \text{Alb}(G_X) \to \text{Alb}(X). \]

Lemma 4.2.1 gives

\[ \dim \text{Alb}(G_X) \leq \dim \text{im}(\alpha_X) \leq \dim X = 2. \]

We have the following cases:

Case 1: \( \dim \text{Alb}(G_X) = \dim \text{Alb}(G_Y) = 0 \), i.e. \( G_X \) and \( G_Y \) are affine algebraic groups.

Since \( \dim \text{Alb}(G_X) = \dim \text{Alb}(G_Y) = 0 \), (5.4) implies that \( \text{Pic}^0(X) \cong \text{Pic}^0(Y) \). If \( \dim \text{Aff}(G_X) = \dim \text{Aff}(G_Y) = 0 \), then \( G_X \cong G_Y = \{\text{id}\} \). If \( \text{Aff}(G_X) \) and \( \text{Aff}(G_Y) \) are of positive dimension, then by Corollary 4.1.6, \( X \) and \( Y \) are ruled surfaces; hence, the theorem follows from Corollary 5.1.3.

Case 2: \( \dim \text{Alb}(G_X) = \dim \text{Alb}(G_Y) = 2 \).

By Proposition 4.2.5, \( X \) and \( Y \) are abelian surfaces, and the theorem follows.
Case 3: \( \dim \text{Alb}(G_X) = \dim \text{Alb}(G_Y) = 1 \).

By Proposition 4.3.1, \( X \) admits a fibration \( p : X \rightarrow \text{Alb}(G_X)/F \) with the fiber \( Z \). In this case, the fiber \( Z \) is a smooth curve. And we proceed by considering the genus of \( Z \).

- **Case 3(i):** \( g(Z) = 0 \)
  
  In this case, \( X \) is a ruled surface. Then \( Y \) is also a ruled surface. Again the theorem then follows from Corollary 5.1.3.

- **Case 3(ii):** \( g(Z) = 1 \)
  
  This is the case when \( \kod(X) = 0 \). It follows that \( X \) is a bi-elliptic surface. Theorem 2.3.3 implies that \( X \cong Y \); hence, the theorem holds.

- **Case 3(iii):** \( g(Z) > 1 \)
  
  In this case, we have \( \kod(X) = 1 \), i.e. \( X \) is a properly elliptic surface. Then (a) follows from Proposition 5.2.5. Corollary 5.2.4 shows that both \( X \) and \( Y \) are either of Albanese dimension 1 or Albanese dimension 2 depending on \( q(X) = q(Y) \); hence, (c) follows. And we obtain (b) by Proposition 5.2.6 and Theorem 5.2.7.
CITED LITERATURE


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